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MOTION OF CHARGED PARTICLES IN TWO-STEP NILPOTENT LIE GROUPS*

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Abstract. We formulate the equation of motion of a charged particle in a Riemannian manifold with a closed two form. Since a two-step nilpotent Lie group has natural left-invariant closed two forms, it is natural to consider the motion of a charged particle in a simply connected two-step nilpotent Lie groups with a left invariant metric. We study the behavior of the motion of a charged particle in the above spaces.

1. Introduction

Let Ω be a closed two-form on a connected Riemannian manifold $(M, \langle\ ,\ \rangle)$, where $\langle\,,\rangle$ is a Riemannian metric on M. We denote by $\bigwedge^m(M)$ the space of m-forms on M. We denote by $\iota(X) : \bigwedge^m(M) \to \bigwedge^{m-1}(M)$ the interior product operator induced from a vector field X on M, and by $\mathcal{L} : T(M) \to T^*(M)$, the Legendre transformation from the tangent bundle $T(M)$ over M onto the cotangent bundle $T^*(M)$ over M, which is defined by

$$\mathcal{L} : T(M) \to T^*(M), \quad u \mapsto \mathcal{L}(u), \quad \mathcal{L}(u)(v) = \langle u, v\rangle\,, \quad u, v \in T(M). \quad (1)$$

A curve $x(t)$ in M is referred as a *motion of a charged particle under electromagnetic field* Ω, if it satisfies the following second order differential equation

$$\nabla_{\dot{x}}\dot{x} = -\mathcal{L}^{-1}(\iota(\dot{x})\Omega) \quad (2)$$

where ∇ is the Levi-Civita connection of M. Here $\nabla_{\dot{x}}\dot{x}$ means the acceleration of the charged particle. Since $-\mathcal{L}^{-1}(\iota(\dot{x})\Omega)$ is perpendicular to the direction $\dot{x}$ of the movement, $-\mathcal{L}^{-1}(\iota(\dot{x})\Omega)$ means the **Lorentz force**. The speed $|\dot{x}|$ is a conservative constant for a charged particle. When $\Omega = 0$, then the motion of a

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charged particle is nothing but a geodesic. The equation (2) originated in the theory of relativity (see [2] for details).

In this paper, we deal with the motion of a charged particles in a simply connected two-step nilpotent Lie group N with a left invariant Riemannian metric. Since a two-step nilpotent Lie group has a non-trivial center Z, we can construct a left-invariant closed two form Ω_{a_0} from an element $a_0 \in Z$ specified below and consider the motion of a charged particle under the electromagnetic field Ω_{a_0}. H. Naitoh and Y. Sakane proved that there are no closed geodesics in a simply connected nilpotent Lie group. In contrast with geodesics, there exist motions of charged particles which are periodic. Kaplan defined a H-type Lie group, which is a kind of two-step nilpotent Lie groups. We study the motion of a charged particle in a H-type Lie group more explicitly than in a general two-step nilpotent Lie group.

2. Charged Particles in Two-step Nilpotent Lie Groups

Let N be a simply connected two-step nilpotent Lie group with a left-invariant Riemannian metric $\langle\, ,\rangle$. Denote by $\mathfrak{n}$ the vector space consisting of all left-invariant vector fields on N. Since $\mathfrak{n}$ is two-step nilpotent, $\mathfrak{n}$ has a non-trivial center $\mathfrak{z}$. Let $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{z}^\perp$ be an orthogonal direct sum decomposition of $\mathfrak{n}$, then $[\mathfrak{z}^\perp, \mathfrak{z}^\perp] \subset \mathfrak{z}$. For $a_0 \in \mathfrak{z}$, we define a linear transformation ϕ_{a_0} on $\mathfrak{z}^\perp$ by

$$\langle \phi_{a_0}(X), Y\rangle = \langle a_0, [X, Y]\rangle, \qquad X, Y \in \mathfrak{z}^\perp.$$

We extend ϕ_{a_0} to a linear transformation on $\mathfrak{n}$ by $\phi = 0$ on $\mathfrak{z}$, which is also denoted by ϕ_{a_0}. We can regard ϕ_{a_0} as a left-invariant $(1,1)$-tensor on N. Then ϕ_{a_0} is skew-symmetric with respect to the left-invariant Riemannian metric $\langle\, ,\rangle$ since

$$\langle \phi_{a_0}(X), Y\rangle + \langle X, \phi_{a_0}(Y)\rangle = \langle a_0, [X, Y]\rangle + \langle a_0, [Y, X]\rangle = 0$$

for any left invariant vector fields $X, Y \in \mathfrak{n}$. If we define a left-invariant two-form Ω_{a_0} on N by

$$\Omega_{a_0}(X, Y) = \langle X, \phi_{a_0}(Y)\rangle, \qquad X, Y \in \mathfrak{n}$$

then a simple calculation implies that Ω_{a_0} is closed. In fact, for any X_1, X_2 and X_3 in $\mathfrak{n}$ we have

$$\begin{aligned} 3!(d\Omega_{a_0})(X_1, X_2, X_3) &= -\mathfrak{S}\ \Omega_{a_0}([X_1, X_2], X_3) \\ &= -\mathfrak{S}\ \langle [X_1, X_2], \phi_{a_0}(X_3)\rangle = 0 \end{aligned}$$

where we denote by $\mathfrak{S}$ the cyclic sum, and the last equality follows from the fact that $[X_1, X_2] \in \mathfrak{z}$ and $\phi(X_3) \in \mathfrak{z}^\perp$. The equation of motion of the charged particle under the electromagnetic field Ω_{a_0} is

$$\nabla_{\dot{x}}\dot{x} = \phi_{a_0}(\dot{x}). \tag{3}$$

Here a curve in a manifold is **simple** if it is a simply closed periodic curve, or if it does not intersect itself. Since N is simply connected, the one dimensional de-Rham cohomology group vanishes. Hence we get the following theorem using Theorem 9 in [2].

Theorem 1. *The motion of a charged particle* (3) *in a simply connected two-step nilpotent Lie group is simple.*

Now we will construct explicitly a simply connected two step nilpotent Lie group with a left-invariant Riemannian metric from an (abstract) two-step nilpotent Lie algebra $\mathfrak{n}$ with an inner product $\langle\,,\rangle$. In order to do this, we recall a **Hausdorff formula** for a Lie group (see [1, p. 106]), which states that

$$\exp X \exp Y = \exp\left(X + Y + \frac{1}{2}[X, Y] + \cdots\right).$$

If the Lie group is two-step nilpotent, then the higher terms $+\cdots$ on the right hand side vanish. Based on the Hausdorff formula, we define a Lie group structure on $\mathfrak{n}$ itself by

$$X \cdot Y = X + Y + \frac{1}{2}[X, Y], \qquad X, Y \in \mathfrak{n}.$$

The identity element in this group is 0, and the inverse element of $x \in \mathfrak{n}$ is equal to $-x$. We denote by $N = (\mathfrak{n}, \cdot)$ the so obtained Lie group. The center of N coincides with $\mathfrak{z}$. Denote by $\mathrm{Lie}(N)$ the Lie algebra consisting of all left-invariant vector fields on N. Then $\mathrm{Lie}(N)$ is identified with $\mathfrak{n}$ as a Lie algebra as mentioned below. Since N is a Euclidean space as a manifold, we can identify $T_0(N)$ with $\mathfrak{n}$ as vector spaces. The identification induces a Lie algebra structure on $T_0(N)$. For $X \in T_0(N)$, we denote by $\tilde{X} \in \mathrm{Lie}(N)$ the left-invariant vector field on N such that $\tilde{X}_0 = X$. The mapping defined by $\mathfrak{n} = T_0(N) \to \mathrm{Lie}(N)$, $X \mapsto \tilde{X}$ gives an isomorphism as Lie algebras. Hence N is a simply connected two-step nilpotent Lie group whose Lie algebra is $\mathfrak{n}$. The inner product $\langle\,,\rangle$ on $\mathfrak{n}$ induces a left-invariant Riemannian metric $\langle\,,\rangle$ on N. Using this notation, we have

$$\Omega_{a_0}(\tilde{X}, \tilde{Y}) = \langle \tilde{X}, \phi\tilde{Y}\rangle = \langle \tilde{a}_0, [\tilde{Y}, \tilde{X}]\rangle = \langle a_0, [Y, X]\rangle.$$

The exponential mapping $\exp : \mathfrak{n} \to N$ is equal to identity mapping. Hence for $X \in T_0(N)$, we have

$$\tilde{X}_x = \frac{d}{dt}(x \cdot tX)_{|t=0} = \frac{d}{dt}\left(x + tX + \frac{t}{2}[x, X]\right)_{|t=0} \in T_x(N).$$

Since the Riemannian metric on N is left-invariant, the left action of N on N itself is isometric. Hence $X \in T_0(N)$ induces a Killing vector field X^* on N by

$$X^*_x = \frac{d}{dt}(\exp tX)x_{|t=0} = \frac{d}{dt}(tX + x + \frac{t}{2}[X, x])_{|t=0} \in T_x(N).$$

The Killing vector field X^* is right-invariant.

Lemma 1. *The mapping defined by*

$$\mathfrak{n} \to \mathfrak{n}, \qquad X \mapsto X + \frac{1}{2}[X, x]$$

is a linear isomorphism.

Proof: Since the mapping is clearly linear, it is sufficient to prove that it is injective. In order to do this, we study the kernel of the mapping. Suppose that $X \in \mathfrak{n}$ satisfy the condition $X + \frac{1}{2}[X, x] = 0$. Decompose X as $X = X_1 + X_2$ where $X_1 \in \mathfrak{z}^{\perp}$ and $X_2 \in \mathfrak{z}$, then $X_1 + (X_2 + \frac{1}{2}[X_1, x]) = 0$. This implies $X_1 = 0$ and $X_2 + \frac{1}{2}[X_1, x] = 0$. Hence we have $X_2 = 0$, hence, $X = 0$. ■

By the lemma above, we have $T_x(N) = \operatorname{span}\{X^*_x \,;\, X \in \mathfrak{n}\}$ for any x in N. The Killing vector field X^* is an infinitesimal automorphism of ϕ.

Lemma 2. *Let* X *be in* $T_0(N) = \mathfrak{n}$. *For a fixed* $x \in N$, *we have* $X^*_x = \tilde{W}_x$, *where we set* $W = X + [X, x]$.

Proof: Since

$$\begin{aligned} \tilde{W}_x &= \frac{d}{dt}\left(x + tX + t[X, x] + \frac{t}{2}[x, X + [X, x]]\right)_{|t=0} \\ &= \frac{d}{dt}\left(x + tX + \frac{t}{2}[X, x]\right)_{|t=0} = X^*_x \end{aligned}$$

we have the assertion. ■

Lemma 3. *Define a one-form* η_{a_0} *on* N *by*

$$\eta_{a_0}(X^*_x) = \langle [x, X], a_0 \rangle, \qquad X \in \mathfrak{n}.$$

Then $\iota(X^*)\Omega_{a_0} = d(\eta_{a_0}(X^*))$ *for any* X *in* $\mathfrak{n}$.

Proof: Let X and Y be in $\mathfrak{n}$. By Lemma 2, we have

$$\begin{aligned} (\iota(X^*_x)\Omega_{a_0})(\tilde{Y}_x) &= \Omega_{a_0}(X^*_x, \tilde{Y}_x) \\ &= \Omega_{a_0}(\tilde{W}_x, \tilde{Y}_x) \\ &= \Omega_{a_0}(X + [X, x]), Y) \\ &= \langle a_0, [Y, X + [X, x]] \rangle = \langle a_0, [Y, X] \rangle. \end{aligned}$$

Using the above equation, we have also

$$\begin{aligned} d(\eta_{a_0}(X^*))(\tilde{Y}_x) &= \tilde{Y}_x(\eta_{a_0}(X^*)) \\ &= \frac{d}{dt}\eta_{a_0}(X^*_{x+tY+\frac{t}{2}[x,Y]})_{|t=0} \\ &= \frac{d}{dt}\langle [x+tY+\frac{t}{2}[x,Y],X],a_0\rangle \\ &= \langle [Y,X],a_0\rangle = (\iota(X^*_x)\Omega_{a_0})(\tilde{Y}_x). \end{aligned}$$

Hence we get $d(\eta_{a_0}(X^*)) = \iota(X^*)\Omega_{a_0}$. ■

Denote by $T_x(N) \to T_0(N); v \mapsto v_{\mathfrak{n}}$ the usual parallel translation in the Euclidean space $\mathfrak{n}$: Take a curve $c(t)$ in N such that $c(0) = x, \dot{c}(0) = v$. Then $v_{\mathfrak{n}} = \frac{d}{dt}(c(t) - x)_{|t=0}$. The following lemma gives a relation between the two linear isomorphisms $L_x^{-1} : T_x(N) \to T_0(N)$ and $T_x(N) \to T_0(N), v \mapsto v_{\mathfrak{n}}$.

Lemma 4. $L_x^{-1}v = v_{\mathfrak{n}} - \frac{1}{2}[x, v_{\mathfrak{n}}]$ *for* $v \in T_x(N)$.

Proof: Take a curve $c(t)$ in N such that $c(0) = x, \dot{c}(0) = v$. Then

$$\begin{aligned} L_x^{-1}v &= L_{-x}v = \frac{d}{dt}\left(-x + c(t) - \frac{1}{2}[x, c(t)]\right)_{|t=0} \\ &= \frac{d}{dt}\left(c(t) - x - \frac{1}{2}[x, c(t) - x]\right)_{|t=0} = v_{\mathfrak{n}} - \frac{1}{2}[x, v_{\mathfrak{n}}]. \end{aligned}$$

Hence we have the assertion. ■

Similarly we define $T_y(\mathfrak{z}^\perp) \to T_0(\mathfrak{z}^\perp), u \mapsto u_{\mathfrak{z}^\perp}$ and $T_z(\mathfrak{z}) \to T_0(\mathfrak{z}), w \mapsto w_{\mathfrak{z}}$. Since $\mathfrak{z}$ is abelian, we have $L_z^{-1}w = w_{\mathfrak{z}}$ for $w \in T_z(\mathfrak{z})$. Hence we can write $w = w_{\mathfrak{z}}$. Let $x \in \mathfrak{n}$ and $v \in T_x(\mathfrak{n})$. Expressing x and v as $x = y + z$ and $v = v_1 + v_2$, where $y \in \mathfrak{z}^\perp, z \in \mathfrak{z}, v_1 \in T_y(\mathfrak{z}^\perp)$ and $v_2 \in T_z(\mathfrak{z})$ we obtain

$$L_x^{-1}v = (v_1)_{\mathfrak{z}^\perp} + \left(v_2 - \frac{1}{2}[y, (v_1)_{\mathfrak{z}^\perp}]\right). \tag{4}$$

Proposition 1. Let $x(t) = y(t) + z(t)$ be a curve in $\mathfrak{n}$, where $y(t) \in \mathfrak{z}^\perp$ and $z(t) \in \mathfrak{z}$. Assume that $y(0) = 0$. Then $x(t)$ describes the motion of a charged particle (3) if and only if

$$\dot{y}(t)_{\mathfrak{z}^\perp} - \phi_{\dot{z}(0)+a_0}y(t) = \dot{y}(0), \qquad \dot{z}(t) - \frac{1}{2}[y(t), \dot{y}(t)_{\mathfrak{z}^\perp}] = \dot{z}(0). \tag{5}$$

Proof: Taking the inner product of (3) and the Killing vector field X^* for $X \in \mathfrak{n}$, we have

$$\frac{d}{dt}\langle \dot{x}, X^*\rangle = \Omega(X^*, \dot{x}) = (\iota(X^*)\Omega)(\dot{x}).$$

Using Lemma 3 we find

$$\frac{d}{dt}\langle \dot{x}, X^*\rangle = (d(\eta(X^*)))(\dot{x}) = \frac{d}{dt}\eta(X^*_{x(t)}).$$

Since $T_x(N) = \mathrm{span}\{X^*_x;\, X \in \mathfrak{n}\}$, the equation (3) is equivalent to

$$\frac{d}{dt}(\langle \dot{x}(t), X^*_{x(t)}\rangle - \eta(X^*_{x(t)})) = 0.$$

By the definition of η, we have

$$\eta(X^*_{x(t)}) = \langle [x(t), X], a_0\rangle = \langle \phi_{a_0}(y(t)), X\rangle.$$

Since $\langle\,,\rangle$ is left invariant

$$\begin{aligned}
\langle \dot{x}, X^*_{x(t)}\rangle &= \langle L_x^{-1}\dot{x}, L_x^{-1}X^*_x\rangle \\
&= \left\langle \dot{y}_{\mathfrak{z}^\perp} + (\dot{z} - \frac{1}{2}[y, \dot{y}_{\mathfrak{z}^\perp}]), X + [X, x]\right\rangle \\
&= \langle \dot{y}_{\mathfrak{z}^\perp}, X\rangle + \left\langle \dot{z} - \frac{1}{2}[y, \dot{y}_{\mathfrak{z}^\perp}], X + [X, x]\right\rangle
\end{aligned}$$

where we have used Lemma 2 and equation (4). Hence the equation (3) is equivalent to

$$\frac{d}{dt}\left(\langle \dot{y}_{\mathfrak{z}^\perp} - \phi_{a_0}(y), X\rangle + \langle \dot{z} - \frac{1}{2}[y, \dot{y}_{\mathfrak{z}^\perp}], X + [X, y]\rangle\right) = 0.$$

Taking $X \in \mathfrak{z}$, we have

$$\dot{z}(t) - \frac{1}{2}[y(t), \dot{y}(t)_{\mathfrak{z}^\perp}] = \dot{z}(0)$$

where we have used the initial condition $y(0) = 0$. Next, taking $X \in \mathfrak{z}^\perp$, we have

$$\frac{d}{dt}\left(\langle \dot{y}_{\mathfrak{z}^\perp} - \phi_{a_0}(y), X\rangle + \langle \dot{z}(0), [X, y]\rangle\right) = 0.$$

Taking into account the initial condition $y(0) = 0$, we finally have

$$\dot{y}(t)_{\mathfrak{z}^\perp} - \phi_{\dot{z}(0)+a_0} y(t) = \dot{y}(0).$$

■

Proposition 2. The motion of a charged particle (3) with $y(0) = 0$ is given by the equations

$$\begin{aligned}
y(t) &= \exp t\phi_{\dot{z}(0)+a_0}\int_0^t \exp(-t\phi_{\dot{z}(0)+a_0})\dot{y}(0)dt \\
z(t) &= z(0) + t\dot{z}(0) + \frac{1}{2}\int_0^t [y(t), (\exp t\phi_{\dot{z}(0)+a_0})\dot{y}(0)]dt.
\end{aligned}$$

Proof: Using the first equation of (5) with $y(0) = 0$, we have

$$y(t) = \exp t\phi_{\dot{z}(0)+a_0} \int_0^t \exp(-t\phi_{\dot{z}(0)+a_0})\dot{y}(0)dt.$$

Hence

$$\phi_{\dot{z}(0)+a_0} y(t) = (\exp t\phi_{\dot{z}(0)+a_0} - 1)\dot{y}(0)$$

which implies that

$$\phi_{\dot{z}(0)+a_0} y(t) + \dot{y}(0) = (\exp t\phi_{\dot{z}(0)+a_0})\dot{y}(0).$$

Using the second and the first equation from (5)

$$\begin{aligned} z(t) &= z(0) + t\dot{z}(0) + \frac{1}{2}\int_0^t [y(t), \dot{y}(t)_{\mathfrak{z}^\perp}]dt \\ &= z(0) + t\dot{z}(0) + \frac{1}{2}\int_0^t [y(t), \phi_{\dot{z}(0)+a_0} y(t) + \dot{y}(0)]dt \\ &= z(0) + t\dot{z}(0) + \frac{1}{2}\int_0^t [y(t), (\exp t\phi_{\dot{z}(0)+a_0})\dot{y}(0)]dt. \end{aligned}$$

Hence we get the assertion. ■

When $\phi_{\dot{z}(0)+a_0} = 0$, then, using the above Proposition, we get $y(t) = t\dot{y}(0)$ and

$$z(t) = z(0) + t\dot{z}(0) + \frac{1}{2}\int_0^t [t\dot{y}(0), \dot{y}(0)]dt = z(0) + t\dot{z}(0).$$

Lemma 5. *The equation of motion* (3) *implies the following relation*

$$\frac{d}{dt}(\langle z(t), \dot{z}(0) + a_0\rangle + \frac{1}{2}\langle y(t), \dot{y}(0)\rangle) = |\dot{z}(0)|^2 + \langle \dot{z}(0), a_0\rangle + \frac{1}{2}|\dot{y}_{\mathfrak{z}^\perp}|^2.$$

Proof: Taking the inner product of the second equation of (5) with $\dot{z}(0) + a_0$, we have

$$\langle \dot{z}, \dot{z}(0) + a_0\rangle - \frac{1}{2}\langle [y, \dot{y}_{\mathfrak{z}^\perp}], \dot{z}(0) + a_0\rangle = |\dot{z}(0)|^2 + \langle \dot{z}(0), a_0\rangle.$$

Using equation (5) again produces

$$\begin{aligned} \langle [y, \dot{y}_{\mathfrak{z}^\perp}], \dot{z}(0) + a_0\rangle &= \langle \phi_{\dot{z}(0)+a_0} y, \dot{y}_{\mathfrak{z}^\perp}\rangle \\ &= \langle \dot{y}_{\mathfrak{z}^\perp} - \dot{y}(0), \dot{y}_{\mathfrak{z}^\perp}\rangle \\ &= |\dot{y}_{\mathfrak{z}^\perp}|^2 - \langle \dot{y}_{\mathfrak{z}^\perp}, \dot{y}(0)\rangle = |\dot{y}_{\mathfrak{z}^\perp}|^2 - \frac{d}{dt}\langle y(t), \dot{y}(0)\rangle. \end{aligned}$$

Hence

$$\frac{d}{dt}(\langle z(t), \dot{z}(0) + a_0\rangle + \frac{1}{2}\langle y(t), \dot{y}(0)\rangle) = |\dot{z}(0)|^2 + \langle \dot{z}(0), a_0\rangle + \frac{1}{2}|\dot{y}_{\mathfrak{z}^\perp}|^2.$$

■

Applying the lemma above for geodesics, we can re-demonstrate the following theorem of Naitoh-Sakane.

Theorem 2. (Naitoh-Sakane [4, Corrolary 3.3]) *Every geodesic in any simply connected two-step nilpotent Lie group with a left-invariant Riemannian metric does not intersect itself.*

Proof: Let $x(t) = y(t) + z(t) \in N$ be a geodesic with $y(0) = 0$. Applying Lemma 5 with $a_0 = 0$

$$\frac{d}{dt}\left(\langle z(t), \dot{z}(0)\rangle + \frac{1}{2}\langle y(t), \dot{y}(0)\rangle\right) = |\dot{z}(0)|^2 + \frac{1}{2}|\dot{y}_{\mathfrak{z}^\perp}|^2 > 0.$$

Hence $\langle z(t), \dot{z}(0)\rangle + \frac{1}{2}\langle y(t), \dot{y}(0)\rangle$ is monotone increasing. Thus $x(t)$ is not periodic. Since we have already proved that $x(t)$ is simple by Theorem 1, we get the assertion. ■

The author thinks that the above proof is easier than the original proof of Naitoh-Sakane.

3. Charged Particles in H-type Lie Groups

In this section, we study the motion of a charged particle in a simply connected H-type Lie group. First we review the definition of H-type Lie algebra according to Kaplan. Let $(U, \langle\, ,\rangle)$ and $(V, \langle\, ,\rangle)$ be finite-dimensional real vector spaces with inner products $\langle\, ,\rangle$. Denote by $\mathrm{End}(V)$ the vector space consisting of all linear transformations on V. We assume that there exists a linear mapping $j : U \to \mathrm{End}(V)$ such that

$$j(a)^2 = -|a|^2 I, \qquad |j(a)x| = |a||x|, \qquad a \in U, \quad x \in V. \tag{6}$$

In other words, V is a Clifford module over the Clifford algebra generated by U. By (6) we have

$$\langle j(a)x, j(b)x\rangle = \langle a, b\rangle |x|^2, \qquad \langle j(a)x, j(a)y\rangle = |a|^2\langle x, y\rangle$$
$$\langle j(a)x, y\rangle + \langle x, j(a)y\rangle = 0, \qquad a, b \in U, \quad x, y \in V.$$

Define a bi-linear mapping $[\, ,] : V \times V \to U$ via the formula

$$\langle a, [x, y]\rangle = \langle j(a)x, y\rangle, \qquad a \in U, \quad x, y \in V. \tag{7}$$

Then $[\, ,]$ is alternative. Substituting $j(b)x$ into y, we have

$$\langle a, [x, j(b)x]\rangle = \langle j(a)x, j(b)x\rangle = \langle a, b\rangle |x|^2.$$

Hence

$$[x, j(b)x] = |x|^2 b, \qquad b \in U, \quad x \in V. \tag{8}$$

We denote by $\mathfrak{n} = U \oplus V$ the orthogonal direct sum of U and V, and define a Lie algebra structure on $\mathfrak{n}$ by

$$[a + x, b + y] = [x, y] \in U, \qquad a, b \in U, \quad x, y \in V.$$

Then the Lie algebra $\mathfrak{n}$ is called **H-type**. Since the H-type Lie algebra $\mathfrak{n}$ is a kind of two-step nilpotent Lie algebra with an inner product, we can define a Lie group structure on $\mathfrak{n}$ with a left-invariant Riemannian metric, whose Lie algebra is $\mathfrak{n}$ itself as we mentioned in the previous section. For $a_0 \in U$, we consider the equation

$$\nabla_{\dot{x}}\dot{x} = j(a_0)\dot{x} \tag{9}$$

of motion of a charged particle. If we express its trajectory as $x(t) = y(t) + z(t)$ where $y(t) \in V, z(t) \in U$, then (9) is equivalent to

$$\dot{y}(t)_V - j(\dot{z}(0) + a_0)y(t) = \dot{y}(0) \tag{10}$$

where $T_y(V) \to V, w \mapsto w_V$ denotes the usual parallel translation in V. Here we have used equation (5).

Theorem 3. *Let* $x(t) = y(t) + z(t) \in N$ (*where* $y(t) \in V, z(t) \in U$) *is a motion of a charged particle* (9) *with* $x(0) = 0$.
1) *When* $\dot{z}(0) + a_0 = 0$*, then* $x(t) = t\dot{x}(0)$.
2) *When* $\dot{z}(0) + a_0 \neq 0$*, then*

$$y(t) = \frac{\sin(t|\dot{z}(0) + a_0|)}{|\dot{z}(0) + a_0|}\dot{y}(0) + \frac{1 - \cos(t|\dot{z}(0) + a_0|)}{|\dot{z}(0) + a_0|^2}j(\dot{z}(0) + a_0)\dot{y}(0)$$

$$z(t) = t\dot{z}(0) + \frac{t|\dot{y}(0)|^2}{2|\dot{z}(0) + a_0|^2}(\dot{z}(0) + a_0) - \frac{\sin(t|\dot{z}(0) + a_0|)}{2|\dot{z}(0) + a_0|^3}|\dot{y}(0)|^2(\dot{z}(0) + a_0).$$

The curve $y(t)$ *is a circle in* V*. The motion of a charged particle is periodic if and only if*

$$a_0 = -\left(\frac{|\dot{y}(0)|^2}{2|\dot{z}(0)|^2} + 1\right)\dot{z}(0).$$

In this case $x(t)$ *is an elliptic motion.*

Remark 1. When $x(t)$ is a geodesic, the condition $a_0 = 0$ implies the theorem of Kaplan [3].

Proof: 1) is clear from (10). We will show 2). Using the first equation of (10), we have

$$y(t) = \frac{\sin(t|\dot{z}(0) + a_0|)}{|\dot{z}(0) + a_0|}\dot{y}(0) + \frac{1 - \cos(t|\dot{z}(0) + a_0|)}{|\dot{z}(0) + a_0|^2}j(\dot{z}(0) + a_0)\dot{y}(0)$$

which implies that

$$\dot{y}(t)_V = \cos(t|\dot{z}(0) + a_0|)\dot{y}(0) + \frac{\sin(t|\dot{z}(0) + a_0|)}{|\dot{z}(0) + a_0|}j(\dot{z}(0) + a_0)\dot{y}(0).$$

Using the equation above, we have

$$[y(t)_V, \dot{y}(t)] = \frac{1-\cos(t|\dot{z}(0)+a_0|)}{|\dot{z}(0)+a_0|^2}[\dot{y}(0), j(\dot{z}(0)+a_0)\dot{y}(0)].$$

Further the second equation of (10) gives

$$\begin{aligned} \dot{z}(t) &= \dot{z}(0) + \frac{1-\cos(t|\dot{z}(0)+a_0|)}{2|\dot{z}(0)+a_0|^2}[\dot{y}(0), j(\dot{z}(0)+a_0)\dot{y}(0)] \\ &= \dot{z}(0) + \frac{1-\cos(t|\dot{z}(0)+a_0|)}{2|\dot{z}(0)+a_0|^2}(\dot{z}(0)+a_0)|\dot{y}(0)|^2 \end{aligned} \tag{11}$$

where we have used the equation (8). Since

$$\begin{aligned} y(t) - \frac{1}{|\dot{z}(0)+a_0|} j\left(\frac{\dot{z}(0)+a_0}{|\dot{z}(0)+a_0|}\right)\dot{y}(0) = {} & \frac{\sin(|\dot{z}(0)+a_0|t)}{|\dot{z}(0)+a_0|}\dot{y}(0) \\ & - \frac{\cos(|\dot{z}(0)+a_0|t)}{|\dot{z}(0)+a_0|} j\left(\frac{\dot{z}(0)+a_0}{|\dot{z}(0)+a_0|}\right)\dot{y}(0) \end{aligned}$$

the curve $y(t)$ is a circle in V whose center is $\frac{1}{|\dot{z}(0)+a_0|} j\left(\frac{\dot{z}(0)+a_0}{|\dot{z}(0)+a_0|}\right)\dot{y}(0)$ and the radius is $\frac{|\dot{y}(0)|}{|\dot{z}(0)+a_0|}$. The periodic condition is as follows

$$\begin{aligned} x(t) \text{ is periodic} &\Leftrightarrow \dot{z}(0) + \frac{|\dot{y}(0)|^2}{2|\dot{z}(0)+a_0|^2}(\dot{z}(0)+a_0) = 0 \\ &\Leftrightarrow a_0 = -\left(\frac{|\dot{y}(0)|^2}{2|\dot{z}(0)|^2}+1\right)\dot{z}(0). \end{aligned}$$

In this case, since

$$\begin{aligned} x(t) + \frac{2|\dot{z}(0)|}{|\dot{y}(0)|^2} j\left(\frac{\dot{z}(0)}{|\dot{z}(0)|}\right)\dot{y}(0) = \frac{2|\dot{z}(0)|}{|\dot{y}(0)|^2}\Bigg(& \sin\left(\frac{|\dot{y}(0)|^2}{2|\dot{z}(0)|}t\right)(\dot{y}(0)+\dot{z}(0)) \\ & + \cos\left(\frac{|\dot{y}(0)|^2}{2|\dot{z}(0)|}t\right) j\left(\frac{\dot{z}(0)}{|\dot{z}(0)|}\right)\dot{y}(0)\Bigg) \end{aligned}$$

the curve $x(t)$ is an elliptic such that the ratio of the long axis to the short axis is equal to $\sqrt{|\dot{y}(0)|^2+|\dot{z}(0)|^2}/|\dot{y}(0)|$. ■

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