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ANALYTIC DESCRIPTION OF THE VISCOUS FINGERING INTERFACE IN A ROTATING HELE-SHAW CELL

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Abstract. The determination of the interface shape between two fluids during the process of viscous fingering (Saffman–Taylor instability) in a Hele-Shaw cell is addressed here. The parametric equations describing this interface are obtained in an explicit analytic form. With these results, a number of interface shapes are presented, including shapes with opposite sides in contact that are considered as the onset of droplet pinch-off.

1. Introduction

Viscous fingering (Saffman–Taylor instability [8]) is the formation of patterns in the interface between two fluids in a Hele-Shaw cell. It occurs during injection when a less viscous fluid displaces a more viscous one in planar or non-planar Hele-Shaw cell [7]. It can also occur due to gravity with or without taking into account chemical or thermal effects [1, 9] if a horizontal interface separates two fluids of different densities and the heavier fluid is above the other one. A closely related problems appear in the study of the collapse of nanotubes [10] and rings exposed to uniform externed presure [2].

Saffman–Taylor instability also occurs in many other frameworks, e.g. in a Hele–Shaw cell subjected to pressure, radial magnetic field or rotation.

Let the interface be given by means of the coordinates $x(s)$, $z(s)$ in a certain Cartesian coordinate frame in the Euclidean plane with s being the interface arclength.

The unit **tangent vector** $\mathbf{t}(s)$ and the unit **normal vector** $\mathbf{n}(s)$ are related to the curvature $\kappa(s)$ through the **Frenet-Serret formulas**

$$\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s), \qquad \mathbf{n}'(s) = -\kappa(s)\mathbf{t}(s).$$

Nye *et al* [5] studied gravitationally driven Saffman-Taylor fingering using oil above air in a Hele-Shaw cell. They derived the expression

$$\kappa = \frac{\mathrm{d}\varphi}{\mathrm{d}s} = -Z$$

for the curvature of the interface between the two fluids and

$$X(\psi) = -\sqrt{2}\int \frac{\psi^2 - c}{(1-(\psi^2-c)^2)^{1/2}}\mathrm{d}\psi$$

where $\psi = (c+\cos\varphi)^{1/2}$ and (X, Z) are the scaled coordinates. It is not a surprize that the relation $\kappa = -Z$ obtained by Nye *et al* [5] gives rise to the prominent *Euler elasticae* curves [11].

It is shown by many authors, see e.g., Leandro *et al* [4] and Oliveira *et al* [6], that in a rotating Hele-Shaw cell the equation, balancing the centrifugal force and the surface tension can be integrated to yield the expression

$$\kappa(r) = \Omega(r^2 - \mathring{r}^2)$$

for the curvature of the interface between the two fluids, where $r = \sqrt{x^2 + z^2}$ is the radius, Ω is the dimensionless angular velocity and $\mathring{r}$ is the radius at which $\kappa(r)$ vanishes.

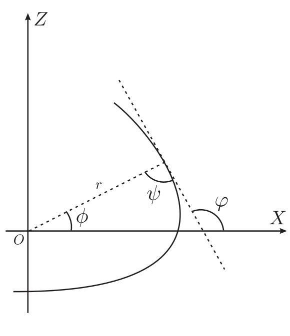

Figure 1. The angles in the approximation by Leandro *et al* [4].

In these papers, the embedding $\phi = \phi(r)$ of the interface is obtained in terms of the angle ψ (see Fig. 1) by the following two integrals

$$\psi = \arcsin\left(\frac{1}{r}\int_{\mathring{r}}^{r} t\kappa(t)\mathrm{d}t + \frac{\mathring{r}}{r}\sin\psi_0\right) \tag{1}$$

$$\phi = \phi_0 + \int_{\mathring{r}}^{r} \frac{1}{t}\tan\psi(t)\mathrm{d}t. \tag{2}$$

However, the integrals are too complicated and the authors of the aforementioned papers proceed the analysis evaluating the foregoing integrals numerically.

Here, we follow another way to determine the parametric equations for the interface in an explicit analytic form by introducing a fictitious dynamical system and present its explicit solutions in terms of Jacobian elliptic functions and elliptic integrals.

2. Parametric Equations of the Interface

It is easy to see that Frenet–Serret formulae can be written as the following system for the coordinates of the position vector $\mathbf{r}(s) = (x(s), z(s))$ of the interface

$$x'' + \kappa(r)z = 0, \qquad z'' - \kappa(r)x = 0, \qquad \kappa(r) = \Omega(r^2 - \mathring{r}^2).$$

Such a dynamical system is studied by the authors of the present analysis within another context in [3, 12]. It is shown in these papers that the coordinates of the position vector can be expressed through the curvature $\kappa(s)$ and the slope angle

$$\varphi(s) = \int \kappa(s)\mathrm{d}s$$

in the form

$$x(s) = \frac{1}{2\Omega}\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\cos\varphi(s) + \frac{1}{4\Omega}(\kappa^2(s) - \mu)\sin\varphi(s)$$

$$z(s) = \frac{1}{2\Omega}\frac{\mathrm{d}\kappa(s)}{\mathrm{d}s}\sin\varphi(s) - \frac{1}{4\Omega}(\kappa^2(s) - \mu)\cos\varphi(s)$$

Hence, if the curvature $\kappa(s)$ and the **slope angle** $\varphi(s)$ are known functions of the arclength, the foregoing expressions for $x(s)$ and $z(s)$ are parametric equations of the interface.

In polar coordinates the foregoing dynamical system transforms to an independent differential equation for $r(s)$ and another equation for the polar angle. Substituting the expression $r^2 = \mathring{r}^2 + \kappa/\Omega$ in the equation for $r(s)$, one easily obtains the following equation for the curvature

$$\kappa'' + \kappa^3 - \mu\kappa - 4\Omega = 0, \qquad \mu = \Omega^2\mathring{r}^2. \tag{3}$$

It possesses an apparent first integral of form

$$\left(\kappa'\right)^2 = P(\kappa(s)), \qquad P(\kappa) = 2E - \frac{1}{4}\kappa^4 + \frac{1}{2}\mu\kappa^2 + 4\Omega\kappa$$

where $E = \frac{1}{4}\Omega^2\mathring{r}^2\left(16 - \Omega^2\mathring{r}^6\right)$.

3. Explicit Formulae for the Curvature and the Slope Angle

We now proceed with description of the solutions to equation (3) for the curvature. Depending on the values of the coefficients μ, Ω and E, two cases for the intrinsic equation of the interface and the corresponding slope angle $\varphi(s)$ are to be considered (see Fig. 2).

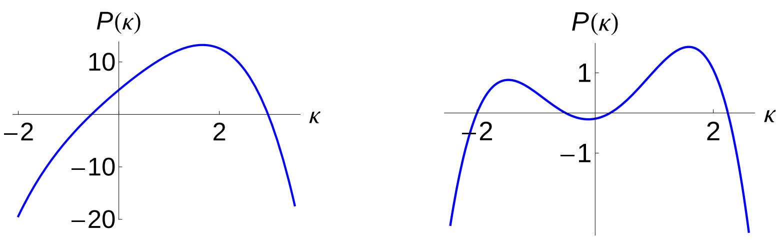

Figure 2. The graphs of the polynomials representing the Case 1-left, and the Case 2-right.

Case 1. The polynomial $P(\kappa)$ has two real roots $\alpha < \beta$ and two complex conjugate roots γ and δ. In this case there exist both periodic and nonperiodic solutions for the curvature $\kappa\,(s)$.

The Periodic solutions exist in the cases when $(3\alpha + \beta)(\alpha + 3\beta) \neq 0$ and $\eta = (\gamma - \delta)/(2i) \neq 0$. They are of the form

$$\kappa_1\,(s) = \frac{(A\beta + B\alpha) - (A\beta - B\alpha)\,\mathrm{cn}(us,k)}{(A + B) - (A - B)\,\mathrm{cn}(us,k)}$$

$$\varphi_1\,(s) = \frac{A\beta - B\alpha}{A - B}s + \frac{(A + B)\,(\alpha - \beta)}{2u\,(A - B)}\Pi\left(-\frac{(A - B)^2}{4AB}, \mathrm{am}(us,k), k\right)$$

$$+ \frac{\alpha - \beta}{2u\sqrt{k^2 + \frac{(A-B)^2}{4AB}}}\arctan\left(\sqrt{k^2 + \frac{(A - B)^2}{4AB}}\,\frac{\mathrm{sn}\,(us,k)}{\mathrm{dn}\,(us,k)}\right)$$

where

$$A = \sqrt{4\eta^2 + (3\alpha+\beta)^2}, \qquad B = \sqrt{4\eta^2 + (\alpha+3\beta)^2}, \qquad u = \frac{1}{4}\sqrt{AB}$$

$$k = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{4\eta^2 + (3\alpha+\beta)(\alpha+3\beta)}{\sqrt{\left[4\eta^2 + (3\alpha+\beta)(\alpha+3\beta)\right]^2 + 16\eta^2(\beta-\alpha)^2}}}.$$

Nonperiodic solutions are obtained in the cases in which

$$(3\alpha+\beta)(\alpha+3\beta) = \eta = 0$$

and are of the form

$$\kappa_2(s) = \zeta - \frac{4\zeta}{1+\zeta^2 s^2}, \qquad \varphi_2(s) = \zeta s - 4\arctan(\zeta s)$$

where $\zeta = \alpha$ if $3\alpha+\beta = 0$ and $\zeta = \beta$ if $\alpha + 3\beta = 0$.

A few examples of simple curves that are appropriate for the observed interface shapes in the fixed Cartesian frame (above) and in the moving co-frame associated with the curve (below) are shown in Fig. 3.

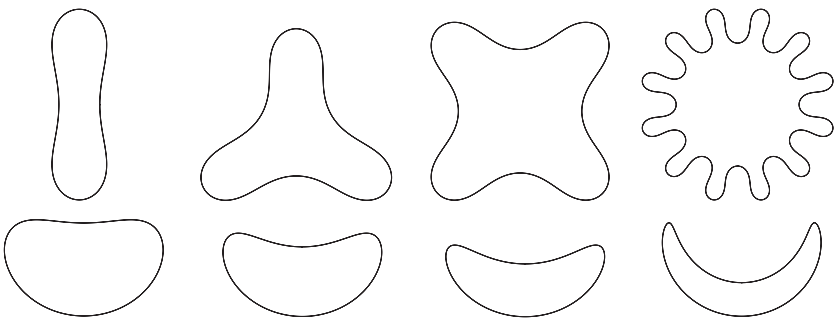

Figure 3. Examples of interface shapes in the fixed Cartesian frame (above) and in the moving co-frame associated with the curve (below).

In a rotating Hele-Shaw cell, there exist values of the angular velocity at which the interface exhibits shapes with points of contact that can be considered as the onset of drop formation and separation (see Fig. 4).

Case 2. The polynomial $P(\xi)$ has four real roots $\alpha < \beta < \gamma < \delta$. Then, two periodic solutions exist

$$\begin{aligned} \kappa_3(s) &= \delta - \frac{(\delta-\alpha)(\delta-\beta)}{(\delta-\beta)+(\beta-\alpha)\,\mathrm{sn}^2(us,k)} \\ \varphi_3(s) &= \delta s - \frac{\delta-\alpha}{u}\Pi\left(\frac{\beta-\alpha}{\beta-\delta}, \mathrm{am}(us,k), k\right) \end{aligned}$$

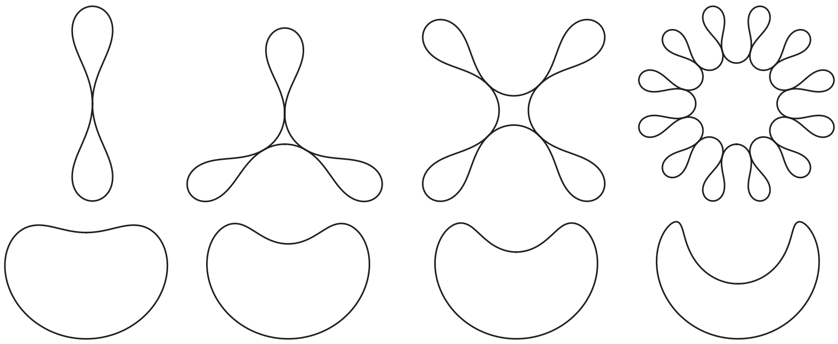

Figure 4. Examples of interface shapes with points of contact in the fixed Cartesian frame (above) and in the moving co-frame associated with the curve (below).

and

$$\begin{aligned}\kappa_4(s) &= \beta + \frac{(\gamma-\beta)(\delta-\beta)}{(\delta-\beta)-(\delta-\gamma)\,\mathrm{sn}^2(us,k)}\\ \varphi_4(s) &= \beta s - \frac{\beta-\gamma}{u}\Pi\left(\frac{\delta-\gamma}{\delta-\beta},\mathrm{am}(us,k),k\right)\end{aligned}$$

where

$$u = \frac{1}{4}\sqrt{(\gamma-\alpha)(\delta-\beta)}, \qquad k = \sqrt{\frac{(\beta-\alpha)(\delta-\gamma)}{(\gamma-\alpha)(\delta-\beta)}}.$$

Curvatures belonging to the **Case 2** always give rise to self-intersecting curves. Fig. 5 comprises an example of four-fold symmetric curves in the fixed Cartesian frame (above) and in the moving co-frame associated with the curve (below). The figure in the most right column clarifies the actual size and positions of the two curves.

In conclusion, the explicit parameterization of the fluid interface in a rotating Hele–Shaw cell presented here gives rise to the equilibrium shapes presented in Fig. 3. At sufficiently high values of the rotation velocity Ω, the shapes in Fig. 3 evolve to the ones shown in Fig. 4 which may be regarded as the formation of drops and the onset of droplet pinch-off.

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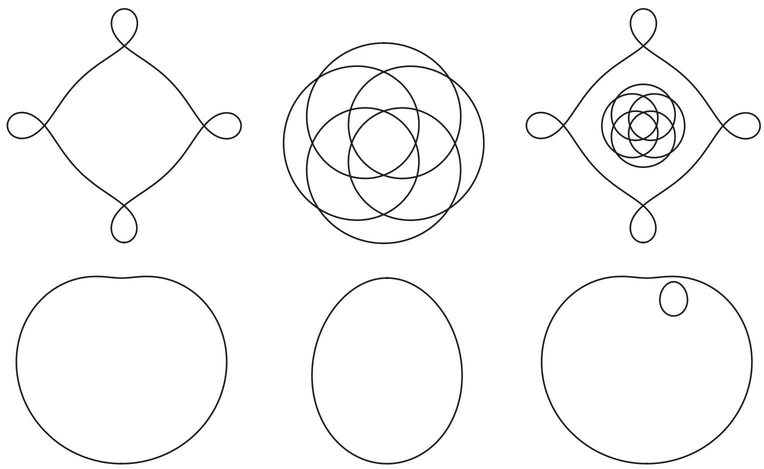

Figure 5. Examples of closed self-intersecting curves of curvatures $\kappa_3\,(s)$ (left) and $\kappa_4\,(s)$ (middle) in the fixed Cartesian frame (above) and in the moving co-frame associated with the curve (below).

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