GREEN'S FUNCTION, WAVEFUNCTION AND WIGNER FUNCTION OF THE MIC-KEPLER PROBLEM

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Abstract. The phase-space formulation of the nonrelativistic quantum mechanics is constructed on the basis of a deformation of the classical mechanics by the $*$-product. We have taken up the MIC-Kepler problem in which Iwai and Uwano have interpreted its wave-function as the cross section of complex line bundle associated with a principal fibre bundle in the conventional operator formalism. We show that its Green's function, which is derived from the $*$-exponential corresponds to unitary operator through the Weyl application, is equal to the infinite series that consists of its wave-functions. Finally, we obtain its Wigner function.

1. Introduction

We come to the reluctant conclusion that in our previous paper [5] we obtained only a piece of the local expression of the Green's function for the MIC-Kepler problem. There (Theorem 12) we have presented two expressions denoted by $G_+(\boldsymbol{r}_f, \boldsymbol{r}_i; E)$ and $G_-(\tilde{\boldsymbol{r}}_f, \tilde{\boldsymbol{r}}_i; E)$ where $\boldsymbol{r} = \tilde{\boldsymbol{r}}$ means the position vector $\boldsymbol{x}$ in $\dot{\mathbb{R}}^3 = \mathbb{R}^3 \backslash \{0\}$ i.e., $\boldsymbol{r} = (x, y, z)$. However, $G_-(\tilde{\boldsymbol{r}}_f, \tilde{\boldsymbol{r}}_i; E)$ is actually identical with $G_+(\boldsymbol{r}_f, \boldsymbol{r}_i; E)$ because the transition function is constant (independent of $\boldsymbol{x}$) and therefore, despite the difference in appearance, τ_- is essentially the same local trivialization as τ_+. This is the reason why $G_-(\tilde{\boldsymbol{r}}_f, \tilde{\boldsymbol{r}}_i; E)$ became equivalent to $G_+(\boldsymbol{r}_f, \boldsymbol{r}_i; E)$ in the case of iii). After that we have succeeded in obtaining the other piece of the local expression denoted by $G_-(\boldsymbol{x}_f, \boldsymbol{x}_i; E)$ via of finding another local trivialization τ_- which is transformed into τ_+ by the transition function of principal S^1 bundle varying with the position (more precisely, the longitudinal angle) of point $\boldsymbol{x}$ (see [4]). We have found, in addition, the wave-function of

the MIC-Kepler problem. In this paper, by turning the right-hand system of orthogonal curvilinear local coordinates on U_- into the left-hand one, we obtain the Green's function and wave-function in a new form. In this way we end up with two left-handed coordinate systems bringing the two local trivializations which are transformed into each other by the transition function of the principal S^1 bundle. Thus it becomes possible to obtain its Wigner function on $T^*(U_+ \cap U_-) \subset T^*\dot{\mathbb{R}}^3$. The energy-eigenfunction on the phase space is called Wigner function, and we found that of the MIC-Kepler problem on the reduced phase-space of $T^*\dot{\mathbb{R}}^4$ by solving the $*$-characteristic equations for its energy and angular momentum where $*$ denotes the Moyal product generated from the canonical coordinates bringing the standard symplectic form on $T^*\dot{\mathbb{R}}^4$ (see [5, Theorem 10]). How they could be expressed in each of the local coordinate systems on $T^*\dot{\mathbb{R}}^3$ is an interesting question, which we have succeeded to answer.

The contents of this paper is as follows. In Section 2 we indicate our conclusive results of the Green's function and wave-function for the MIC-Kepler problem, where it comes to be apparent that the Green's function is not a function existing globaly on the configuration space $\dot{\mathbb{R}}^3$ but a cross section in the complex line bundles over $\dot{\mathbb{R}}^3$ which have been introduced by Iwai and Uwano [3]. In Section 3, we express the Hamiltonian system of the MIC-Kepler problem in terms of the spherical coordinates and their conjugate momentums to obtain its Wigner function on the phase space $T^*\dot{\mathbb{R}}^3$ without z-axis.

2. Green's Function and Wavefunction

The MIC-Kepler problem is the reduced Hamiltonian system of the four-dimensional conformal Kepler problem by an S^1 action, if the associated momentum mapping equals to some fixed value μ which stands for the strength of Dirac's monopole field [2]. Then the Green's function of the MIC-Kepler problem is obtained by reducing that one of the conformal Kepler problem which have been already shown in [4] and [5]. Here we have found another kind of local coordinate system especially reconsidering one side of the local trivializations, denoted by τ_-, derived from the open subset U_- in $\dot{\mathbb{R}}^3 = U_+ \cup U_-$ as it is necessary to make it transformable into the other side τ_+ through the transition function of the principal fibre bundle $g_{-+} : U_+ \cap U_- \ni \boldsymbol{x} \longmapsto \mathrm{e}^{\mathrm{i}\phi(\boldsymbol{x})} = \mathrm{e}^{-\mathrm{i}\tilde{\phi}(\boldsymbol{x})} \in S^1$ (see [4]) and it is also necessary to alter the orientation of the orthogonal curvilinear local coordinate system on U_- as that on one U_+ i.e., anti-clockwise. The details of the latest local coordinate system are as follows (see Fig. 1).

Let U_+ be an open subset without negative z-axis such that

$$U_+ = \left\{ \boldsymbol{x}(r, \theta, \phi) \in \dot{\mathbb{R}}^3 \,;\, r > 0 \,,\, 0 \le \theta < \pi \,,\, 0 \le \phi < 2\pi \right\}$$
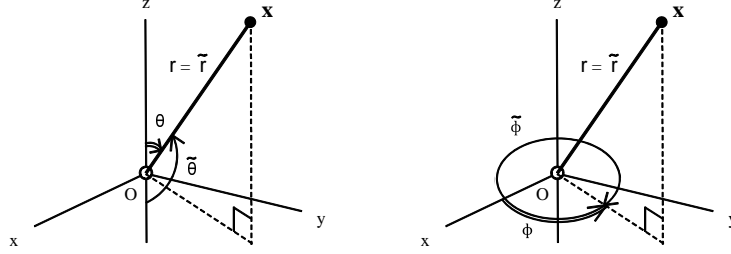
Figure 1. The configuration space $\dot{\mathbb{R}}^3 = \mathbb{R}^3 \backslash \{0\}$.

bringing the following local trivialization where π is the bundle projection and ν has the range of values $0 \leq \nu < 4\pi$

$$\tau_+ : \pi^{-1}(U_+) \ni \boldsymbol{u} \longmapsto (\pi(\boldsymbol{u}), \varphi_+(\boldsymbol{u})) = (\boldsymbol{x}(r, \theta, \phi), \exp(i\nu/2)) \in U_+ \times S^1$$

$$
\begin{array}{l}
x = r \sin\theta \cos\phi \\
y = r \sin\theta \sin\phi \\
z = r \cos\theta
\end{array}
\left\{
\begin{array}{ll}
u_1 = \sqrt{r} \cos\dfrac{\theta}{2} \cos\dfrac{\nu + \phi}{2}, & u_2 = \sqrt{r} \cos\dfrac{\theta}{2} \sin\dfrac{\nu + \phi}{2} \\[2ex]
u_3 = \sqrt{r} \sin\dfrac{\theta}{2} \cos\dfrac{\nu - \phi}{2}, & u_4 = \sqrt{r} \sin\dfrac{\theta}{2} \sin\dfrac{\nu - \phi}{2}.
\end{array}
\right.
$$

Similarly U_- is an other open subset without positive z-axis such that

$$U_- = \left\{ \boldsymbol{x}(\tilde{r}, \tilde{\theta}, \tilde{\phi}) \in \dot{\mathbb{R}}^3 \,;\, \tilde{r} > 0 \,,\, 0 \leq \tilde{\theta} < \pi \,,\, 0 < \tilde{\phi} \leq 2\pi \right\}$$

bringing the following local trivialization where $0 \leq \tilde{\nu} < 4\pi$

$$\tau_- : \pi^{-1}(U_-) \ni \boldsymbol{u} \longmapsto (\pi(\boldsymbol{u}), \varphi_-(\boldsymbol{u})) = (\boldsymbol{x}(\tilde{r}, \tilde{\theta}, \tilde{\phi}), \exp(i\tilde{\nu}/2)) \in U_- \times S^1$$

$$
\begin{array}{l}
x = \tilde{r} \sin\tilde{\theta} \cos\tilde{\phi} \\
y = -\tilde{r} \sin\tilde{\theta} \sin\tilde{\phi} \\
z = -\tilde{r} \cos\tilde{\theta}
\end{array}
\left\{
\begin{array}{ll}
u_1 = -\sqrt{\tilde{r}} \sin\dfrac{\tilde{\theta}}{2} \cos\dfrac{\tilde{\nu} + \tilde{\phi}}{2}, & u_2 = -\sqrt{\tilde{r}} \sin\dfrac{\tilde{\theta}}{2} \sin\dfrac{\tilde{\nu} + \tilde{\phi}}{2} \\[2ex]
u_3 = -\sqrt{\tilde{r}} \cos\dfrac{\tilde{\theta}}{2} \cos\dfrac{\tilde{\nu} + 3\tilde{\phi}}{2}, & u_4 = -\sqrt{\tilde{r}} \cos\dfrac{\tilde{\theta}}{2} \sin\dfrac{\tilde{\nu} + 3\tilde{\phi}}{2}.
\end{array}
\right.
$$

We suppose that the real parameter $E < 0$ denoting the energy of the MIC-Kepler problem do not coinside with some of its eigenvalues as given in [6] and [5, Theorem 10]

$$E \neq E_N = -\frac{2mk^2}{\hbar^2 (N+2)^2}, \qquad N = 0, 1, 2, \ldots$$

The positive constant m is the mass of the electron. The charge of the electron equals $-e$ where $e > 0$ is the charge of the proton called an 'elementary charge',

and k is the positive constant defined as

$$k \equiv \frac{e^2}{4\pi\varepsilon}$$

where $\varepsilon > 0$ means electric permittivity. The positive parameter $\hbar$ is defined as $\hbar \equiv h/2\pi$ where h is Planck's constant. Then we reduce the Green's function of the conformal Kepler problem denoted by $G(\boldsymbol{u}_f, \boldsymbol{u}_i; 4k)$ to the Green's functions of the MIC-Kepler problem on U_+ and U_- denoted by $G_+(\boldsymbol{x}_f, \boldsymbol{x}_i; E)$ and $G_-(\boldsymbol{x}_f, \boldsymbol{x}_i; E)$ respectively. The subscripts i and f say that the points with each of them denote initial and final points of motion in a proper configuration space respectively. The final results are in the following proposition where l is an arbitrary integer quantizing μ by the relation of $\mu = l\hbar/2$, and $J_l(\upsilon)$ denote the **Bessel functions**.

Proposition 1. i) *When* $\boldsymbol{x}_i, \boldsymbol{x}_f \in U_+$, *the Green's function of the* MIC-Kepler *problem is*

$$G_+(\boldsymbol{x}_f, \boldsymbol{x}_i; E = -m\omega^2/8)$$

$$= r_f \lim_{\chi \to 4\pi - 0} \int_0^\chi G(\boldsymbol{u}_f, \boldsymbol{u}_i; 4k) \exp\left(\mathrm{i}\,l\,\frac{\nu_i - \nu_f}{2}\right) \mathrm{d}\nu_i$$

$$= -\frac{\mathrm{i}^{l+1} m^2 \omega^2}{16\pi\hbar^3} \lim_{y' \to +0} \int_0^\infty \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(4k - \mathrm{i}y')(t + \mathrm{i}y')} \operatorname{cosec}^2(\omega t + \mathrm{i}\omega y')$$

$$\times \exp\left[-\mathrm{i}\,\frac{m\omega}{2\hbar}(r_i + r_f)\cot(\omega t + \mathrm{i}\omega y') - \mathrm{i}\,l \cdot \frac{\Theta}{2}\right]$$

$$\times J_l\left(\frac{m\omega}{2\hbar}\sqrt{2\boldsymbol{x}_i \cdot \boldsymbol{x}_f + 2r_i r_f}\operatorname{cosec}(\omega t + \mathrm{i}\omega y')\right) \mathrm{d}t$$

where $\quad \dfrac{\Theta}{2} \equiv \tan^{-1}\left[\dfrac{\sin\dfrac{\phi_i - \phi_f}{2}}{\cos\dfrac{\phi_i - \phi_f}{2}} \cdot \dfrac{\cos\dfrac{\theta_i + \theta_f}{2}}{\cos\dfrac{\theta_i - \theta_f}{2}}\right].$

ii) *When* $\boldsymbol{x}_i, \boldsymbol{x}_f \in U_-$, *the Green's function is*

$$G_-(\boldsymbol{x}_f, \boldsymbol{x}_i; E = -m\omega^2/8)$$

$$= \tilde{r}_f \lim_{\chi \to 4\pi - 0} \int_0^\chi G(\boldsymbol{u}_f, \boldsymbol{u}_i; 4k) \exp\left(\mathrm{i}\,l\,\frac{\tilde{\nu}_i - \tilde{\nu}_f}{2}\right) \mathrm{d}\tilde{\nu}_i$$

$$= -\frac{\mathrm{i}^{l+1} m^2 \omega^2}{16\pi\hbar^3} \lim_{y' \to +0} \int_0^\infty \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(4k - \mathrm{i}y')(t + \mathrm{i}y')} \operatorname{cosec}^2(\omega t + \mathrm{i}\omega y')$$

$$\times \exp\left[-\mathrm{i}\,\frac{m\omega}{2\hbar}(\tilde{r}_i + \tilde{r}_f)\cot(\omega t + \mathrm{i}\omega y') - \mathrm{i}\,l \cdot \frac{\tilde{\Theta}}{2}\right]$$

$$\times J_l\left(\frac{m\omega}{2\hbar}\sqrt{2\boldsymbol{x}_i \cdot \boldsymbol{x}_f + 2\tilde{r}_i \tilde{r}_f}\operatorname{cosec}(\omega t + \mathrm{i}\omega y')\right) \mathrm{d}t$$

where

$$
\frac{\tilde{\Theta}}{2} \equiv \tan^{-1} \left[\frac{\sin \dfrac{\tilde{\phi}_i - \tilde{\phi}_f}{2}}{\cos \dfrac{\tilde{\phi}_i - \tilde{\phi}_f}{2}} \cdot \frac{2 \cos (\tilde{\phi}_i - \tilde{\phi}_f) \cos \dfrac{\tilde{\theta}_i}{2} \cos \dfrac{\tilde{\theta}_f}{2} + \cos \dfrac{\tilde{\theta}_i - \tilde{\theta}_f}{2}}{2 \cos (\tilde{\phi}_i - \tilde{\phi}_f) \cos \dfrac{\tilde{\theta}_i}{2} \cos \dfrac{\tilde{\theta}_f}{2} - \cos \dfrac{\tilde{\theta}_i + \tilde{\theta}_f}{2}} \right] .
$$

iii) *When* $\boldsymbol{x}_i,\ \boldsymbol{x}_f \ \in U_+ \cap U_-,\ \theta = \pi - \tilde{\theta},\ \phi = 2\pi - \tilde{\phi}$ *and* $\tilde{\Theta}$ *is given by*

$$
\frac{\tilde{\Theta}}{2} = \tan^{-1} \left[-\frac{\sin \dfrac{\phi_i - \phi_f}{2}}{\cos \dfrac{\phi_i - \phi_f}{2}} \cdot \frac{2 \cos (\phi_i - \phi_f) \sin \dfrac{\theta_i}{2} \sin \dfrac{\theta_f}{2} + \cos \dfrac{\theta_i - \theta_f}{2}}{2 \cos (\phi_i - \phi_f) \sin \dfrac{\theta_i}{2} \sin \dfrac{\theta_f}{2} + \cos \dfrac{\theta_i + \theta_f}{2}} \right] .
$$

As it can be seen, the Green's functions on $U_+ \cap U_-$ are not equivalent with each other since the difference between G_+ and G_- is the one between Θ and $\tilde{\Theta}$ because of the equality $r = \tilde{r}$. Besides $\tilde{\Theta}$ is not equal to Θ obviously. Right then, what is the concrete relation between G_+ and G_- on the common part? An answer to this question is the following proposition established by similar methods as that in the earlier paper [4, §5].

Proposition 2. i) *When* $\boldsymbol{x} \in U_+$ *, the wave function of the* MIC-Kepler *problem is*

$$
\Psi_N^+ (\boldsymbol{x}) = \frac{m\omega}{2\sqrt{\pi}\hbar} \left(\sqrt{\frac{m\omega}{\hbar}} \right)^N \frac{\mathcal{P} \left(r\cos^2 \frac{\theta}{2} ,\ r\sin^2 \frac{\theta}{2} \right)}{\sqrt{k_1! k_2! k_3! k_4!}} \ \exp \left(-\frac{m\omega}{2\hbar} r \right)
$$
$$
\left(\sqrt{r} \cos \frac{\theta}{2} \right)^{k_1 + k_3} \left(\sqrt{r} \sin \frac{\theta}{2} \right)^{k_2 + k_4} \exp \left[-\mathrm{i} \left(k_1 - k_2 - k_3 + k_4 \right) \frac{\phi}{2} \right]
$$

and if $\boldsymbol{x}_i,\ \boldsymbol{x}_f \ \in U_+$ *, the Green's function of the* MIC-Kepler *problem is also written by*

$$
G_+(\boldsymbol{x}_f,\ \boldsymbol{x}_i;\ E) \ = \ \sum_{N=0}^{\infty} \frac{1}{4k - (N+2)\hbar\omega} \ \Psi_N^+(\boldsymbol{x}_f) \overline{\Psi_N^+(\boldsymbol{x}_i)}
$$

where $\omega \equiv \sqrt{-8E/m}$ *, the infinite sum of* N *includes the finite sum of all terms whose non-negative integers* $(k_1,\ k_2,\ k_3,\ k_4)$ *that satisfy the following conditions for a fixed* $N \in \mathbb{N} \cup \{0\}$

$$
k_1 + k_2 + k_3 + k_4 = N, \qquad k_1 + k_2 - k_3 - k_4 = -l, \qquad \mathbb{Z} \ni l = 2\mu/\hbar
$$

and $\mathcal{P}(X,\ Y)$ *is the polynomial*

$$
\mathcal{P}(X, Y) = \sum_{j=0}^{k_1} \sum_{s=0}^{k_2} j! s! \left(-\frac{\hbar}{m\omega} \right)^{j+s} {}_{k_1}C_j \cdot {}_{k_3}C_j \cdot {}_{k_2}C_s \cdot {}_{k_4}C_s \, X^{-j} \, Y^{-s} .
$$

ii) *When $\boldsymbol{x} \in U_-$, the wave function of the* MIC-Kepler *problem is*

$$\Psi_N^- (\boldsymbol{x}) = \frac{m\omega}{2\sqrt{\pi}\hbar} \left(-\sqrt{\frac{m\omega}{\hbar}}\right)^N \frac{\mathcal{P}\left(\tilde{r}\sin^2\frac{\tilde{\theta}}{2}, \tilde{r}\cos^2\frac{\tilde{\theta}}{2}\right)}{\sqrt{k_1!k_2!k_3!k_4!}} \exp\left(-\frac{m\omega}{2\hbar}\tilde{r}\right)$$

$$\left(\sqrt{\tilde{r}}\sin\frac{\tilde{\theta}}{2}\right)^{k_1+k_3} \left(\sqrt{\tilde{r}}\cos\frac{\tilde{\theta}}{2}\right)^{k_2+k_4} \exp\left[-i\left(k_1 + 3k_2 - k_3 - 3k_4\right)\frac{\tilde{\phi}}{2}\right]$$

and for $\boldsymbol{x}_i$, $\boldsymbol{x}_f \in U_-$, the Green's function of the MIC-Kepler *problem is*

$$G_-(\boldsymbol{x}_f, \boldsymbol{x}_i; E) = \sum_{N=0}^{\infty} \frac{1}{4k - (N+2)\hbar\omega} \Psi_N^-(\boldsymbol{x}_f) \bar{\Psi}_N^-(\boldsymbol{x}_i).$$

iii) *When $\boldsymbol{x} \in U_+ \cap U_-$, the relation between Ψ_N^+ and Ψ_N^- is*

$$\Psi_N^- (\boldsymbol{x}) = \Psi_N^+ (\boldsymbol{x})\, e^{-i\, l\, \phi}$$

and if $\boldsymbol{x}_i$, $\boldsymbol{x}_f \in U_+ \cap U_-$, the relation between G_+ and G_- is

$$G_-(\boldsymbol{x}_f, \boldsymbol{x}_i; E) = G_+(\boldsymbol{x}_f, \boldsymbol{x}_i; E)\, e^{-i\, l\, (\phi_f - \phi_i)}.$$

3. Wigner Function

When the z-axis is excluded from the configuration space $\dot{\mathbb{R}}^3$ the Hamiltonian system of the **conformal Kepler problem** $(T^*(\pi^{-1}(U_+ \cap U_-)), \mathrm{d}\vartheta, H)$ where ϑ denotes the canonical one-form of $T^*\dot{\mathbb{R}}^4 \supset T^*(\pi^{-1}(U_+ \cap U_-))$ is described in the above-mentioned local trivializations τ_+ and τ_- (see also §2) as follows[1]

$$H(r, \theta, \phi, \nu, \rho_r, \rho_\theta, \rho_\phi, \rho_\nu) = \frac{1}{2m}\left(\rho_r^2 + \frac{\rho_\theta^2}{r^2} + \frac{\rho_\phi^2 + \rho_\nu^2 - 2\rho_\phi\rho_\nu\cos\theta}{r^2\sin^2\theta}\right) - \frac{k}{r}$$

where ρ_r, ρ_θ, ρ_ϕ and ρ_ν are the conjugate momentums of r, θ, ϕ and ν respectively such that $\vartheta \equiv \rho_1 \mathrm{d}u_1 + \rho_2 \mathrm{d}u_2 + \rho_3 \mathrm{d}u_3 + \rho_4 \mathrm{d}u_4 = \rho_r \mathrm{d}r + \rho_\theta \mathrm{d}\theta + \rho_\phi \mathrm{d}\phi + \rho_\nu \mathrm{d}\nu$, or

$$H(\tilde{r}, \tilde{\theta}, \tilde{\phi}, \tilde{\nu}, \rho_{\tilde{r}}, \rho_{\tilde{\theta}}, \rho_{\tilde{\phi}}, \rho_{\tilde{\nu}})$$

$$= \frac{1}{2m}\left(\rho_{\tilde{r}}^2 + \frac{\rho_{\tilde{\theta}}^2}{\tilde{r}^2} + \frac{\rho_{\tilde{\phi}}^2 + \rho_{\tilde{\nu}}^2(4\cos\tilde{\theta} + 5) - 2\rho_{\tilde{\phi}}\rho_{\tilde{\nu}}(\cos\tilde{\theta} + 2)}{\tilde{r}^2\sin^2\tilde{\theta}}\right) - \frac{k}{\tilde{r}}$$

---

[1]The forms written by the trivialization τ_+ equal the ones given by Iwai [1, §.3].

where $\rho_{\tilde{r}}$, $\rho_{\tilde{\theta}}$, $\rho_{\tilde{\phi}}$ and $\rho_{\tilde{\nu}}$ are the conjugate momentums of $\tilde{r}$, $\tilde{\theta}$, $\tilde{\phi}$ and $\tilde{\nu}$ respectively such that $\vartheta \equiv \rho_1 \mathrm{d}u_1 + \rho_2 \mathrm{d}u_2 + \rho_3 \mathrm{d}u_3 + \rho_4 \mathrm{d}u_4 = \rho_{\tilde{r}} \mathrm{d}\tilde{r} + \rho_{\tilde{\theta}} \mathrm{d}\tilde{\theta} + \rho_{\tilde{\phi}} \mathrm{d}\tilde{\phi} + \rho_{\tilde{\nu}} \mathrm{d}\tilde{\nu}$. Moreover verifying the following equalities

$$\tilde{r} = r, \qquad \tilde{\theta} + \theta = \pi, \qquad \tilde{\phi} + \phi = 2\pi, \qquad \tilde{\nu} - \nu = 2\phi$$
$$\rho_{\tilde{r}} = \rho_r, \qquad \rho_{\tilde{\theta}} = -\rho_\theta, \qquad \rho_{\tilde{\phi}} = 2\rho_\nu - \rho_\phi, \qquad \rho_{\tilde{\nu}} = \rho_\nu$$

we have the following equivalence on the common part $T^*(\pi^{-1}(U_+ \cap U_-)) \subset T^*\dot{\mathbb{R}}^4$

$$H(r, \theta, \phi, \nu, \rho_r, \rho_\theta, \rho_\phi, \rho_\nu) = H(\tilde{r}, \tilde{\theta}, \tilde{\phi}, \tilde{\nu}, \rho_{\tilde{r}}, \rho_{\tilde{\theta}}, \rho_{\tilde{\phi}}, \rho_{\tilde{\nu}}).$$

Additionally $\rho_{\tilde{\nu}} = \rho_\nu$ coincides with the associated momentum mapping $\psi(\boldsymbol{u}, \boldsymbol{\rho})$

$$\rho_{\tilde{\nu}} = \rho_\nu = \frac{1}{2}(-u_2\rho_1 + u_1\rho_2 - u_4\rho_3 + u_3\rho_4) = \psi(\boldsymbol{u}, \boldsymbol{\rho})$$

and therefore restricting the Hamiltonian system $(T^*(\pi^{-1}(U_+ \cap U_-)), \mathrm{d}\vartheta, H)$ on the subset $\psi^{-1}(\mu) \subset T^*_{\boldsymbol{u}}\dot{\mathbb{R}}^4$

$$\psi^{-1}(\mu) = \left\{ (\boldsymbol{u}, \boldsymbol{\rho}) \in T^*_{\boldsymbol{u}}\dot{\mathbb{R}}^4 ; \psi(\boldsymbol{u}, \boldsymbol{\rho}) = \mu \right\}$$

is easily done by setting each of the conjugate momentum ρ_ν or $\rho_{\tilde{\nu}}$ to the fixed value μ. Further, according to Iwai & Uwano, $T^*\dot{\mathbb{R}}^3$ is diffeomorphic with the quotient space $\psi^{-1}(\mu)/S^1$ (see [2, Lemma 2.4]) i.e., $\pi_\mu^* \sigma_\mu = \iota_\mu^* \mathrm{d}\vartheta$ and $\pi_\mu^* H_\mu = \iota_\mu^* H$ where $\iota_\mu : \psi^{-1}(\mu) \to T^*\dot{\mathbb{R}}^4$ is the inclusion and $\pi_\mu : \psi^{-1}(\mu) \ni (\boldsymbol{u}, \boldsymbol{\rho}) \to (\boldsymbol{x}, \boldsymbol{p}) \in T^*\dot{\mathbb{R}}^3$ is the map which provides a principal S^1 bundle such that $\boldsymbol{x} = \pi(\boldsymbol{u})$ and

$$\begin{pmatrix} p_x \\ p_y \\ p_z \\ -\mu/r \end{pmatrix} = \frac{1}{4r} \begin{pmatrix} 2u_3 & 2u_4 & 2u_1 & 2u_2 \\ -2u_4 & 2u_3 & 2u_2 & -2u_1 \\ 2u_1 & 2u_2 & -2u_3 & -2u_4 \\ 2u_2 & -2u_1 & 2u_4 & -2u_3 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \end{pmatrix}.$$

Then, we have the following equations

$$\rho_r = p_r, \qquad \rho_\theta = p_\theta, \qquad \rho_\phi = p_\phi + \mu \cos\theta$$
$$\rho_{\tilde{r}} = p_{\tilde{r}}, \qquad \rho_{\tilde{\theta}} = p_{\tilde{\theta}}, \qquad \rho_{\tilde{\phi}} = p_{\tilde{\phi}} + \mu \cos\tilde{\theta} + 2\mu$$

where p_r, p_θ, p_ϕ, $p_{\tilde{r}}$, $p_{\tilde{\theta}}$ and $p_{\tilde{\phi}}$ are the conjugate momentums of r, θ, ϕ, $\tilde{r}$, $\tilde{\theta}$ and $\tilde{\phi}$ defined by

$$p_x \mathrm{d}x + p_y \mathrm{d}y + p_z \mathrm{d}z = p_r \mathrm{d}r + p_\theta \mathrm{d}\theta + p_\phi \mathrm{d}\phi = p_{\tilde{r}} \mathrm{d}\tilde{r} + p_{\tilde{\theta}} \mathrm{d}\tilde{\theta} + p_{\tilde{\phi}} \mathrm{d}\tilde{\phi}.$$

In this way the reduced Hamiltonian system $(T^*(U_+ \cap U_-), \sigma_\mu, H_\mu)$ that is referred from now on as the **MIC-Kepler problem** is

$$\begin{cases} \sigma_\mu = \mathrm{d}p_r \wedge \mathrm{d}r + \mathrm{d}p_\theta \wedge \mathrm{d}\theta + \mathrm{d}(p_\phi + \mu\cos\theta) \wedge \mathrm{d}\phi \\ H_\mu(r, \theta, \phi, p_r, p_\theta, p_\phi) = \dfrac{1}{2m}\left(p_r^2 + \dfrac{p_\theta^2}{r^2} + \dfrac{p_\phi^2}{r^2\sin^2\theta}\right) - \dfrac{k}{r} + \dfrac{\mu^2}{2mr^2} \end{cases}$$

or

$$\begin{cases} \sigma_\mu = \mathrm{d}p_{\tilde{r}} \wedge \mathrm{d}\tilde{r} + \mathrm{d}p_{\tilde{\theta}} \wedge \mathrm{d}\tilde{\theta} + \mathrm{d}(p_{\tilde{\phi}} + \mu\cos\tilde{\theta} + 2\mu) \wedge \mathrm{d}\tilde{\phi} \\ H_\mu(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}) = \dfrac{1}{2m}\left(p_{\tilde{r}}^2 + \dfrac{p_{\tilde{\theta}}^2}{\tilde{r}^2} + \dfrac{p_{\tilde{\phi}}^2}{\tilde{r}^2\sin^2\tilde{\theta}}\right) - \dfrac{k}{\tilde{r}} + \dfrac{\mu^2}{2m\tilde{r}^2}. \end{cases}$$

Moreover provided that the following equalities

$$p_{\tilde{r}} = p_r, \qquad p_{\tilde{\theta}} = -p_\theta, \qquad p_{\tilde{\phi}} = -p_\phi$$

are satisfied we have the following equivalence on the common part $T^*(U_+ \cap U_-) \subset T^*\dot{\mathbb{R}}^3$.

$$H_\mu(r, \theta, \phi, p_r, p_\theta, p_\phi) = H_\mu(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}})$$

Similarly, the Wigner function of the MIC-Kepler problem (see [5, Theorem 10]) can be rewritten as a function on $T^*(U_+ \cap U_-)$ which is almost the phase space $T^*\dot{\mathbb{R}}^3$. Finally if N is an arbitrary non-negative integer called 'principal quantum number', l is an arbitrary integer quantizing μ by the relation of $\mu = l\hbar/2$ and $L_n(X)$ denotes the **Laguerre polynomial** of degree n, i.e.,

$$L_n(X) = \sum_{\alpha=0}^{n}(-1)^\alpha \frac{n!}{(\alpha!)^2(n-\alpha)!}\, X^\alpha$$

$$\sum_{n=0}^{\infty} L_n(X)\, \xi^n = \frac{1}{1-\xi}\exp\left(-\frac{\xi}{1-\xi}\, X\right)$$

we can formulate the following

Proposition 3. *Suppose that the point* $x \in \mathbb{R}^3$ *is not on the z-axis. Then the Wigner function of the* MIC-Kepler *problem is given as follows*

$$f_N(r, \theta, \phi, p_r, p_\theta, p_\phi) = \frac{(-1)^N}{(\pi\hbar)^4}\, \mathrm{e}^{-2(N+2)}$$

$$\times L_{n_a}\left(\frac{N+2}{2mk}\left[\mathcal{A}^2 + \frac{\mathcal{C}^2}{r(1+\cos\theta)}\right]\right) L_{n_b}\left(\frac{N+2}{2mk}\left[\mathcal{A}^2 + \frac{\mathcal{D}^2}{r(1+\cos\theta)}\right]\right)$$

$$\times L_{n_c}\left(\frac{N+2}{2mk}\left[\mathcal{B}^2 + \frac{\mathcal{E}^2}{r(1-\cos\theta)}\right]\right) L_{n_d}\left(\frac{N+2}{2mk}\left[\mathcal{B}^2 + \frac{\mathcal{F}^2}{r(1-\cos\theta)}\right]\right)$$

or

$$f_N(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}) = \frac{(-1)^N}{(\pi\hbar)^4} \, e^{-2(N+2)}$$

$$\times L_{n_a}\left(\frac{N+2}{2mk}\left[\tilde{\mathcal{A}}^2 + \frac{\tilde{\mathcal{C}}^2}{\tilde{r}(1-\cos\tilde{\theta})}\right]\right) L_{n_b}\left(\frac{N+2}{2mk}\left[\tilde{\mathcal{A}}^2 + \frac{\tilde{\mathcal{D}}^2}{\tilde{r}(1-\cos\tilde{\theta})}\right]\right)$$

$$\times L_{n_c}\left(\frac{N+2}{2mk}\left[\tilde{\mathcal{B}}^2 + \frac{\tilde{\mathcal{E}}^2}{\tilde{r}(1+\cos\tilde{\theta})}\right]\right) L_{n_d}\left(\frac{N+2}{2mk}\left[\tilde{\mathcal{B}}^2 + \frac{\tilde{\mathcal{F}}^2}{\tilde{r}(1+\cos\tilde{\theta})}\right]\right)$$

where n_a, n_b, n_c and n_d are non-negative integers such that

$$\begin{cases} 2(n_a + n_d) = N + l \\ 2(n_b + n_c) = N - l \end{cases} \quad i.e., \quad \begin{cases} |l| \leq N \\ N \text{ and } l \text{ are simultaneously even or odd} \end{cases}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{C}}, \tilde{\mathcal{D}}, \tilde{\mathcal{E}}$ and $\tilde{\mathcal{F}}$ are given by the formulas

$$\mathcal{A}(r, \theta, \phi, p_r, p_\theta, p_\phi) = p_r\sqrt{r(1+\cos\theta)} - p_\theta\sqrt{\frac{1-\cos\theta}{r}}$$

$$\mathcal{B}(r, \theta, \phi, p_r, p_\theta, p_\phi) = p_r\sqrt{r(1-\cos\theta)} + p_\theta\sqrt{\frac{1+\cos\theta}{r}}$$

$$\mathcal{C}(r, \theta, \phi, p_r, p_\theta, p_\phi) = p_\phi + r(1+\cos\theta)\left(\frac{2mk}{\hbar(N+2)} + \frac{\mu}{r}\right)$$

$$\mathcal{D}(r, \theta, \phi, p_r, p_\theta, p_\phi) = p_\phi - r(1+\cos\theta)\left(\frac{2mk}{\hbar(N+2)} - \frac{\mu}{r}\right)$$

$$\mathcal{E}(r, \theta, \phi, p_r, p_\theta, p_\phi) = p_\phi + r(1-\cos\theta)\left(\frac{2mk}{\hbar(N+2)} - \frac{\mu}{r}\right)$$

$$\mathcal{F}(r, \theta, \phi, p_r, p_\theta, p_\phi) = p_\phi - r(1-\cos\theta)\left(\frac{2mk}{\hbar(N+2)} + \frac{\mu}{r}\right)$$

$$\tilde{\mathcal{A}}(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}) = p_{\tilde{r}}\sqrt{\tilde{r}(1-\cos\tilde{\theta})} + p_{\tilde{\theta}}\sqrt{\frac{1+\cos\tilde{\theta}}{\tilde{r}}}$$

$$\tilde{\mathcal{B}}(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}) = p_{\tilde{r}}\sqrt{\tilde{r}(1+\cos\tilde{\theta})} - p_{\tilde{\theta}}\sqrt{\frac{1-\cos\tilde{\theta}}{\tilde{r}}}$$

$$\tilde{\mathcal{C}}(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}) = p_{\tilde{\phi}} - \tilde{r}(1 - \cos\tilde{\theta})\left(\frac{2mk}{\hbar(N+2)} + \frac{\mu}{\tilde{r}}\right)$$

$$\tilde{\mathcal{D}}(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}) = p_{\tilde{\phi}} + \tilde{r}(1 - \cos\tilde{\theta})\left(\frac{2mk}{\hbar(N+2)} - \frac{\mu}{\tilde{r}}\right)$$

$$\tilde{\mathcal{E}}(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}) = p_{\tilde{\phi}} - \tilde{r}(1 + \cos\tilde{\theta})\left(\frac{2mk}{\hbar(N+2)} - \frac{\mu}{\tilde{r}}\right)$$

$$\tilde{\mathcal{F}}(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}) = p_{\tilde{\phi}} + \tilde{r}(1 + \cos\tilde{\theta})\left(\frac{2mk}{\hbar(N+2)} + \frac{\mu}{\tilde{r}}\right).$$

Furthermore we have the equalities

$$\tilde{\mathcal{A}} = \mathcal{A}, \quad \tilde{\mathcal{B}} = \mathcal{B}, \quad \tilde{\mathcal{C}} = -\mathcal{C}, \quad \tilde{\mathcal{D}} = -\mathcal{D}, \quad \tilde{\mathcal{E}} = -\mathcal{E}, \quad \tilde{\mathcal{F}} = -\mathcal{F}$$

leading to the following equivalence

$$f_N(r, \theta, \phi, p_r, p_\theta, p_\phi) = f_N(\tilde{r}, \tilde{\theta}, \tilde{\phi}, p_{\tilde{r}}, p_{\tilde{\theta}}, p_{\tilde{\phi}}).$$

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