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PRE-SYMPLECTIC STRUCTURE ON THE SPACE OF CONNECTIONS

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Abstract. Let X be a four-manifold with boundary three-manifold M. We shall describe (i) a pre-symplectic structure on the sapce $\mathcal{A}(X)$ of connections on the bundle $X \times \mathrm{SU}(n)$ that comes from the canonical symplectic structure on the cotangent space $T^*\mathcal{A}(X)$. By the boundary restriction of this pre-symplectic structure we obtain a pre-symplectic structure on the space $\mathcal{A}_0^\flat(M)$ of flat connections on $M \times \mathrm{SU}(n)$ that have null charge.

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1. Introduction

Let X be an oriented Riemannian four-manifold with boundary $M = \partial X$. For the trivial principal bundle $P = X \times \mathrm{SU}(n)$ we denote by $\mathcal{A}(X)$ the space of irrreducible connections on X. The following theorems are proved.

Theorem 1. *Let $P = X \times \mathrm{SU}(n)$ be the trivial $\mathrm{SU}(n)-$principal bundle on a four-manifold X. There exists a canonical pre-symplectic structure on the space of irreducible connections $\mathcal{A}(X)$ given by the two-form*

$$\sigma_A^s(a,b) = \frac{1}{8\pi^3}\int_X \mathrm{Tr}[(ab-ba)F_A] - \frac{1}{24\pi^3}\int_M \mathrm{Tr}[(ab-ba)A]$$

for $a, b \in T_A\mathcal{A}(X)$.

Theorem 2. *Let ω be a two-form on $\mathcal{A}(M)$ defined by*

$$\omega_A(a,b) = -\frac{1}{24\pi^3}\int_M \mathrm{Tr}[(ab-ba)A]$$

for $a, b \in T_A\mathcal{A}(M)$. *Let*

$$\mathcal{A}_0^\flat(M) = \left\{ A \in \mathcal{A}(M) ;\ F_A = 0\,,\ \int_M \mathrm{Tr}\ A^3\ = 0 \right\}.$$

Then $\left(\mathcal{A}_0^\flat(M),\, \omega|_{\mathcal{A}_0^\flat(M)}\right)$ *is a pre-symplectic manifold.*

In this exposition we shall explain Theorem 1 and its background comming from the symplectic structure on the cotangent space of $\mathcal{A}(X)$. The proof of Theorem2 needs an observation about the extension of flat connections from the boundary three manifold M to the four manifold X, and needs a long discussion. The detailed discussion was developed in the arXiv note [4]. This is a part of the author's research on geometric quantization of connection spaces over a four manifold. The previous results are appeared in [3]. There we proved the following theorems

Theorem 3. *Let* $\mathcal{G}_0(X)$ *be the group of gauge transformations on* X *that are identity on the boundary* M. *The action of* $\mathcal{G}_0(X)$ *on* $\mathcal{A}(X)$ *is a Hamiltonian action and the corresponding moment map is given by*

$$\Phi\ \ :\ \ \mathcal{A}(X) \longrightarrow (Lie\,\mathcal{G}_0)^* = \Omega^4(X, Lie\,G) :\ A \longrightarrow F_A^2$$

$$\langle \Phi(A), \xi \rangle\ =\ \Phi^\xi(A)\ =\ \frac{1}{8\pi^3} \int_X \mathrm{Tr}(F_A^2 \xi), \qquad \xi \in Lie\,\mathcal{G}_0(X)\,.$$

For a manifold X endowed with a closed two-form σ, we call a *pre-quantization* of (X, σ) a hermitian line bundle $(\mathbf{L}, \langle\,,\rangle)$ over X equiped with a hermitian connection ∇ whose curvature is σ, [2] .

Theorem 4. *There exists a pre-quantization of the moduli space of the pre-symplectic manifold* $\left(\mathcal{M}^\flat = \mathcal{A}^\flat(X)/\mathcal{G}_0(X),\, \omega\,\right)$, *that is, there exists a hermitian line bundle with connection* $\mathcal{L}^\flat \longrightarrow \mathcal{M}^\flat$, *whose curvature is equal to the pre-symplectic form* $\mathrm{i}\,\omega$,
where

$$\mathcal{A}^\flat(X) = \{ A \in \mathcal{A}(X) ;\ F_A = 0\, \}$$

and the closed two-form ω *on* $\mathcal{M}^\flat$ *is induced from that on* $\mathcal{A}^\flat(X)$ *as the boundary value of* σ^s (*and then as the quotient*)

$$\omega_A(a, b) =\ -\frac{1}{24\pi^3} \int_M \mathrm{Tr}[(ab - ba)A]\,.$$

2. Space of Connections

Let M be a compact, connected and oriented m-dimensional riemannian manifold with boundary ∂M. Let $G = \mathrm{SU}(N)$, $N \geq 2$ and let $P \xrightarrow{\pi} M$ be a principal G-bundle. $\mathcal{A} = \mathcal{A}(M)$ denotes the space of *irreducible* connections over P. $T_A\mathcal{A} =$

$\Omega^1(M, Lie\,G)$ is the tangent space at $\mathcal{A}$. Let $T^*_A\mathcal{A} = \Omega^{m-1}(M, Lie\,G)$ be the cotangent space at $A \in \mathcal{A}$. The pairing of $\alpha \in T^*_A\mathcal{A}$ and $a \in T_A\mathcal{A}$ is given by

$$\langle \alpha,\, a\rangle_A = \int_M \mathrm{tr}(\, a \wedge \alpha\,).$$

For a function $F = F(A)$ on $\mathcal{A}$ valued in a vector space V, the derivation $\partial_A F : T_A\mathcal{A} \longrightarrow V$ is defined by the functional variation of $A \in \mathcal{A}$

$$(\partial_A F)a = \lim_{t\longrightarrow 0} \frac{1}{t}\,(\,F(A+ta) - F(A)\,)\,, \qquad a \in T_A\mathcal{A}.$$

For example, $(\partial_A A)\,a = a$. The curvature of $A \in \mathcal{A}$ is by definition $F_A = \mathrm{d}A + \frac{1}{2}[A\wedge A] \in \Omega^2_{s-2}(M, Lie\,G)$, and we have $(\partial_A F_A)a = \mathrm{d}_A a$. The derivations of a vector field and a one-form φ are defined similarly. We have the following formulas

$$[\,\mathbf{v},\, \mathbf{w}\,]_A \;=\; (\partial_A \mathbf{v})\mathbf{w}_A - (\partial_A \mathbf{w})\mathbf{v}_A$$

$$\mathbf{v}\langle \varphi, \mathbf{u}\rangle(A) \;=\; \langle \varphi_A, (\partial_A \mathbf{u})\mathbf{v}_A\rangle + \langle (\partial_A \varphi)\,\mathbf{v}_A, \mathbf{u}_A\rangle.$$

The exterior derivative $\widetilde{\mathrm{d}}$ on $\mathcal{A}(M)$ is defined as follows. For a function F on $\mathcal{A}(M)$ $(\widetilde{\mathrm{d}}F)_A\, a = (\partial_A F)\, a$. For a one-form Φ on $\mathcal{A}(M)$

$$\begin{aligned}(\widetilde{\mathrm{d}}\,\Phi)_A(\mathbf{a},\mathbf{b}) \;&=\; (\partial_A\langle \Phi, \mathbf{b}\rangle)\mathbf{a} - (\partial_A\langle \Phi, \mathbf{a}\rangle)\mathbf{b} - \langle \Phi, [\mathbf{a},\mathbf{b}]\rangle \\ &=\; \langle (\partial_A \Phi)\mathbf{a}, \mathbf{b}\rangle - \langle (\partial_A \Phi)\mathbf{b}, \mathbf{a}\rangle.\end{aligned}$$

For a two-form φ

$$(\widetilde{\mathrm{d}}\,\varphi)_A(\mathbf{a},\mathbf{b},\mathbf{c}) = (\partial_A\varphi(\mathbf{b},\mathbf{c}))\mathbf{a} + (\partial_A\varphi(\mathbf{c},\mathbf{a}))\mathbf{b} + (\partial_A\varphi(\mathbf{a},\mathbf{b}))\mathbf{c}\,.$$

For a function $\Phi = \Phi(A,\lambda)$ of $(A,\lambda) \in T^*\mathcal{A}$, the directional derivative $\delta_A\Phi \in T^*\mathcal{A}$ at $(A,\lambda) \in T^*\mathcal{A}$ to the direction $a \in T_A\mathcal{A}$ is given by

$$\langle \delta_A\Phi,\, a\rangle_A \;= \lim_{t\longrightarrow 0} \frac{1}{t}(\Phi(A+ta,\lambda) - \Phi(A,\lambda)).$$

Similarly $\delta_\lambda\Phi \in T^*\mathcal{A}$ to the direction $\lambda \in T_A\mathcal{A}$ is defined. Then the exterior differential of Φ on $T^*\mathcal{A}$ is given by

$$(\,\widetilde{\mathrm{d}}\,\Phi\,)_{(A,\lambda)}\begin{pmatrix} a \\ \alpha \end{pmatrix} \;=\; \langle \delta_A\Phi,\, a\rangle_A \;+\; \langle \alpha\,,\, \delta_\lambda\Phi\rangle_A\,, \qquad \begin{pmatrix} a \\ \alpha \end{pmatrix} \in T_{(A,\lambda)}T^*\mathcal{A}\,. \tag{1}$$

3. Canonical One-Form and Two-Form on $T^*\mathcal{A}$

The followings are *standard facts on the cotangent bundle of any manifold* applied to our space of connections $\mathcal{A}$. Tangent space to the cotangent bundle $T^*\mathcal{A}$ at the point $(A,\lambda) \in T^*\mathcal{A}$ is

$$T_{(A,\lambda)}T^*\mathcal{A} = T_A\mathcal{A} \oplus T^*_\lambda\mathcal{A} = \Omega^1(M,\, Lie\,G) \oplus \Omega^{m-1}(M,\, Lie\,G).$$

The canonical one-form θ on $T^*\mathcal{A}$ is defined by

$$\theta_{(A,\lambda)}(\begin{pmatrix} a \\ \alpha \end{pmatrix}) = \langle\, \lambda, \pi_* \begin{pmatrix} a \\ \alpha \end{pmatrix} \rangle_A = \int_M \mathrm{tr}\ a \wedge \lambda, \quad \begin{pmatrix} a \\ \alpha \end{pmatrix} \in T_{(A,\lambda)} T^*\mathcal{A}.$$

For a one-form ϕ on $\mathcal{A}$, we have $\phi^*\theta = \phi$, and the derivation of the one-form θ is given by

$$\partial_{(A,\lambda)}\, \theta \begin{pmatrix} a \\ \alpha \end{pmatrix} = \langle \alpha\,, a\rangle, \qquad (a,\, \alpha) \in T_{(A,\lambda)} T^*\mathcal{A}\,.$$

The canonical two-form is defind by

$$\sigma = \widetilde{\mathrm{d}}\, \theta.$$

We have

$$\sigma_{(A,\lambda)} \left(\begin{pmatrix} a \\ \alpha \end{pmatrix}, \begin{pmatrix} b \\ \beta \end{pmatrix} \right) = \langle\, \alpha, b\,\rangle_A - \langle\, \beta, a\,\rangle_A = \int_M \mathrm{tr}[\, b \wedge \alpha - a \wedge \beta\,]\,.$$

σ is a *non-degenerate* closed two-form on the cotangent space $T^*\mathcal{A}$.
For a function $\Phi = \Phi(A, \lambda)$ on $T^*\mathcal{A}$ corresponds the Hamitonian vector field X_Φ

$$(\,\widetilde{\mathrm{d}}\,\Phi\,)_{(A,\lambda)} \;=\; \sigma(\,(X_\Phi)_{(A,\lambda)},\,\cdot\,).$$

The formula (1) implies that the Hamiltonian vector field of Φ is given by

$$X_\Phi = \begin{pmatrix} -\delta_\lambda \Phi \\ \delta_A \Phi \end{pmatrix}.$$

Now let $\mathcal{G}(M)$ be the group of (pointed) gauge transformations

$$\mathcal{G}(M) = \,\{ g \in \Omega^0_s(M, G)\ ;\ g(p_0) = 1\}.$$

The group $\mathcal{G}(M)$ acts freely on $\mathcal{A}(M)$ by

$$g \cdot A = g^{-1}\mathrm{d}g + g^{-1}Ag = A + g^{-1}\mathrm{d}_A g.$$

Then $\mathcal{G}(M) = \Omega^0_s(M,\, Lie\, G)$ acts on $T_A\mathcal{A}$ by ; $a \longrightarrow \mathrm{Ad}_{g^{-1}}\, a = g^{-1}ag$, and acts on $T^*_A\mathcal{A}$ by its dual $\alpha \longrightarrow g\alpha g^{-1}$. Hence the canonical one-form θ and two-form σ are $\mathcal{G}(M)$-invariant.
The infinitesimal action of $\xi \in Lie\,\mathcal{G}(M)$ on $T^*\mathcal{A}$ gives a vector field $\xi_{T^*\mathcal{A}}$ (called fundamental vector field) on $T^*\mathcal{A}$.

$$\xi_{T^*\mathcal{A}}(A, \lambda) = \frac{\mathrm{d}}{\mathrm{d}t}\ \exp t\xi \cdot \begin{pmatrix} A \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathrm{d}_A \xi \\ [\xi, \lambda] \end{pmatrix}$$

at $(A, \lambda) \in T^*\mathcal{A}$. For each $\xi \in Lie\,\mathcal{G}$ we define the function

$$\mathbf{J}^\xi(A, \lambda) = \,\theta_{(A,\lambda)}\,(\xi_{T^*\mathcal{A}}) \;=\int_M \mathrm{tr}\,(\,\mathrm{d}_A\xi \wedge\, \lambda)\,. \tag{2}$$

Then $\mathbf{J}(A, \lambda) \in (\,Lie\,\mathcal{G}\,)^*$ and (2) yields

$$\widetilde{\mathrm{d}}\,\mathbf{J}^{\xi} \;=\; \sigma(\,\xi_{T^*\mathcal{A}},\,\cdot\,), \qquad \xi \in Lie\,\mathcal{G}.$$

Theorem 5. *The action of the group of gauge transformations $\mathcal{G}(M)$ on the symplectic space $(\,T^*\mathcal{A}(M),\,\sigma\,)$ is an hamiltonian action and the moment map is given by*

$$\mathbf{J}^{\xi}(A, \lambda) = \int_M \mathrm{tr}\,(\,\mathrm{d}_A\xi \wedge\,\lambda)\,.$$

4. Generating Functions

Let $\tilde{s}\,:\,\mathcal{A}\,\longrightarrow\,T^*\mathcal{A}$ is a local section of $T^*\mathcal{A}$. We write it by $\tilde{s}(A) = (A,\,s(A))$ with $s(A) \in T^*_A\mathcal{A}$.
The pullback of the canonical one-form θ by $\tilde{s}$ defines a one-form θ^s on $\mathcal{A}$

$$\theta^s_A(a) = \,(\widetilde{s}^*\,\theta)_A a, \qquad a \in T_A\mathcal{A}\,.$$

Lemma 6.

$$\theta^s = s.$$

That is

$$(\theta^s)_A a = \langle s(A),\,a\rangle$$

for $a \in T_A\mathcal{A}$.

Let $\sigma^s = \tilde{s}^*\sigma$ be the pullback by $\tilde{s}$ of the canonical two-form σ.

$$\sigma^s_A(a, b) = \sigma_{\tilde{s}(A)}(\tilde{s}_*a,\,\tilde{s}_*b) = \sigma_{(A,s(A))}\left(\begin{pmatrix} a \\ (s_*)_A a\end{pmatrix}, \begin{pmatrix} b \\ (s_*)_A b\end{pmatrix}\right)$$

σ^s is a closed two-form on $\mathcal{A}$. From Lemma 6 we see that

$$\sigma^s = \widetilde{\mathrm{d}}\,s\,.$$

Example 1 (Atiyah-Bott [1])**.** Let M be a surface (two-dimensional manifold) and

$$T_A\mathcal{A} \simeq\, T^*_A\mathcal{A} \simeq \Omega^1(M, LieG).$$

Define the generating function

$$s:\,\mathcal{A} \ni A \longrightarrow s(A) = A \in \Omega^1(M, LieG) = T^*_A\mathcal{A}.$$

Then

$$(\theta^s)_A a = \int_M \mathrm{tr}(\,Aa\,)$$

$$\begin{aligned} \omega_A(a, b) \;\equiv\; \sigma^s_A(a, b) &= (\widetilde{\mathrm{d}}\,\theta^s)_A(a, b) = \langle(\partial_A\theta^s)a, b\rangle - \langle(\partial_A\theta^s)b, a\rangle \\ &= \int_M \mathrm{tr}(ba)\,-\,\int_M \mathrm{tr}(ab) = 2\int_M \mathrm{tr}(ba). \end{aligned}$$

Then $(\mathcal{A}(M),\,\omega\,)$ is a symplectic manifold, in fact ω is non-degenerate.

5. Pre-symplectic Structure on the Space of Connections on a Four-Manifold

Let X is a Riemannian four-manifold with boundary $M = \partial X$ that may be empty, $P = X \times \mathrm{SU}(n)$ is the trivial principal bundle and $\mathcal{A}(X)$ is the tangent space to the space of irreducible $L^2_{s-\frac{1}{2}}$-connections

$$T_A\mathcal{A}(X) = \Omega^1_{s-\frac{1}{2}}(X, Lie\, G).$$

Let $\tilde{s}$ be a section of the cotangent bundle

$$\tilde{s}(A) = (A, s(A)) = \Big(A\,,\, q(\, AF_A + F_A A - \frac{1}{2}A^3) \Big)$$

where $s(A) = q(AF_A + F_A A - \frac{1}{2}A^3)$ is a three-form on X valued in $\mathfrak{su}(n)$, $q_3 = \frac{1}{24\pi^3}$.

Lemma 7. *Let $\theta^s = \tilde{s}^*\theta$ and $\sigma^s = \tilde{s}^*\sigma$ be the pullback of the canonical one and two forms by $\tilde{s}$. Then we have*

$$\theta^s_A(a) = \frac{1}{24\pi^3}\int_X \mathrm{Tr}[\,(AF + FA - \frac{1}{2}A^3)\,a\,], \qquad a \in T_A\mathcal{A}$$

and

$$\sigma^s_A(a,b) = \frac{1}{8\pi^3}\int_X \mathrm{Tr}[(ab-ba)F] - \frac{1}{24\pi^3}\int_{\partial M} \mathrm{Tr}[(ab-ba)A]\,.$$

Proof: The first equation follows from the very definition, $(\widetilde{s}^*\theta)_A a = \langle s(A), a\rangle$. For $a, b \in T_A\mathcal{A}$

$$\begin{aligned}(\widetilde{\mathrm{d}}\,\theta^s)_A(a,b) &= \langle(\partial_A\theta^s)a, b\rangle - \langle(\partial_A\theta^s)b, a\rangle \\ &= \frac{1}{24\pi^3}\int_X \mathrm{Tr}[\,2(ab-ba)F - (ab-ba)A^2 \\ &\qquad\qquad - (b\,\mathrm{d}_A a + \mathrm{d}_A ab - \mathrm{d}_A ba - a\,\mathrm{d}_A b)A\,].\end{aligned}$$

Since

$$\mathrm{d}\ \mathrm{Tr}[(ab-ba)A] = \mathrm{Tr}[(b\,\mathrm{d}_A a + \mathrm{d}_A a\, b - \mathrm{d}_A b\, a - a\,\mathrm{d}_A b)A] + \mathrm{Tr}[(ab-ba)(F+A^2)]$$

we have

$$\sigma^s_A(a,b) = \frac{1}{8\pi^3}\int_X \mathrm{Tr}[\,(ab-ba)F\,] - \frac{1}{24\pi^3}\int_M \mathrm{Tr}[\,(ab-ba)A\,]$$

for $a, b \in T_A\mathcal{A}$. Thus Theorem 1 is proved. ∎

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