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THE SPINNING GAS CLOUDS WITH PRECESSION: THE SYMMETRY GENERATORS

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Abstract. The Dyson model of a spinning ellipsoidal gas cloud expanding into a vacuum has been found to be Liouville integrable under certain additional assumptions, such as the absence of either vorticity or of angular momentum. Here we present a new formulation in the form of a 4x4 matrix equation, which generalizes a similar result obtained in rotationless cases. This implies to consider an extended affine space of seven dimensions, in which the seven coordinates of the point-mass representative of the cloud obey differential equations of the same general form as those defining the elliptic functions. This leads very directly to the linearization of the system in the so-called degenerate cases. We obtain also explicit expressions for the symmetry generators, a prerequisite in the task of constructing a Backlund transformation.

1. Introduction

Here we consider the model of a spinning cloud of gas of ellipsoidal shape, expanding into a vacuum, proposed by Dyson [5]. This belongs to a more general class of self-gravitating models, studied in particular by Dirichlet [4], Riemann [19], Chandrasekhar [3] in the case of incompressible fluids, and by Ovsiannikov [18], Dyson and by Fujimoto [6] in the compressible case.

The Dyson model becomes completely integrable by quadratures (Gaffet [8], hereafter Paper I) in the absence of either vorticity or of angular momentum, when the cloud is constituted of an ideal gas with the adiabatic index $\gamma = 5/3$ characteristic of monatomic gases.

Under the restriction of rotation about a fixed principal axis, it has been found (Gaffet [7, 9]) that the equations of motion are amenable to the puzzling form

$$M_{ij} x'^{j} = 0 \tag{1}$$

in which M is a 4×4 matrix, x^i (i= 1, 2, 3) are the basic functions, x^4 is a quadratic function of the latter, and a prime denotes derivation. The remarkable facts are that the matrix elements of M are simply linear functions of the x^i, and that the latter are very closely related to a coordinate system, which makes the equations of motion manifestly separable. One of the motivations of the present work was to extend these results to another sub-case of interest, namely that of minimal energy with precession, introduced in references [11] and [12] (hereafter Paper II). In Section 3 we propose for it a matrix formulation closely related with (1), and show that it admits a natural formulation in terms of Wronskians constructed from the four unknown functions.

In Section 4, through an extension of the space of functions to eight dimensions - extension suggested by our earlier works, Paper II and [14] (hereafter Paper III) - we find that these new unknowns satisfy a system with properties reminiscent of those of Riccati systems, and in fact *analogous to the equations defining the elliptic functions of order two*. This turns out to be a powerful tool for a deeper study of its properties, and leads us, first to obtain a simple expression of the integrating factor (Section 4.3) associated with each Liouville torus of this Liouville integrable Hamiltonian system (Whittaker [22]), then to an explicit determination of the second symmetry generator (Section 5). We remark that the finding of the second generator is an unavoidable step (see Section 5.2) in the task of generalizing to the minimal energy cases the Backlund transformation obtained by Gaffet [7] in the cases without rotation.

Another remarkable by-product of the new formulation presented here is the simplicity with which it leads to the linearization in the sub-cases with minimal energy called "degenerate" (Section 6).

2. The Equations of Motion in General, and in the Block-Diagonal Cases

2.1. The Equations of Motion and their First-Integrals

In the case of the Dyson model with the restricting conditions mentioned above, the equations of motion have already been derived in earlier works [9, 10] and involve a 3×3 symmetric and traceless velocity matrix v, together with a diagonal matrix Δ with unit determinant. The diagonal part of v represents the rate of deformation of the cloud, and its off-diagonal part is related with the angular velocity matrix. The symmetric nature of v is a consequence of the assumed absence of vorticity, and the vanishing of the trace results from the fact that the expanding motion of the cloud can be (and has been) treated separately.

The diagonal matrix $\Delta = (\Delta_1, \Delta_2, \Delta_3)$ has components which are proportional to the squares of the principal axes of the cloud. The equations of motion are derivable from a Hamiltonian (I, Appendix A) with time coordinate τ say - which is a function of the physical time t (Anisimov and Lysikov [2], see also [10], equations (4.3), (4.4) therein) but the appropriate independent variable - with respect to which the Painleve property is conjectured to hold - is

$$u = \int T_c \,\mathrm{d}t = \int X_0 \,\mathrm{d}\tau$$

where T_c is the temperature of the cloud, and X_0 = TrΔ. The equations of motion then assume the form

$$\frac{\mathrm{d}}{\mathrm{d}u} \ln \Delta_i = 2v_{ii}, \qquad \frac{\mathrm{d}v}{\mathrm{d}u} + v^2 + [v, \omega] - \frac{1}{\Delta} = kI \tag{2}$$

where ω is the (antisymmetric) angular velocity matrix

$$\omega_{ij} = \frac{\Delta_i + \Delta_j}{\Delta_i - \Delta_j} v_{ij}$$

k an a priori arbitrary scalar function (which is defined by (2) itself) and I the unit matrix.

A useful equivalent formulation involves the following set of variables (Paper I)

$$X_n = \mathrm{Tr}(v^n \Delta), \qquad Y_n = \mathrm{Tr}(v^n / \Delta)$$

in which n= 0,1,2, and

$$T = -1/2 \mathrm{Tr}(v^2), \qquad P = \det(v)$$

and reads

$$T'(u) = 3P - Y_1, \qquad P'(u) = Y_2 + 2/3T(Y_0 - T) \tag{3}$$

$$\begin{aligned} X_0'(u) &= 2X_1 \\ X_1'(u) &= X_2 + 3 - X_0(Y_0 + 2S/3) \\ X_2'(u) &= -2/3X_1(2T + Y_0) \end{aligned} \tag{4}$$

$$\begin{aligned} Y_0'(u) &= -2Y_1 \\ Y_1'(u) &= 2/3Y_0(Y_0 - T) - 3Y_2 - 2X_0 \\ Y_2'(u) &= 4/3Y_1(2T + Y_0) - 4PY_0 + 2X_1. \end{aligned} \tag{5}$$

These eight variables in addition satisfy an algebraic constraint (Paper I, Section 5.2 therein)

$$K_6(X_n, Y_n, T, P) = 0.$$

The energy constant m which is canonically conjugate to the time τ is given by

$$9m = X_0 X_2 - X_1^2 + 3X_0$$

and the total angular momentum j^2 by

$$j^2 = X_0 X_2 - X_1^2 + 3Y_2 + 4TY_0$$

where j is the angular momentum vector in the rotating frame

$$j_k = (\Delta_i - \Delta_j)v_{ij}, \qquad i, j, k = \text{circ. perm. of } 1, 2, 3.$$

There are in addition two integrals of motion of the sixth degree in velocities, whose expression involves a three-vector $\tilde{j}$ related to the angular momentum vector j, as

$$\tilde{j} = -\Delta j.$$

The first one, L_6, can be written compactly in terms of a pair of triple products

$$L_6 = L_{66} + L_{64}, \qquad L_{66} = (\tilde{j}, v\tilde{j}, v^2\tilde{j}), \qquad L_{64} = -3(j, \tilde{j}, v\tilde{j}) \tag{6}$$

where the lower indices are a reminder of the degree in velocities. The triple product

$$K_{66} = (j, vj, v^2 j) \tag{7}$$

will also be of interest in what follows.

The remaining constant of motion ϵ admits a compact expression as well, involving two matrices U and V (Paper III)

$$-4\epsilon = \det U + \mathrm{Tr}(2UV + U + 2V) + \tilde{j}^2/3 \tag{8}$$

with

$$U = \Delta^{-1}(v^2 + 4T/3)\Delta^{-1}, \qquad V = \Delta(v^2 + T/3).$$

We note that $\det U = \det V = P^2 + 4T^3/27$.

2.2. The Minimal Energy Cases

Given a particular choice of all other integrals of motion, the energy constant m cannot be less than a minimum value m_0 if at least a part of the Liouville torus is to remain real. At this point the exterior product of the four one-forms corresponding to the constants of motion becomes zero, since m is an extremum

$$\mathrm{d}m \wedge \mathrm{d}j^2 \wedge \mathrm{d}\epsilon \wedge \mathrm{d}L_6 = 0. \tag{9}$$

The resulting relation between the integrals has been derived explicitly (see Gaffet [13]), and admits a rational parametrization in terms of three parameters h, l, K

$$\begin{aligned} m &= \frac{(2y+1)(1-K)}{3p} \\ j^2 &= \frac{6K(1-y)}{p} \\ \epsilon &= \frac{2K(x-1)}{x} \\ L_6 &= 27\frac{K^2}{xy}(K-1+2y-xy) \end{aligned} \tag{10}$$

where $x = h^3$, $y = l^3$, $p = hl$.

The corresponding Liouville tori are two-dimensional surfaces in the space of the eight coordinates (X_n, Y_n, T, P). The extra condition on the Liouville tori imposed by (9) can be expressed in a simple way in terms of the vectors j and $\tilde{j}$ [11]

$$j \cdot \tilde{j} = f_{12} = -3K(S + X_0/p) \tag{11}$$

where

$$S = T - Y_0. \tag{12}$$

Although there is only one independent condition, the following one holds as well

$$\tilde{j}^2 = f_{22} = c_X\left(\frac{X_0^2}{1-K} - 2Y_0\right) - 9K \tag{13}$$

with $c_X = 3K/p$.

Using the above conditions (11) and (13), together with the conditions obtained by their differentiation, one gets an algebraic system (see [11], Section 3.2 therein) for the variables X_1, Y_1, P (which are the variables of odd degree in velocities). Choosing for definiteness the case $m = 5$, $j^2 = 12$, they read

$$\begin{aligned} 3X_1^2 - X_0Y_1^2 &= A_1(X_0, Y_0, T) \\ 3X_1Y_1 + Y_0Y_1^2 &= -B_1 \\ (Y_0^2 - 3X_0)(P + Y_1) - \tilde{A}_1X_1 + C_1Y_1 &= 0 \end{aligned}$$

where $A_1, \tilde{A}_1, B_1, C_1$ are polynomials in (X_0, Y_0, T)

$$
\begin{aligned}
\tilde{A}_1 &= T + \frac{c_X X_0}{K-1} \\
A_1 &= 4/3TX_0(Y_0^2 - 3X_0) + X_0Y_0(33 + \frac{30}{K-1} - X_0) \\
&\quad + c_X X_0^3/(1-K) + 9X_0(1-K) - 135 \\
B_1 &= T[4/3Y_0^3 - 5X_0Y_0 + 9(1-K)] - Y_0^2[X_0 - 33 + 30/(1-K)] \\
&\quad + c_X X_0^2 Y_0/(1-K) + 3X_0^2 + 45X_0/(K-1) \\
C_1 &= Y_0(T/3 - Y_0) + 2X_0 + 15/(1-K).
\end{aligned}
$$

Taking into account the condition imposed by the constancy of the integral ϵ (see equation (8)), which is the form

$$(P + Y_1)^2 = S_3(X_0, Y_0, T)$$

one obtains explicit expressions for the products X_1^2, X_1Y_1, Y_1^2 in terms of X_0, Y_0, T only

$$X_1^2 = P_7/D_4, \qquad X_1Y_1 = Q_7/D_4, \qquad Y_1^2 = R_7/D_4$$

where P_7, Q_7, R_7 are polynomials and

$$D_4 = X_0\tilde{A}_1{}^2 + 2Y_0\tilde{A}_1C_1 + 3C_1^2. \tag{14}$$

We note that when $D_4 = 0$ on the Liouville torus, the polynomials P_7, Q_7, R_7 also vanish and the variables X_1, Y_1 determined by (2.23) are no longer single-valued. The line $D_4 = 0$ is in fact a double line on the torus, when considered in the coordinate system (X_0, Y_0, T).

2.3. The Block-Diagonal Cases

In the cases of rotation about a fixed principal axis, the velocity matrix v becomes block-diagonal, and it has been found [9] that the equations of motion can be written in matrix form

$$M_{ij}x'^j(u) = 0 \tag{15}$$

where M is a 4×4 symmetric matrix with coefficients linear in the variables x^i, and has, of course, zero determinant. The variable x^4 is not independent, and is a quadratic combination of the other x^i. As a result, the equation of the Liouville torus, $\det M = 0$, is a quartic polynomial in (x^1, x^2, x^3). Equation (15) only determines the relative scale of the derivatives x'^i, but their absolute scale can also be found, as

$$x'^i x'^j = C^{ij} \tag{16}$$

where C^{ij} is the cofactor matrix.

There are at least three remarkable things to be pointed out about equation (15). The first is that the lack of linear independence of the four equations that it represents, precisely gives rise to one of the equations of the Liouville torus. A second is that the dependence of the matrix M on the four variables should be so simple. A third is that the associated equation (16) has a very simple form too, when the independent variable is chosen to be the one (u) with respect to which the Painleve property holds (Weiss *et al* [21], Kowalevski [16], [17], Ince [15]).

We show in the next section that a formulation analogous to (15) also holds in the minimal energy cases with precession.

3. A 4×4 Matrix Formulation Applicable to the Precessing Cases

3.1. A 3×3 Matrix Equation

We now go back to the minimal energy cases with precession defined in Section 2.2. For any given values of the integrals m and j^2, the remaining integrals L_6 and ϵ are rational functions of the parameter K. As a first step, we look for a 3×3 matrix relation similar to (15), where the matrix coefficients are functions of X_0, Y_0, T and of X_2, and the column vector is constituted by the variables (X_1, Y_1, P) which are of odd degree (respectively 1, 1 and 3) in velocities. The remaining Y_2 can be eliminated using the values chosen for m and j^2

$$Y_2 = -4/3TY_0 + X_0 + j^2/3 - 3m.$$

Using the general expressions of j^2, $j\cdot,\tilde{j}$, $\tilde{j}^2$ given in Paper I (equation (4.6) therein), we obtain by differentiation of equation (11) the following first component of a matrix equation of the form: $A_{ij}x^j = 0$, where x^j stands for the column vector (X_1, Y_1, P)

$$-A_{1j}x^j = (Y_2 + 2c_X)X_1 + (X_2 + 3K)Y_1 + [X_0Y_0 + 9(K-1)]P = 0.$$

Similarly, differentiation of equation (13) gives

$$A_{2j}x^j = (T + \frac{c_X X_0}{K-1})X_1 + (Y_2 + TY_0 - c_X)Y_1 + (3X_0 - Y_0^2)P = 0.$$

We remark that A_{13} and A_{23} may be identified with the first two components of the cross-product

$$\begin{pmatrix} -X_0 & 3(1-K) & Y_0 \\ -Y_0 & X_0 & 3 \end{pmatrix} \tag{17}$$

of which the third component is $A_{33} = 3Y_0(1-K) - X_0^2$. This suggests choosing as third row of our matrix the linear combination satisfying

$$-Y_0A_{1j} + X_0A_{2j} + 3A_{3j} = 0.$$

Now, as a result of the general expressions of j^2, $j\cdot,\tilde{j}$, $\tilde{j}^2$ mentioned above, and of equations (11), (13), the quadratic combinations formed with X_1 and Y_1 satisfy the following relations

$$\begin{aligned} X_1^2 &= F_{11} = X_0X_2 + 3Y_2 + 4TY_0 - j^2 \\ X_1Y_1 &= F_{12} = -[X_0Y_2 + Y_0X_2 + T(X_0Y_0 + 3)] - f_{12} \\ Y_1^2 &= F_{22} = Y_0Y_2 + 3X_2 + 4TX_0 - f_{22}. \end{aligned} \tag{18}$$

It then turns out that, taking account of the identity $F_{22}X_1 - F_{12}Y_1 = 0$, the expression of A_{3j} can be simplified significantly, and the third equation becomes

$$A_{3j}x^j = (X_2 + TX_0 + 3K)X_1 + (1-K)TY_1 + [3Y_0(1-K) - X_0^2]P = 0.$$

This completes the determination of the 3×3 matrix A. The compatibility condition $\det A = 0$ is of the second degree in X_2, and may be viewed as determining X_2 as an implicit function of (X_0, Y_0, T). The eigenvector (X_1, Y_1, P) is fully determined, apart from its sign, by the relations (18). Finally, the requirement that the derivatives of (X_0, Y_0, T) satisfy the equations (3)-(5), namely

$$X_0' = +2X_1, \qquad Y_0' = -2Y_1, \qquad T' = 3P - Y_1$$

makes the equation $A_{ij}x^j = 0$ a closed differential system, of the third order.

A related formulation, which involves a partially symmetric matrix A_S, may also be found by forming the linear combinations, inspired by (17)

$$\begin{aligned} A_{S1j} &= -X_0A_{1j} + 3(1-K)A_{2j} + Y_0A_{3j} \\ A_{S2j} &= -Y_0A_{1j} + X_0A_{2j} + 3A_{3j} \\ A_{S3j} &= A_{3j}. \end{aligned}$$

The result is the following matrix

$$A_S = \begin{pmatrix} F_{22} & -F_{12} & 0 \\ -F_{12} & F_{11} & 0 \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

It then follows that

$$\det A = -\frac{\det A_S}{A_{33}} = F_{12}^2 - F_{11}F_{22}.$$

3.2. The Additional Constraint ϵ = constant

In order that the above system should describe the evolution of a spinning cloud, it is necessary to incorporate an additional constraint, so that the integral of motion ϵ assumes a constant value (given by equation (10)). This gives rise to a new constraint on the variables (X_0, Y_0, T, X_2) in addition to that implied by $\det A = 0$, in agreement with the two-dimensional nature of the Liouville torus. The derivative of X_2 then coincides with that given by equation (4), namely

$$X_2'(u) = -2/3 X_1(2T + Y_0).$$

With equation (4) incorporated, the matrix equation becomes four-dimensional, and may be written

$$B_{ij} z'^j = 0 \tag{19}$$

where B_{ij} is a 4×4 matrix and z^j may be chosen to be the column vector (X_0, Y_0, T, X_2).

3.3. The Matrix Equation as Simple Relations Between Wronskians

For reasons that will become apparent in the next Section 4, a choice of variable more appropriate than the above z^j turns out to be (X_0, Y_0, S, U), where S has already been defined (equation (12)) and

$$-U = X_2 + 3 + TX_0/3$$

and, as a result, the derivative of U is given by

$$3U' = 2X_1(T + Y_0) + X_0(Y_1 - 3P).$$

Defining the **Wronskian of two functions** F, G which are linear combinations of (X_0, S, U) as

$$[F, G] = FG' - GF'$$

whereas, by definition, when one of the functions is Y_0

$$[Y_0, F] = -[F, Y_0] = 2Y_0 F' - FY_0'$$

the four equations of motion encapsulated in equation (19) are found to express simple relations between Wronskians

$$\begin{aligned} SU' + 6(1-K)S' + S/3[X_0, S] + [Y_0, U] &= (X_0 + \frac{j^2}{3} - 3m + 2c_X)X_0' \\ 2X_0 S' + 6(T + \frac{c_X X_0}{1-K})X_0' - Y_0/3[Y_0, S] &= (\frac{j^2}{3} - 3m - c_X)Y_0' \qquad (20) \\ (K-1)(3X_0' - [Y_0, S]) + [X_0, U] - X_0/3[X_0, S] &= 0 \end{aligned}$$

$$3U' = [S, X_0] + [Y_0, X_0]. \tag{21}$$

The equations (20) correspond to the rows 1,2,3 of the matrix equation $A_{ij}x^j = 0$, and equation (21) to the fourth line of the matrix equation (19).

4. The Eight-Dimensional Space of Functions Z

In the preceding Section 3, we have obtained a new formulation of the equations of motion (equation (19)) in close analogy with that (equation (15)) that was found in the block-diagonal cases. We also observed that it may be written in the equivalent form (20)-(21) where the Wronskians play a major role. This strongly suggests that X_0, S, U are, in some sense, variables of the first degree, while Y_0 itself (and hence also T) is of the second degree. This view is supported by the expressions of $X_0'^2, X_0'Y_0', Y_0'^2$, which are of the form (see equation (18))

$$\begin{aligned} X_0'^2 &= -4/3TX_0^2 + ... \\ X_0'Y_0' &= -8/3TX_0Y_0 + ... \\ Y_0'^2 &= -16/3TY_0^2 + ... \end{aligned} \tag{22}$$

where the dots on the right sides stand for polynomial terms of lower degree.

4.1. The Seven Variables of the First Degree

The expressions of the remaining quadratic combinations $X_0'S', X_0'U'$, etc. do not however admit polynomial expressions in terms of the four basic variables X_0, S, U, T. This suggests enlarging this set of variables so as to restore a polynomial behaviour, and a natural candidate is the set of eight variables S_n first considered in Paper II (Section 4 therein), which are seventh degree polynomials in X_0, Y_0, S - or, rather the seven ratios S_n/S_1 ($S_1 = D_4$, see equation (14)).

It has been shown in Paper III that $(X_0, S, U, L_{66}, K_{66})$ (see equations (6), (7)) and another variable denoted by Z_{W0} (equation (3.19) in Paper III), all are linear combinations of the ratios S_n/S_1 (we note that U is a linear function - with constant coefficients - of X_0 and of the variable denoted by (jVj) (Paper III, equation (3.7) therein).

If we then consider $X_0, S, U, L_{66}, K_{66}, Z_{W0}$ to be quantities of the first degree, then we find that not only the products $X_0'S', X_0'U', S'^2, S'U', U'^2$ all admit polynomial expressions in that extended space, but in addition to that, their leading terms are $-4/3TX_0S, -4/3TX_0U$ etc., in agreement with the general form of equation (22).

More generally, letting

$$Z_0 = 1, \qquad Z_1 = X_0, \qquad Z_2 = S, \qquad Z_3 = U$$
$$Z_4 = L_{66}, \qquad Z_5 = K_{66}, \qquad Z_6 = Z_{W0}$$

and letting Z_7 be a seventh, linearly independent, linear combination of the ratios S_n/S_1 (see also the next Section 4.2 for an independent definition of Z_7), it is found that, for all i, j = 1,...,7

$$Z_i'(u)Z_j'(u) = P_{ij}(Z_n) = -4/3TZ_iZ_j + ... \tag{23}$$

where P_{ij} is a quartic polynomial. (The variable T, or $Y_0 = T - S$, coincides with a quadratic combination of Z_n).

Moreover, the second derivatives $Z_i''(u)$ are found to be given by

$$Z_i''(u) = K_i(Z_n) = -8/3TZ_i + ... \tag{24}$$

where K_i is a cubic polynomial.

The fundamental properties of equations (23) and (24) are characteristic of certain systems of Riccati equations, and fully account for the particularly simple form of the equations of motion in terms of Wronskians (Section 3.3), and for the associated matrix formulation of Section 3.2. The similarity with Riccati equations is clearly related with the Painleve property exhibited by this differential system. Moreover, equations (23) and (24) may be viewed as constituting *a generalization, to a seventh dimensional space, of the equations defining the elliptic functions* of order two.

4.2. The Ten Quadratic Relations Satisfied by Z_n

It should be noted that *there are* 28 *differential equations of the form* (Paper II, equation (4.3))*, although the number of unknown functions is only seven*. The associated compatibility condition is the following set of ten linearly independent quadratic relations

$$Z_3Z_5 = cZ_1Z_5 + Q_1(Z_1, \dots, Z_4) \tag{25}$$

$$\begin{aligned} Z_6 &= Q_2(Z_1, \dots, Z_5) \\ Z_1Z_6 &= Q_3(Z_1, \dots, Z_5) \\ Z_3Z_6 &= Q_4(Z_1, \dots, Z_5) \\ Z_4Z_6 &= Q_5(Z_1, \dots, Z_5) \end{aligned} \tag{26}$$

$$\begin{aligned} Z_7 &= Q_6(Z_1, \dots, Z_5) \\ Z_1Z_7 &= Q_7(Z_1, \dots, Z_5) \\ Z_3Z_7 &= Q_8(Z_1, \dots, Z_6) \\ Z_4Z_7 &= Q_9(Z_1, \dots, Z_6) \\ Z_5Z_7 &= Q_{10}(Z_1, \dots, Z_6) \end{aligned} \tag{27}$$

where c is a constant and the Q_n are quadratic functions.

We note that the variable Z_7 may thus be defined as a quadratic combination of $Z_1, ..., Z_5$ such that $Z_5 Z_7$ is effectively of the second degree only - that product coincides, on the Liouville torus, with some quadratic combination of $Z_1, ..., Z_6$. Of course, Z_7 is thus only determined up to an additive linear combination of $Z_0, ..., Z_6$.

Although the ten relations (25)-(27) are linearly independent, only five of them are functionally independent and, taken together, they constitute the equations of the two-dimensional Liouville torus, i.e., the torus is the intersection of ten quadrics in the seven-dimensional space with coordinates Z_n. It may also be viewed as the intersection of the five quadrics (25), (26) in the six-dimensional space spanned by $Z_1, ..., Z_6$.

In Paper II (equation (4.6) therein) we actually found sixteen different ways of writing the following equation, valid all over the Liouville torus

$$B^2 - AC = 0$$

where A, B, C are certain linear combinations of the eight Z_n and the preceding results show that at most ten of these expressions can be linearly independent.

In the same work (equation (4.8) therein) we also obtained sixteen different ways of expressing the Liouville torus in the form

$$P(A, B, C, D) = 0$$

where P is an homogeneous quartic polynomial, and A, B, C, D are certain linear combinations of the eight Z_n.

4.3. The Integrating Factor

In general, integral curves of a differential system reducible to the first order, located on a two-dimensional surface $F(x, y, z) = 0$, are given by an equation of the form

$$Q\mathrm{d}\Phi = \frac{z'\mathrm{d}y - y'\mathrm{d}z}{F_x} = \frac{x'\mathrm{d}z - z'\mathrm{d}x}{F_y} = \frac{y'\mathrm{d}x - x'\mathrm{d}y}{F_z}$$

where Φ is a constant on each curve (or "trajectory"), and Q may be termed the integrating factor.

In the block-diagonal cases ([9], equation (4.14) therein) it was found that, using the 4×4 matrix formulation (see Section 2.3) and letting $F = \det M$, the integrating factor is unity. In the present case where precession is included, letting $F = \det A$ (see Section 3), this is no longer true. We note, however, that in addition to X_0, Y_0, S, the determinant $\det A$ also involves the variable U, which is a non-polynomial function of X_0, Y_0, S

$$U = S_U / D_4$$

(in the above expression, the numerator is a linear combination of the eight polynomials $S_n(X_0, Y_0, S)$, and D_4 has already been defined (14)). This suggests that the integrating factor may involve some power of the denominator D_4. As it turns out, the trajectories are determined by the equation

$$D_4 \mathrm{d}\Phi = \frac{S' \mathrm{d}X_0 - X_0' \mathrm{d}S}{\partial(\det A)/\partial Y_0} \tag{28}$$

where $\mathrm{d}\Phi$ is an exact differential, i.e., the integrating factor is just $Q = D_4$. It is worth noting that the denominator of $\mathrm{d}\Phi$, namely $D_4 \partial(\det A)/\partial Y_0$, coincides with *a cubic combination of the variables Z_n*, as was the case in the block-diagonal cases, where the determinant of $M(x^1, x^2, x^3)$ is a quartic.

5. The Second Symmetry Generator

Up to now, the independent variable u has only been defined on trajectories, but it is of course possible to define all over the Liouville torus a variable, $\tilde{u}$ say, that will coincide with u on each trajectory. In the present problem, which is known to be Liouville integrable, there must exist an explicit expression for its exact differential $\mathrm{d}\tilde{u}$. (From now on we will always write u in place of $\tilde{u}$ for simplicity).

There are thus two basic symmetry generators: $\partial/\partial u|_\Phi$, whose explicit expression is just the set of the equations of motion, and $\partial/\partial\Phi|_u$, whose explicit expression is yet to be found. The two generators of course commute.

Let us denote by δ the operation of the second generator

$$\delta F = \partial F/\partial\Phi|_u.$$

Using for definiteness the coordinate system (X_0, S), the following relations between exact differentials hold

$$\mathrm{d}X_0 = X_0' \mathrm{d}u + \delta X_0 \mathrm{d}\Phi, \qquad \mathrm{d}S = S' \mathrm{d}u + \delta S \mathrm{d}\Phi$$

and, conversely

$$\mathrm{d}\Phi = \frac{S' \mathrm{d}X_0 - X_0' \mathrm{d}S}{S' \delta X_0 - X_0' \delta S}, \qquad \mathrm{d}u = \frac{-\delta S \mathrm{d}X_0 + \delta X_0 \mathrm{d}S}{S' \delta X_0 - X_0' \delta S}. \tag{29}$$

The first equation in (29) merely states that $S'\delta X_0 - X_0' \delta S = \partial(X_0, S)/\partial(\Phi, u)$ is the **integrating factor**.

5.1. The Canonical Time τ and the Second Generator

We now point out a useful connection between the definition (2) of the time τ canonically conjugated to the energy m, and the second generator. In the same way as in the case of the variable u, the definition of τ may be extended so as to make $\mathrm{d}\tau$ an exact differential on the Liouville torus

$$\mathrm{d}\tau = \frac{\mathrm{d}u - F\mathrm{d}\Phi}{X_0} \tag{30}$$

and the function F must satisfy the compatibility condition

$$\frac{\mathrm{d}}{\mathrm{d}u}(F/X_0) + \frac{\mathrm{d}}{\mathrm{d}\Phi}(1/X_0) = 0$$

which we rewrite, using the notation $[F, G]$ for the Wronskian $FG' - GF'$

$$\delta X_0 = [X_0, F]. \tag{31}$$

That is, knowing the explicit expression (30) of $\mathrm{d}\tau$, is the same as knowing the second generator's transformation formula for the variable X_0. Moreover, it also gives us the transformation formulae for the u-derivatives of X_0 of all order, since the two generators commute. This leads us to consider the new coordinate system $X_0, X_1 = X_0'/2$, in which the expression (29) of $\mathrm{d}\Phi$ becomes

$$\mathrm{d}\Phi = \frac{(X_1'\mathrm{d}X_0 - X_0'\mathrm{d}X_1)}{\partial(X_0, X_1)/\partial(\Phi, u)}.$$

The new integrating factor is thus

$$\frac{\partial(X_0, X_1)}{\partial(\Phi, u)} = \delta X_0 X_1' - \delta X_1 X_0' = 1/2(X_0''\delta X_0 - X_0'\frac{\mathrm{d}}{\mathrm{d}u}\delta X_0). \tag{32}$$

Comparing this equation with equations (29) and (28)), $\dfrac{\partial(X_0, S)}{\partial(\Phi, u)} = D_4\dfrac{\partial(\det A)}{\partial Y_0}$ $= \hat{Q}$ say, we get a linear ordinary differential equation for δX_0

$$X_0''\delta X_0 - X_0'\frac{\mathrm{d}}{\mathrm{d}u}\delta X_0 = \hat{Q}\frac{\partial X_0'}{\partial S}|_{X_0}.$$

Now, we have seen that $F'(u)$ is a quantity of degree $n+1$ when F is of degree n, in the sense that F'^2 can be identified with a polynomial in Z_n of degree $2(n+1)$. It is reasonable to assume that this is also true of δF, and that consequently δX_0 ought to be a quantity of second degree, which in turn suggests that the function F of equation (31) may be a linear combination of Z_n: $F = \sum c_n Z_n$. If so, the equation (34) becomes easily solvable, as it amounts to a linear algebraic equation for the eight constants c_n. We obtain the following simple result

$$F = L_{66}.$$

Thus the exact differential of the time τ is found as

$$\mathrm{d}\tau = \frac{(\mathrm{d}u - L_{66}\mathrm{d}\Phi)}{X_0} \tag{33}$$

and at the same time the action of the second generator on X_0 and all its derivatives is also obtained.

The transformation formula for S then follows from the relation (see equations (29) and (32))

$$S'\delta X_0 - X_0'\delta S = \hat{Q}$$

and δS turns out to be of the second degree in the same sense as δX_0 is. It admits an expansion of the form quite analogous to the Wronskian relations (20)-(21) of Section 3.3

$$\delta S = \sum_1^5 c_n Z_n' + c_6[X_0, U] + c_7[S, U] + (c_8 X_0 + c_9 S)[X_0, S].$$

Given δX_0, the action of the second generator on any other variable can be determined similarly. Of course, its action on derivatives of S of arbitrary order can be found by mere u-differentiation, as already mentioned in the case of the derivatives of X_0.

5.2. The Second Generator as an Unavoidable Intermediate Step in the Search for a Backlund Transformation

In the context of partial differential equations (PDE), a Backlund transformation (BT) involves a spectral function, which is the solution of a linear equation depending on one parameter - the spectral parameter - and establishes a correspondence between solutions of the PDE (Scott [20], Ablowitz [1] *et al*). In the present case, a BT will therefore change any trajectory on the Liouville torus into some other. Now, in the case of an equation possessing the Painleve property with respect to an independent variable denoted by u, the BT is expected to preserve the value of u along a trajectory. More precisely, since u is in principle only determined modulo an additive constant in our case, the BT may be said to amount to a finite translation of u, together with, of course, a finite translation of the integration constant Φ, i.e., a finite translation in the (u, Φ) plane (reference [7], Section 4, p.8351). In particular, the infinitesimal BT, which usually corresponds to the limit of an infinite value of the spectral parameter, must be a linear combination of the two generators $\partial_u, \partial_\Phi$. This shows that the determination of the second generator is a necessary step in the process of finding a Backlund transformation.

Let us note finally that a BT has indeed been found in the rotationless cases (reference [7] , Section 4 and the “Note Added in Proof”, p.8353), which gives hope that it may also be found in the cases incorporating precession.

6. The Linearization of the Degenerate Cases

The linearizability in the degenerate cases - which are characterized by the presence of a singular line on the Liouville torus, which is also a particular solution - is the result of the following two circumstances

- The decomposability of the equations (23) into pairs of equations of Riccati, along a family of sections - (K_x) say - parametrized by x, of the Liouville torus.
- The fact that the independent variable u may be chosen to be constant along each section (K_x).

Thus the differential system reduces to equations of the form

$$Z_n'(u) = \Omega Z_n + Z_{nL} \tag{34}$$

where the second degree part on the right-hand-side is ΩZ_n. As a consequence of the form of equations (23) $\Omega^2 = -4/3T$ modulo linear terms, so that the factor

$$\Omega = \sum_0^7 \omega_k(x) Z_k$$

may be chosen to be independent of n. The last term in (34)

$$Z_{nL} = \sum_0^7 \zeta_{nk}(x) Z_k$$

is of course linear.

The singular line is found to be located on one (or several) hyperplane(s)

$$Z_s = \sum_0^7 \sigma_k Z_k = 0$$

and, keeping in mind that the singular line is a particular solution, there exist at least one such combination Z_s whose derivative assumes the form

$$Z_s'(u) = \Omega Z_s + \lambda(x) Z_s.$$

We note that the λ term on the right-hand-side is reducible and may be eliminated through an appropriate redefinition of the factor Ω. The Wronskians with Z_s then reduce to

$$[Z_s, Z_n] = Z_s Z_{nL}$$

and, upon transformation to the new set of variables $X_n = Z_n/Z_s$ the differential system is *linearized*

$$X_n'(u) = [Z_s, Z_n]/Z_s^2 = Z_{nL}/Z_s = \sum_0^7 \zeta_{nk}(x)X_k$$

(we recall that x is a function of u only).

The above method of linearization is applicable to all three cases of degeneracy defined in [13] (Section 7 therein), and not only to the cases of vanishing L_6.

7. Conclusion

In the block-diagonal cases, which are the cases of cloud rotation about a fixed axis, the equations of motion were found to be amenable to the puzzling form (15), involving a 4×4 matrix M with zero determinant. It was one of the motivations of the present study to see whether a similar formulation could also exist in the cases with precession. A close analogue has indeed been found (of Section 3), and we have shown that its existence is related with the possibility of rewriting the equations of motion in the form of a differential system presenting the Riccati-like properties (23) and (24), where the unknown functions are the coordinates in a seven-dimensional affine space. These properties are in turn related with the Painleve property exhibited by the system.

This new 4×4 matrix formulation directly leads to a simple and explicit expression of the *integrating factor* associated with the integral curves on the Liouville tori (Section 4.3). Further, the consideration of the seven-dimensional space naturally leads to a determination of the second symmetry generator, $\partial/\partial\Phi$, whose action on the variable X_0 (representing the temperature of the cloud, see equation (2)) is found to be particularly simple (equation (31), where $F = L_{66}$). This in turn entails a correspondingly simple and general expression of the differential of the time τ (canonical conjugate of the energy constant m) in the form of an exact differential on the Liouville torus (equation (33)). It is worth mentioning here that an explicit knowledge of the second generator is a required step in the search for a Backlund transformation that relates the various integral curves of the system (see Section 5.2).

Finally, the Riccati-like properties of the system are found to provide a direct shortcut to the *linearization* of the system in the so-called degenerate cases (Section 6).

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