

RECURSION OPERATORS AND REDUCTIONS OF INTEGRABLE EQUATIONS ON SYMMETRIC SPACES*

VLADIMIR S. GERDJKOV, ALEXANDER V. MIKHAILOV[†] and
TIHOMIR I. VALCHEV

*Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy
of Sciences, 1784 Sofia, Bulgaria*

[†]*Applied Mathematics Department, University of Leeds, Leeds, LS2 9JT, UK*

Abstract. We study certain classes of integrable nonlinear differential equations related to the type symmetric spaces. Our main examples concern equations related to **A.III**-type symmetric spaces. We use the Cartan involution corresponding to this symmetric space as an element of the reduction group and restrict generic Lax operators to this symmetric space. Next we outline the spectral theory of the reduced Lax operator L and construct its fundamental analytic solutions. Analyzing the Wronskian relations we introduce the ‘squared solutions’ of L and derive the recursion operators by three different methods.

CONTENTS

1. Introduction	12
2. Preliminaries	13
2.1. Generalized Zakharov-Shabat Systems	13
2.2. Gauge Transformations and Generalized Heisenberg Ferromagnets ..	17
2.3. The Group of Reductions	19
3. Reductions of Generalized HF Equation	20
3.1. Lax Representation	20
3.2. Spectral Properties of L	22
4. Recursion Operators	25

*Reprinted from *J. Geom. Symmetry Phys.* **20** (2010) 1–34.

4.1. Recursion Operators Through GKS Approach.....	25
4.2. Recursion Operators Through the Recursion Relations.....	28
5. Wronskian Relations and ‘Squared Solutions’ of L	31
5.1. The Skew-Scalar Product.....	32
5.2. The Mapping $\mathfrak{M}_{L_1} \rightarrow \mathfrak{T}_j$	33
6. Recursion Operators – An Alternative Approach.....	34
7. The Class of NLEE and the Recursion Operators.....	36
8. Conclusions.....	37
Acknowledgements.....	38
Appendix A. Derivation of $\text{ad}_{L_1}^{-1}$	38
Appendix B. Derivation of the ‘Adjoint’ Recursion Operators.....	40
References.....	41

1. Introduction

One of classical models of statistical physics is provided by **Heisenberg’s equation**

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx}, \quad \mathbf{S}^2 = 1 \quad (1)$$

which describes the behavior of an one-dimensional ferromagnet characterized by a spin vector $\mathbf{S}(x, t)$ in a closest neighbors approximation. By making use of the Lie algebras isomorphism

$$e_i \leftrightarrow \sigma_i, \quad i = 1, 2, 3$$

where σ_i are **Pauli’s matrices**

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

one is able to set equation (1) into a matrix form

$$iS_t = \frac{1}{2}[S, S_{xx}], \quad S(x, t) = \sum_{k=1}^3 S_k(x, t)\sigma_k, \quad S^2 = \mathbb{1}. \quad (2)$$

Heisenberg’s equation is integrable in the sense of inverse scattering transform [25] and moreover it is gauge equivalent to the nonlinear Schrödinger equation. Its Lax representation reads

$$L(\lambda) \equiv i\partial_x - \lambda S, \quad A(\lambda) \equiv i\partial_t + \frac{i\lambda}{2}[S, S_x] + 2\lambda^2 S. \quad (3)$$

The purpose of the present paper is to derive nonlinear evolution equations to generalize Heisenberg’s model and study some of the properties of their Lax operators.

We are going to focus our attention on equations whose Lax representation is related to $\mathfrak{su}(3)$.

This paper is a natural continuation of our previous paper [10]. In Section 2 below we give some of the necessary preliminaries. In Section 3 we study the \mathbb{Z}_2 -reductions of the generalized Heisenberg ferromagnets related to symmetric spaces. There we outline the spectral properties of the relevant Lax operator and the construction of its fundamental analytic solutions. Section 4 is devoted to the derivation of the recursion operators Λ . Here we first derive Λ using the Gürses-Karasu-Sokolov (GKS) method [15]. We also outline a second way of deriving Λ based on the recursion relations for the coefficients of the A operator in the Lax representation. In Section 5 we analyze the Wronskian relations as basic tool in the inverse scattering method [2, 3]. From them there naturally arise the ‘squared solutions’, which play a fundamental role also in the analysis of the mapping between the set of allowed potentials and the minimal sets of scattering data. It is well known [1, 11] that for a wide class of Lax operators such mappings can be viewed as generalized Fourier transforms, in which the ‘squared solutions’ play the role of generalized exponents. In Section 6 we recalculate the recursion operators, now considering them as the operators, whose eigenfunctions are the ‘squared solutions’. Thus we are able to show that all three definitions for Λ are compatible. In Section 7 we briefly outline some of the fundamental properties of the NLEE. We end with some conclusions in Section 8.

2. Preliminaries

2.1. Generalized Zakharov-Shabat Systems

We are going to deal with nonlinear evolution equations (NLEE) to represent the compatibility condition $[L, A] = 0$ of two differential operators of the form

$$\begin{aligned} L\psi(x, t, \lambda) &\equiv (i\partial_x + U(x, t, \lambda)) \psi(x, t, \lambda) = 0 \\ A\psi(x, t, \lambda) &\equiv (i\partial_t + V(x, t, \lambda)) \psi(x, t, \lambda) = \psi(x, t, \lambda)f(\lambda) \\ U(x, t, \lambda) &= q(x, t) - \lambda J_0 \end{aligned} \quad (4)$$

where U and V take values in some simple Lie algebra and λ is a **spectral parameter**. The fundamental solutions to (4) then take values in the corresponding Lie group. Since the compatibility condition of L and A holds identically with respect to λ one obtains a sequence of recurrent differential relations. Solving these relations generates the very NLEE.

From now on we assume that $J_0 = \text{diag}(J_{0,1}, J_{0,2}, \dots, J_{0,N})$ is a real constant regular matrix to fulfill $J_{0,1} > J_{0,2} > \dots > J_{0,N}$ while the potential q is smooth

function obeying zero boundary conditions

$$\lim_{x \rightarrow \pm\infty} |x|^k q = 0, \quad \text{for all } k \in \mathbb{N}.$$

It can be shown that the spectrum of a generic operator L consists of a continuous and a discrete part. Due to the requirements imposed on q and J the continuous spectrum of L coincides with the real axis in the complex λ -plane.

A significant role in the scattering theory of operator L is played by fundamental solutions ψ_+ and ψ_- called **Jost solutions**. They are normalized at infinity as follows

$$\lim_{x \rightarrow \pm\infty} \psi_{\pm}(x, t, \lambda) e^{i\lambda J_0 x} = \mathbb{1}, \quad \lambda \in \mathbb{R}. \quad (5)$$

One assures that the definition above is correct for all t by choosing $f(\lambda)$ in (4) in a appropriate way, namely

$$f(\lambda) = \lim_{x \rightarrow \pm\infty} V(x, t, \lambda).$$

The transition matrix between Jost solutions

$$\psi_-(x, t, \lambda) = \psi_+(x, t, \lambda) T(t, \lambda) \quad (6)$$

is called **scattering matrix**. Its time evolution is driven by the dispersion law $f(\lambda) = \lim_{x \rightarrow \pm\infty} V(x, t, \lambda)$ of NLEE as follows

$$T(t, \lambda) = e^{if(\lambda)t} T(0, \lambda) e^{-if(\lambda)t}.$$

Jost solutions do not permit analytic continuation in $\mathbb{C} \setminus \mathbb{R}$. However, it is possible to construct another pair of **fundamental solutions** χ^+ and χ^- with analytic properties in the upper half plane \mathbb{C}_+ and the lower half plane \mathbb{C}_- respectively by using the factors in the **Gauss decomposition** of T as follows

$$T(t, \lambda) = T^{\mp}(t, \lambda) D^{\pm}(\lambda) (S^{\pm}(t, \lambda))^{-1}$$

where the matrices $S^{\pm}(\lambda)$, $D^{\pm}(\lambda)$ and $T^{\pm}(\lambda)$ are of the form

$$S^+(\lambda) = \begin{pmatrix} 1 & S_{12}^+ & \dots & S_{1n}^+ \\ 0 & 1 & \dots & S_{2n}^+ \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}, \quad T^+(\lambda) = \begin{pmatrix} 1 & T_{12}^+ & \dots & T_{1n}^+ \\ 0 & 1 & \dots & T_{2n}^+ \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \quad (7)$$

$$D^+(\lambda) = \text{diag}(D_1^+, \dots, D_n^+), \quad D^-(\lambda) = \text{diag}(D_1^-, \dots, D_n^-) \quad (8)$$

$$S^-(\lambda) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ S_{21}^- & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S_{n1}^- & S_{n2}^- & \dots & 1 \end{pmatrix}, \quad T^-(\lambda) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ T_{21}^- & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1}^- & T_{n2}^- & \dots & 1 \end{pmatrix}. \quad (9)$$

It is well known how given $T(\lambda)$ one can construct explicitly its Gauss decomposition, see [6,27]. Here we give only the expressions for $D^\pm(\lambda)$ which are analytic functions for $\lambda \in \mathbb{C}_\pm$ respectively

$$D_j^+(\lambda) = \frac{m_j^+(\lambda)}{m_{j-1}^+(\lambda)}, \quad D_j^-(\lambda) = \frac{m_{n-j+1}^-(\lambda)}{m_{n-j}^-(\lambda)} \quad (10)$$

where m_j^\pm are the principal upper and lower minors of $T(\lambda)$ of order j .

Then the **fundamental analytic solutions** (FAS) of L are related to the Jost solutions [27] by

$$\chi^\pm(x, \lambda) = \psi_-(x, \lambda)S^\pm(\lambda) = \psi_+(x, \lambda)T^\mp(\lambda)D^\pm(\lambda). \quad (11)$$

It follows immediately that χ^+ and χ^- are solutions of a local Riemann-Hilbert problem

$$\chi^+(x, \lambda) = \chi^-(x, \lambda)G(\lambda), \quad \lambda \in \mathbb{R} \quad (12)$$

where the sewing function G is given by

$$G(\lambda) = [S^-(\lambda)]^{-1}S^+(\lambda).$$

It is important to note that the sewing function determines the minimal set of scattering data of L on the continuous spectrum. Analysis based on the Wronskian relations [1–3], (see also [11] and numerous references therein) allows one to interpret the inverse scattering method as a generalized Fourier transform. Skipping the details we formulate the main results

$$\begin{aligned} \mathfrak{T}_1 &\equiv \{\tau_\alpha^\pm(\lambda); \alpha > 0, \lambda \in \mathbb{R}\} \\ \mathfrak{T}_2 &\equiv \{\rho_\alpha^\pm(\lambda); \alpha > 0, \lambda \in \mathbb{R}\} \end{aligned} \quad (13)$$

where

$$\begin{aligned} \tau_\alpha^\pm(\lambda) &= i[[\text{ad}_{J_0}^{-1}[B_0, q(x)], e_{\mp\alpha}^\pm(x, \lambda)]]], \quad \alpha > 0 \\ \rho_\alpha^\pm(\lambda) &= -i[[\text{ad}_{J_0}^{-1}[B_0, q(x)], e_{\pm\alpha}^\pm(x, \lambda)]]], \quad \alpha > 0. \end{aligned} \quad (14)$$

The notations that we have used above are as follows. We have assumed that J_0 and $q(x)$ belong to the simple Lie algebra \mathfrak{g} (in our case $\mathfrak{g} \simeq \mathfrak{sl}(N)$, the positive root α belongs to the root system Δ (in our case, $\alpha = e_i - e_j$ with $i < j$) and E_α is the corresponding **Weyl generator** (in our case $E_\alpha = E_{ij}$, where the $N \times N$ matrix E_{ij} has only one non-vanishing matrix element at position (i, j) , i.e., $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$), for the other simple Lie algebras see [16]. We have also made use of the skew-scalar product

$$[[X, Y]] = \int_{-\infty}^{\infty} dx \langle X(x), [J_0, Y(x)] \rangle \quad (15)$$

where $X(x)$ and $Y(x)$ are smooth functions vanishing for $x \rightarrow \pm\infty$ and taking values in the co-adjoint orbit of \mathfrak{g} passing through J_0 . We will also need the projector π_{J_0} onto the image of ad_{J_0}

$$\text{ad}_{J_0} X \equiv [J_0, X], \quad \pi_{J_0} = \text{ad}_{J_0}^{-1} \text{ad}_{J_0} \quad (16)$$

i.e., $X = \pi_{J_0} X$ and $Y = \pi_{J_0} Y$. We will denote the linear space of all such functions by \mathfrak{M}_{J_0} . It is well known that this will be the phase space of the NLEE related to L . Finally we have to introduce the squared solutions

$$e_{\pm\alpha}^{\pm}(x, \lambda) = \chi^{\pm} E_{\alpha}(\chi^{\pm})^{-1}(x, \lambda). \quad (17)$$

Each squared solution takes values in \mathfrak{g} , satisfies the equation

$$i \frac{\partial e_{\pm\alpha}^{\pm}}{\partial x} + [q - \lambda J_0, e_{\pm\alpha}^{\pm}(x, \lambda)] = 0 \quad (18)$$

and can be naturally split into

$$e_{\pm\alpha}^{\pm}(x, \lambda) = \pi_{J_0} e_{\pm\alpha}^{\pm}(x, \lambda) + (\mathbb{1} - \pi_{J_0}) e_{\pm\alpha}^{\pm}(x, \lambda). \quad (19)$$

It is also easy to see that it is only $e_{\pm\alpha}^{\pm}(x, \lambda) = \pi_{J_0} e_{\pm\alpha}^{\pm}(x, \lambda)$ that contributes to the skew-scalar products in (14).

It can be proved [5, 9] (see also the review paper [6] and the monograph [11]), that the set of squared solutions $e_{\pm\alpha}^{\pm}(x, \lambda)$ form complete sets of functions in the space \mathfrak{M}_{J_0} . This fact allows one to prove that the inverse scattering method is in fact a generalized Fourier transform. The completeness relation for the squared solutions can be viewed also as the spectral decomposition of the recursion operators Λ_{\pm} for which the squared solutions are eigenfunctions

$$\Lambda_+ e_{\mp\alpha}^{\pm}(x, \lambda) = \lambda e_{\mp\alpha}^{\pm}(x, \lambda), \quad \Lambda_- e_{\pm\alpha}^{\pm}(x, \lambda) = \lambda e_{\pm\alpha}^{\pm}(x, \lambda), \quad \alpha > 0. \quad (20)$$

Inserting the splitting (19) into equation (18) one derives the following explicit form of the recursion operator [5, 9]

$$\Lambda_{\pm} X = \text{ad}_{J_0}^{-1} \left\{ i \frac{\partial X}{\partial x} + \pi_{J_0} [q(x), X] + i \pi_{J_0} \left[q(x), (\mathbb{1} - \pi_{J_0}) \int_{\pm\infty}^x [q(y), X(y)] \right] \right\}.$$

The recursion operators Λ_{\pm} and their conjugate Λ_{\pm}^* with respect to the skew-scalar product

$$[[X, \Lambda_{\pm} Y]] = [[\Lambda_{\pm}^* X, Y]], \quad \Lambda_{\pm}^* = \Lambda_{\mp} \quad (21)$$

allow one to derive all fundamental properties of the NLEE. For example, the hierarchy of the NLEE related to L in (4) have the form

$$i \text{ad}_{J_0}^{-1} \frac{\partial q}{\partial t} + c_k (\Lambda_{\pm}^*)^k \text{ad}_{J_0}^{-1} [B_0, q(x, t)] = 0 \quad (22)$$

are characterized by dispersion laws $f(\lambda) = c_k \lambda^k B_0$ and are equivalent to the following linear evolution equations for the scattering data

$$i \frac{\partial S^\pm}{\partial t} + c_k \lambda^k [B_0, S^\pm(\lambda, t)] = 0. \quad (23)$$

2.2. Gauge Transformations and Generalized Heisenberg Ferromagnets

In this subsection we are going to discuss how the properties of Lax operators are affected by the action of a gauge transformation g . For more details we refer the reader to [8, 11, 25, 27].

If we apply a **gauge transformation**, say

$$g : \psi(x, t, \lambda) \rightarrow \tilde{\psi}(x, t, \lambda) = g^{-1}(x, t) \psi(x, t, \lambda)$$

then the Lax representation remains intact

$$0 = [\tilde{L}, \tilde{A}] = [L, A], \quad \tilde{L} = g^{-1} L g, \quad \tilde{A} = g^{-1} A g. \quad (24)$$

Thus one can associate different Lax pairs to equivalent NLEE but written in terms of changed dependent variables

$$\tilde{U} = i g^{-1} \partial_x g + g^{-1} U g, \quad \tilde{V} = i g^{-1} \partial_t g + g^{-1} V g.$$

Our further study concerns NLEEs which are gauge equivalent to these whose auxiliary linear problem $L\psi = 0$ is generalized Zakharov-Shabat system. In what follows we fix up

$$g(x, t) = \psi_+(x, t, \lambda = 0), \quad \text{i.e.,} \quad \lim_{x \rightarrow \infty} g(x, t) = \mathbb{1}. \quad (25)$$

Then the corresponding Lax operator takes the form

$$\tilde{L}\tilde{\psi} \equiv i \frac{\partial \tilde{\psi}}{\partial x} - \lambda S(x, t) \tilde{\psi}(x, t, \lambda) = 0 \quad (26)$$

where

$$S(x, t) = g^{-1} J_0 g(x, t). \quad (27)$$

We also impose a constraint on $S(x, t)$ by requesting that

$$\lim_{x \rightarrow \infty} S(x, t) = \lim_{x \rightarrow -\infty} S(x, t) = J_0. \quad (28)$$

From this condition and from equation (25) there follows that $\lim_{x \rightarrow -\infty} g(x, t) = T_0^{-1} \equiv T^{-1}(0)$ must be diagonal matrix.

The Jost solutions $\tilde{\psi}_\pm(x, t, \lambda)$ and the fundamental analytic solutions $\tilde{\chi}^\pm(x, t, \lambda)$ of \tilde{L} are related to the ones for L as follows (here we skip the variable t)

$$\begin{aligned} \tilde{\psi}_+(x, \lambda) &= g^{-1}(x, t) \psi_+(x, \lambda), & \tilde{\psi}_-(x, \lambda) &= g^{-1}(x) \psi_-(x, \lambda) T_0^{-1} \\ \tilde{\chi}^+(x, \lambda) &= g^{-1}(x) \chi^+(x, \lambda) T_0^{-1}, & \tilde{\chi}^-(x, \lambda) &= g^{-1}(x) \chi^-(x, \lambda) T_0^{-1}. \end{aligned} \quad (29)$$

The relations between the scattering matrices and their Gauss factors read

$$\begin{aligned} \tilde{T}(\lambda, t) &= T(\lambda, t)T_0^{-1}, & \tilde{S}^\pm(\lambda, t) &= T_0S^\pm(\lambda, t)T_0^{-1} \\ \tilde{D}^\pm(\lambda) &= D^\pm(\lambda)T_0^{-1}, & \tilde{T}^\pm(\lambda, t) &= T^\pm(\lambda, t). \end{aligned} \quad (30)$$

The FAS satisfy slightly different Riemann-Hilbert problem

$$\tilde{\psi}^+(x, t, \lambda) = \tilde{\psi}^-(x, t, \lambda)\tilde{G}(\lambda, t), \quad \tilde{G}(\lambda, t) = T_0G(\lambda, t)T_0^{-1}. \quad (31)$$

Next we start and apply consecutively the gauge transformation to all formulae in Subsection 2.1. This is possible because in all steps above we used explicitly gauge covariant formulations on all steps.

The minimal sets of scattering data are

$$\begin{aligned} \tilde{\mathfrak{I}}_1 &\equiv \{\tilde{\tau}_\alpha^\pm(\lambda); \alpha > 0, \lambda \in \mathbb{R}\}, & \tilde{\tau}_\alpha^\pm(\lambda) &= \tau_\alpha^\pm(\lambda)\tau_{0,\alpha}^\pm \\ \tilde{\mathfrak{I}}_2 &\equiv \{\tilde{\rho}_\alpha^\pm(\lambda); \alpha > 0, \lambda \in \mathbb{R}\}, & \tilde{\rho}_\alpha^\pm(\lambda) &= \rho_\alpha^\pm(\lambda) \\ \tau_{0,\alpha}^\pm &= \langle T_0^{-1}E_{\mp\alpha}T_0, E_{\pm\alpha} \rangle. \end{aligned} \quad (32)$$

In deriving the Wronskian relations some modifications occurs, which lead to the necessity of a modified skew-scalar product of the form

$$[[\tilde{X}, \tilde{Y}]] = \int_{-\infty}^{\infty} dx \langle \tilde{X}(x), [S(x), \tilde{Y}(x)] \rangle \quad (33)$$

where $\tilde{X}(x)$ and $\tilde{Y}(x)$ are smooth functions vanishing for $x \rightarrow \pm\infty$ and taking values in the co-adjoint orbit of \mathfrak{g} passing through $S(x)$. We will also need the projector π_S onto the image of ad_S

$$\text{ad}_S \tilde{X} \equiv [S, \tilde{X}], \quad \pi_S = \text{ad}_S^{-1} \text{ad}_S \quad (34)$$

i.e., $\tilde{X} = \pi_S \tilde{X}$ and $\tilde{Y} = \pi_S \tilde{Y}$. We will denote the linear space of all such functions by $\tilde{\mathfrak{M}}_S$ which is considered as the phase space of the NLEE related to \tilde{L} .

The squared solutions also get modified

$$\tilde{e}_\alpha^\pm(x, \lambda) = \tilde{\chi}^\pm E_\alpha(\tilde{\chi}^\pm)^{-1}(x, \lambda). \quad (35)$$

Each squared solution takes values in \mathfrak{g} and satisfies the equation

$$i \frac{\partial \tilde{e}_{\pm\alpha}^\pm}{\partial x} - \lambda [S(x), \tilde{e}_{\pm\alpha}^\pm(x, \lambda)] = 0 \quad (36)$$

and can be naturally split into

$$\tilde{e}_{\pm\alpha}^\pm(x, \lambda) = \pi_S \tilde{e}_{\pm\alpha}^\pm(x, \lambda) + (\mathbf{1} - \pi_S) \tilde{e}_{\pm\alpha}^\pm(x, \lambda). \quad (37)$$

Again it is only $\mathbf{e}_{\pm\alpha}^\pm(x, \lambda) = \pi_S \tilde{e}_{\pm\alpha}^\pm(x, \lambda)$ that contributes to the skew-scalar products in (38).

With all this from the Wronskian relations there follow

$$\begin{aligned}\tilde{\tau}_\alpha^\pm(\lambda) &= -\llbracket \text{ad}_{\bar{S}}^{-1} S_x, \tilde{e}_{\mp\alpha}^\pm(x, \lambda) \rrbracket, & \alpha > 0 \\ \tilde{\rho}_\alpha^\pm(\lambda) &= \llbracket \text{ad}_{\bar{S}}^{-1} S_x, \tilde{e}_{\pm\alpha}^\pm(x, \lambda) \rrbracket, & \alpha > 0.\end{aligned}\quad (38)$$

It remains to derive the recursion operators $\tilde{\Lambda}_\pm$ defined by

$$\tilde{\Lambda}_+ \tilde{e}_{\mp\alpha}^\pm(x, \lambda) = \lambda \tilde{e}_{\mp\alpha}^\pm(x, \lambda), \quad \tilde{\Lambda}_- \tilde{e}_{\pm\alpha}^\pm(x, \lambda) = \lambda \tilde{e}_{\pm\alpha}^\pm(x, \lambda). \quad (39)$$

In doing this we will make use not only of the splitting (37) but also of the covariant derivative [12]

$$\nabla_x \cdot \equiv \partial_x \cdot - [g^{-1} g_x, \cdot] = \partial_x \cdot - [\text{ad}_{\bar{S}}^{-1} S_x, \cdot]. \quad (40)$$

The advantage of the covariant derivative is that it commutes with the projector π_S . Skipping the details, we give the answer

$$\begin{aligned}\tilde{\Lambda}_\pm X &= \text{ad}_{\bar{S}}^{-1} \left\{ i \frac{\partial \tilde{X}}{\partial x} - i \pi_S [\text{ad}_{\bar{S}}^{-1} S_x, \tilde{X}] \right. \\ &\quad \left. - i \pi_S [\text{ad}_{\bar{S}}^{-1} S_x, (\mathbb{1} - \pi_S) \nabla_{x, \pm}^{-1} [\text{ad}_{\bar{S}}^{-1} S_y, \tilde{X}(y)]] \right\}.\end{aligned}\quad (41)$$

At the end of this Section we provide the description of the main series of NLEE, related to \tilde{L}

$$i \frac{\partial S}{\partial t} + c_k (\tilde{\Lambda}_\pm^*)^{k-1} \text{ad}_{\bar{S}}^{-1} \frac{\partial S}{\partial x} = 0 \quad (42)$$

where $\tilde{\Lambda}_\pm^*$ is conjugate to $\tilde{\Lambda}_\pm$ with respect to the skew-scalar product

$$\llbracket \tilde{X}, \tilde{\Lambda}_\pm \tilde{Y} \rrbracket = \llbracket \tilde{\Lambda}_\pm^* \tilde{X}, \tilde{Y} \rrbracket, \quad \tilde{\Lambda}_\pm^* = \tilde{\Lambda}_{\mp}. \quad (43)$$

These NLEE are characterized by dispersion laws $f(\lambda) = c_k \lambda^k J_0$ and are equivalent to the following linear evolution equations for the scattering data

$$i \frac{\partial \tilde{S}^\pm}{\partial t} + c_k \lambda^k [J_0, \tilde{S}^\pm(\lambda, t)] = 0. \quad (44)$$

For details concerning the derivation of $\text{ad}_{J_0}^{-1}$ and $\text{ad}_{\bar{S}}^{-1}$ see [13] and Appendix A.

2.3. The Group of Reductions

A very important concept in the theory of integrable equations is that of reduction. Typically the integrable NLEE with physical applications are not derived from generic Lax pairs but from those to obey certain extra symmetries. A formalization of this notion is provided by Mikhailov's reduction group [18–20]. Let G_R is a finite group acting in the set of fundamental solutions $\psi(x, \lambda)$ in the following manner

$$\psi(x, \lambda) \rightarrow \tilde{\psi}(x, \lambda) \equiv \tilde{C}(\psi(x, c^{-1}(\lambda)))$$

where \tilde{C} is a group automorphism and $c : \mathbb{C} \rightarrow \mathbb{C}$ is a conformal mapping. Hence the induced action on the Lax operators reads

$$\tilde{L}(\lambda) = \tilde{C}L(c^{-1}(\lambda))\tilde{C}^{-1}, \quad \tilde{A}(\lambda) = \tilde{C}A(c^{-1}(\lambda))\tilde{C}^{-1}.$$

It preserves the commutation conditions of L and A

$$[\tilde{L}, \tilde{A}] = [L, A] = 0.$$

From the natural requirement that the set of fundamental solutions is G_R -invariant certain symmetry conditions on U and V are obtained. Thus we have a reduction of the number of independent equations.

Example 1. Let us consider the following action of \mathbb{Z}_2 group

$$\psi \rightarrow \tilde{\psi}(x, \lambda) = K \left[\psi^\dagger(x, \lambda^*) \right]^{-1} K^{-1} \quad (45)$$

where K is a diagonal matrix to satisfy $K^2 = \mathbb{1}$. As a result U and V must obey the equalities

$$KU^\dagger(x, \lambda^*)K = U(x, \lambda), \quad \Rightarrow \quad Kq^\dagger K = q, \quad KJ^\dagger K = J \quad (46)$$

$$KV^\dagger(x, \lambda^*)K = V(x, \lambda). \quad (47)$$

3. Reductions of Generalized HF Equation

3.1. Lax Representation

In this section we aim to derive a two-component system of NLEE to represent a reduction of the generalized Heisenberg ferromagnet equation related to the $SU(3)/S(U(1) \times U(2))$ symmetric space and study some properties of its Lax operator L .

In order to obtain a Lax pair related to $SU(3)/S(U(1) \times U(2))$ we start from a linear bundle Lax representation associated with the algebra $\mathfrak{sl}(3, \mathbb{C})$

$$L \equiv i\partial_x \psi + \lambda L_1 \psi = 0, \quad A \equiv i\partial_t \psi + (\lambda A_1 + \lambda^2 A_2) \psi = \psi C(\lambda). \quad (48)$$

Next one extracts the real compact form of $\mathfrak{sl}(3, \mathbb{C})$ by introducing a \mathbb{Z}_2 reduction of the form

$$\psi \rightarrow \tilde{\psi}(x, \lambda) = \left[\psi^\dagger(x, \lambda) \right]^{-1}. \quad (49)$$

It restricts the Lax pair in the following manner

$$L^\dagger(\lambda^*) = -\check{L}(\lambda), \quad A^\dagger(\lambda^*) = -\check{A}(\lambda) \quad \Rightarrow \quad L_1^\dagger = L_1, \quad A_{1,2}^\dagger = A_{1,2}. \quad (50)$$

Thus the matrix coefficients in L and A are tightly connected with the $\mathfrak{su}(3)$ algebra (they differ from the elements of $\mathfrak{su}(3)$ by an imaginary unit which is not essential for our further considerations). Further we impose another \mathbb{Z}_2 reduction

$$\psi \rightarrow \tilde{\psi}(x, \lambda) = J_1 \psi(x, -\lambda) J_1, \quad J_1 = \text{diag}(-1, 1, 1) \quad (51)$$

which leads to the following demand on the Lax operators

$$J_1 L(-\lambda) J_1 = L(\lambda), \quad J_1 A(-\lambda) J_1 = A(\lambda). \quad (52)$$

The latter reduction represents Cartan's involutive automorphism [16] involved in the definition of the symmetric space $SU(3)/S(U(1) \times U(2))$. Cartan's involutive automorphism induces a \mathbb{Z}_2 grading in the Lie algebra

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1, \quad \mathfrak{g}^\sigma = \{X \in \mathfrak{g}; J_1 X J_1 = (-1)^\sigma X\}. \quad (53)$$

As a result of the simultaneous action of (50) and (52) one obtains

$$L_1 = \begin{pmatrix} 0 & u & v \\ u^* & 0 & 0 \\ v^* & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & a & b \\ a^* & 0 & 0 \\ b^* & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{23}^* & a_{33} \end{pmatrix} \quad (54)$$

for the matrix coefficients which are related to the classical Cartan symmetric space $SU(3)/S(U(1) \times U(2))$.

The NLEE associated with the Lax pair (48) is obtained by comparing the coefficients before the equal powers of λ

$$\lambda^3 : [A_2, L_1] = 0 \quad (55)$$

$$\lambda^2 : iA_{2,x} + [L_1, A_1] = 0 \quad (56)$$

$$\lambda : A_{1,x} - L_{1,t} = 0. \quad (57)$$

Due to (55) A_2 is a polynomial of L_1 but since the degree of its characteristic polynomial is three then A_2 is simply a quadratic polynomial of the form

$$A_2 \equiv L_2 = \frac{1}{3} \operatorname{tr} L_1^2 \mathbb{1} - L_1^2 = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} - |u|^2 & -u^* v \\ 0 & -uv^* & \frac{2}{3} - |v|^2 \end{pmatrix}. \quad (58)$$

In order to find A_1 one needs to invert the commutator¹ in (56) to get

$$A_1 = -i \operatorname{ad}_{L_1}^{-1} L_{2,x} + \alpha L_1, \quad \alpha \in \mathbb{R}. \quad (59)$$

At this point we impose one additional constraint

$$|u|^2 + |v|^2 = 1 \quad (60)$$

that is the vector (u, v) lives in a three-dimensional sphere in \mathbb{R}^4 . This is quite analogous to condition (1) in the case of the classical Heisenberg ferromagnet and simplifies our further calculations. It can be proven (see Appendix A) that the following equality holds true

$$\operatorname{ad}_{L_1}^{-1} = \frac{1}{4} (5 \operatorname{ad}_{L_1} - \operatorname{ad}_{L_1}^3). \quad (61)$$

¹One should keep in mind that this operation is well defined only on the quotient space $\mathfrak{su}(3)/\ker \operatorname{ad}_{L_1}$. This is why one needs to add a term proportional to L_1 in order to recover the complete expression for A_1

After performing all necessary computations one could write A_1 in the form (54) with a and b given by

$$\begin{aligned} a &\equiv iu_x + i(uu_x^* + vv_x^*)u + \alpha u \\ b &\equiv iv_x + i(uu_x^* + vv_x^*)v + \alpha v. \end{aligned} \quad (62)$$

Finally one substitutes (62) into (57) and derives the following system of coupled equations

$$\begin{aligned} iu_t + u_{xx} + (uu_x^* + vv_x^* - i\alpha)u_x + (uu_x^* + vv_x^*)_x u &= 0 \\ iv_t + v_{xx} + (uu_x^* + vv_x^* - i\alpha)v_x + (uu_x^* + vv_x^*)_x v &= 0. \end{aligned} \quad (63)$$

A slightly more general system of equations than (63) along with its Lax representation are found in [14]. The latter system is related to $\mathfrak{sl}(N, \mathbb{R})$ and the Lax operators are generic, i.e., they are not reduced.

Remark 1. In what follows we will often need to split given matrix-valued function into part commuting with L_1 and a part ‘orthogonal’ to L_1 . Note that in our case L_1 does not have Jordan cells and satisfies the characteristic equation

$$L_1^3 = L_1. \quad (64)$$

This means that any matrix-valued function of L_1 is at most polynomial of second order on L_1 . Therefore for a generic matrix valued function $Z(x, \lambda)$ the above-mentioned splitting takes the form

$$Z(x, \lambda) = \frac{1}{2}L_1 \langle L_1, Z(x, \lambda) \rangle + \frac{3}{2}L_2 \langle L_2, Z(x, \lambda) \rangle + Z^\perp(x, \lambda) \quad (65)$$

where $\langle X, Y \rangle = \text{tr}(XY)$ and we have used the relations

$$L_2 = A_2, \quad \langle L_1, L_1 \rangle = 2, \quad \langle L_1, L_2 \rangle = 0, \quad \langle L_2, L_2 \rangle = \frac{2}{3}. \quad (66)$$

In addition $\langle Z^\perp(x, \lambda), L_j \rangle = 0$, $j = 1, 2$. Note also that $L_1 \in \mathfrak{g}^{(1)}$ and $L_2 \in \mathfrak{g}^{(0)}$.

3.2. Spectral Properties of L

The spectral properties of the Lax operator crucially depend on the choice of the class of admissible potentials. Below we consider the constant boundary conditions

$$\lim_{x \rightarrow \pm\infty} u(x, t) = 0, \quad \lim_{x \rightarrow \pm\infty} v(x, t) = e^{i\phi_\pm}. \quad (67)$$

This choice of boundary conditions ensures that the asymptotic potentials $U_\pm(\lambda) = \lim_{x \rightarrow \pm\infty} \lambda L_1$ satisfy

$$\begin{aligned} U_{\pm, \text{as}}(\lambda) &= \psi_{0, \pm} \lambda J_0 \psi_{0, \pm}^{-1}, \quad \psi_{0, \pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -e^{i\phi_\pm} \\ 0 & \sqrt{2} & 0 \\ e^{-i\phi_\pm} & 0 & 1 \end{pmatrix} \\ J_0 &= \text{diag}(1, 0, -1). \end{aligned} \quad (68)$$

The Jost solutions are fundamental solutions defined as follows

$$\lim_{x \rightarrow \pm\infty} \psi_{\pm}(x, \lambda) e^{-i\lambda J_0 x} \psi_{0, \pm}^{-1} = \mathbb{1}. \quad (69)$$

Due to the existence of reductions the Jost solutions satisfy the symmetry relations

$$\left[\psi_{\pm}^{\dagger}(x, \lambda^*) \right]^{-1} = \psi_{\pm}(x, \lambda), \quad (70)$$

$$J_1 \psi_{\pm}(x, -\lambda) J_1 = \psi_{\pm}(x, \lambda), \quad J_1 = \text{diag}(1, -1, -1). \quad (71)$$

The Jost solutions are well defined on the real axis in the complex λ -plane. Once the Jost solutions are introduced one defines their transition matrix $T(\lambda)$

$$\psi_{-}(x, \lambda) = \psi_{+}(x, \lambda) T(\lambda). \quad (72)$$

As a consequence of symmetries (70) and (71) the scattering matrix $T(\lambda)$ obeys the following conditions

$$\left[T^{\dagger}(\lambda^*) \right]^{-1} = T(\lambda), \quad J_1 T(-\lambda) J_1 = T(\lambda). \quad (73)$$

From the Lax representation there follows, that the scattering matrix evolves according to the differential equation

$$i\partial_t T + [f(\lambda), T] = 0 \quad \Rightarrow \quad T(t, \lambda) = e^{if(\lambda)t} T(0, \lambda) e^{-if(\lambda)t} \quad (74)$$

where

$$f(\lambda) = \lim_{x \rightarrow \pm\infty} \psi_{0, \pm}^{-1} (\lambda A_1(x) + \lambda^2 A_2(x)) \psi_{0, \pm} \quad (75)$$

is the dispersion law of nonlinear equation. In our case it is

$$f(\lambda) = \frac{\lambda^2}{3} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \lambda \alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In what follows we will restrict ourselves with the special case $\phi_{+} = \phi_{-} = 0$. Then we have $U_{+, as} = U_{-, as}$ and $U_{+}(x, \lambda) = U_{-}(x, \lambda)$.

The main tools in studying the spectral theory of a Lax operator are fundamental analytic solutions. We construct the solutions $\chi^{+}(x, \lambda)$ and $\chi^{-}(x, \lambda)$ which are analytic functions in the upper half plane \mathbb{C}_{+} and lower half plane \mathbb{C}_{-} respectively. In order to construct $\chi^{\pm}(x, \lambda)$ firstly we introduce the auxiliary functions $\eta^{\pm}(x, \lambda)$

$$\eta_{\pm}(x, \lambda) = \psi_{0, \pm}^{-1} \psi_{\pm}(x, \lambda) e^{-i\lambda J_0 x}. \quad (76)$$

Obviously $\eta_{\pm}(x, \lambda)$ are solutions to the associated system

$$i \frac{d\eta_{\pm}}{dx} + U_{\pm}(x, \lambda) \eta_{\pm}(x, \lambda) - \lambda \eta_{\pm}(x, \lambda) J_0 = 0 \quad (77)$$

where

$$U_{\pm}(x, \lambda) = \lambda \psi_{0, \pm}^{-1} L_1(x) \psi_{0, \pm} \quad (78)$$

and satisfies the boundary conditions $\lim_{x \rightarrow \pm\infty} \eta_{\pm}(x, \lambda) = \mathbb{1}$.

Equivalently $\eta_{\pm}(x, \lambda)$ are solutions of the following Volterra-type integral equations

$$\eta_{\pm}(x, \lambda) = \mathbb{1} + i \int_{\pm\infty}^x dy e^{i\lambda J_0(x-y)} [U_{\pm}(y, \lambda) - \lambda J_0] \eta_{\pm}(y, \lambda) e^{-i\lambda J_0(x-y)}. \quad (79)$$

Next we introduce $\xi^+(x, \lambda)$ as a solution to the following set of integral equations

$$\xi_{kl}^+(x, \lambda) = \delta_{kl} + i \int_{-\infty}^x dy e^{i\lambda(J_{0,kk} - J_{0,uu})(x-y)} [(U_-(y, \lambda) - \lambda J_0) \xi^+(y, \lambda)]_{kl} \quad (80)$$

for $k \leq l$ and

$$\xi_{kl}^+(x, \lambda) = i \int_{\infty}^x dy e^{i\lambda(J_{0,kk}(\lambda) - J_{0,uu})(x-y)} [(U_-(y, \lambda) - \lambda J_0) \xi^+(y, \lambda)]_{kl} \quad (81)$$

for $k > l$. It is easy to check that ξ^+ has the proper analytic properties in \mathbb{C}_+ due to the appropriate choice of the lower integration limits in equations (80) and (81). The fundamental analytic solution $\chi^+(x, \lambda)$ of the Lax operator L is obtained from $\xi^+(x, \lambda)$ by applying the simple transformation

$$\chi^+(x, \lambda) = \psi_{0,-} \xi^+(x, \lambda) e^{i\lambda J_0 x}. \quad (82)$$

The fundamental analytic solution $\chi^-(x, \lambda)$ analytic for $\lambda \in \mathbb{C}_-$ is obtained by applying analogous transformation

$$\chi^-(x, \lambda) = \psi_{0,-} \xi^-(x, \lambda) e^{i\lambda J_0 x} \quad (83)$$

where $\xi^-(x, \lambda)$ is a solution to the equations

$$\xi_{kl}^-(x, \lambda) = \delta_{kl} + i \int_{\infty}^x dy e^{i\lambda(J_{0,kk} - J_{0,uu})(x-y)} ((U_-(y, \lambda) - \lambda J_0) \xi^-(y, \lambda))_{kl} \quad (84)$$

for $k \leq l$ and

$$\xi_{kl}^-(x, \lambda) = i \int_{-\infty}^x dy e^{i\lambda(J_{0,kk} - J_{0,uu})(x-y)} ((U_-(y, \lambda) - \lambda J_0) \xi^-(y, \lambda))_{kl} \quad (85)$$

for $k > l$.

The fundamental analytic solutions are linearly related to the Jost solutions for $\lambda \in \mathbb{R}$. These relations are expressed through the factors of Gauss decomposition

$$T(t, \lambda) = T^{\mp} D^{\pm} (S^{\pm})^{-1} \quad (86)$$

of $T(t, \lambda)$ and have the form

$$\chi^{\pm}(x, \lambda) = \psi_{-}(x, \lambda) S^{\pm} = \psi_{+}(x, \lambda) T^{\mp}(\lambda) D^{\pm}(\lambda). \quad (87)$$

From the reduction conditions (73) and equation (86) there follows

$$\begin{aligned} (S^+(\lambda^*))^{\dagger} &= (S^-(\lambda))^{-1}, & (T^+(\lambda^*))^{\dagger} &= (T^-(\lambda))^{-1} \\ J_1 S^{\pm}(-\lambda) J_1 &= S^{\pm}(\lambda), & J_1 T^{\pm}(-\lambda) J_1 &= T^{\pm}(\lambda) \\ (D^+(\lambda^*))^{\dagger} &= (D^-(\lambda))^{-1}, & D^{\pm}(-\lambda) &= D^{\pm}(\lambda). \end{aligned} \quad (88)$$

The reductions have also impact on the FAS, namely

$$(\chi^+)^\dagger(x, \lambda^*) = [\chi^-(x, \lambda)]^{-1}, \quad J_1 \chi^+(x, -\lambda) J_1 = \chi^-(x, \lambda). \quad (89)$$

From the relations (82) and (87) one easily obtains that ξ^+ and ξ^- are interrelated in the continuous spectrum through a Riemann-Hilbert problem

$$\xi^+(x, \lambda) = \xi^-(x, \lambda) G(x, \lambda), \quad G(x, \lambda) = e^{i\lambda J_0 x} (S^-)^{-1} S^+(\lambda) e^{-i\lambda J_0 x}. \quad (90)$$

Thus the inverse spectral problem can be reduced to a Riemann-Hilbert problem to find matrix functions analytic in the upper and lower half plains of λ and satisfying (90) on the real axis.

Remark 2. The Riemann-Hilbert problem allows singular solutions as well. The simplest types of singularities are simple poles and zeroes of the FAS and generically correspond to discrete eigenvalues of the Lax operator L . Due to the reduction symmetries the discrete eigenvalues must form orbits of the reduction group. Generic orbits contain quadruplets, so if μ is an eigenvalue, then $-\mu$ and $\pm\mu^*$, are eigenvalues too. However, we can have degenerate orbits too. If the eigenvalue lies on the imaginary axis we will have doublets of eigenvalues.

4. Recursion Operators

4.1. Recursion Operators Through GKS Approach

In this section we aim to construct recursion operator for the NLEE (63), i.e., a pseudo-differential operator \mathcal{R} to map a symmetry of (63) into another symmetry. For that purpose we are applying the method proposed by Gürses, Karasu and Sokolov (GKS) [15].

Firstly, let us remark that the zero curvature condition of the Lax operators

$$L = i\partial_x + \lambda L_1, \quad A = i\partial_t + V(x, t, \lambda) \quad (91)$$

could be written in the following manner

$$iL_t \equiv [L, V] \quad (92)$$

where $L_t \equiv \lambda L_{1,t}$ is the Freschét derivative of L . In order to derive the recursion operator one interrelates two adjacent flows \check{A} and A [15, 26] through the equality

$$\check{V} = \kappa(\lambda)V + B. \quad (93)$$

All quantities above are invariant under the \mathbb{Z}_2 reductions (50) and (52) and the function $\kappa(\lambda) = \lambda^2$ is the primitive automorphic function invariant under the transform $\lambda \rightarrow -\lambda$. The operator B is chosen to be a quadratic polynomial of λ in analogy with the second Lax operator in (4)

$$B = \lambda B_1 + \lambda^2 B_2 \quad (94)$$

where the hermitian matrices involved above have the form

$$B_1 \equiv \begin{pmatrix} 0 & \mathbf{c}^T \\ \mathbf{c}^* & 0 \end{pmatrix}, \quad B_2 \equiv \begin{pmatrix} d & 0 \\ 0 & D \end{pmatrix}, \quad D \equiv \begin{pmatrix} \alpha & \beta \\ \beta^* & \delta \end{pmatrix}. \quad (95)$$

After substituting (93) in the analog of (92) when the evolution parameter t is replaced by another evolution parameter τ we obtain the following basic equation

$$iL_\tau = i\lambda^2 L_t + [L, B]. \quad (96)$$

At this point we introduce an auxiliary quantity $J = \alpha + \delta$. Since all matrices are traceless we have

$$d = -\operatorname{tr} D = -(\alpha + \delta) = -J.$$

Equation (96) splits into the following system of recurrence relations

$$\lambda^3 : iL_{1,t} + [L_1, B_2] = 0, \quad \Rightarrow \quad i\mathbf{u}_t + (D^* - d)\mathbf{u} = 0 \quad (97)$$

$$\lambda^2 : iB_{2,x} + [L_1, B_1] = 0, \quad \Rightarrow \quad \begin{aligned} iD_x + \mathbf{u}^* \mathbf{c}^T - \mathbf{c}^* \mathbf{u}^T &= 0 \\ id_x + \mathbf{u}^T \mathbf{c}^* - \mathbf{c}^T \mathbf{u}^* &= 0 \end{aligned} \quad (98)$$

$$\lambda : L_{1,\tau} = B_{1,x}, \quad \Rightarrow \quad \mathbf{u}_\tau = \mathbf{c}_x. \quad (99)$$

where $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$. The system (97) is linear for the matrix elements of D . A solution to (97) is given by

$$\begin{aligned} \alpha &= i(uu_t^* + v^*v_t) + J(2|v|^2 - |u|^2), & \beta &= i(vu_t^* - u^*v_t) - 3Ju^*v \\ \delta &= -i(uu_t^* + v^*v_t) + J(2|u|^2 - |v|^2). \end{aligned} \quad (100)$$

The system (98) is underdetermined since if \mathbf{c} is a solution then $\mathbf{c} + a\mathbf{u}$ for any $a \in \mathbb{R}$ represents a solution too. Thus we impose an additional constraint, namely

$$\mathbf{u}^* \mathbf{c}^T + \mathbf{c}^* \mathbf{u}^T = 0. \quad (101)$$

It is correct iff the following compatibility condition holds

$$\alpha_x |v|^2 + \delta_x |u|^2 - \beta_x uv^* - \beta_x^* u^* v = 0. \quad (102)$$

The answer for \mathbf{c} reads

$$\mathbf{c} = \begin{pmatrix} c \\ s \end{pmatrix} = \frac{i}{2} \begin{pmatrix} u(\delta_x - \alpha_x) - 2v\beta_x^* \\ -v(\delta_x - \alpha_x) - 2u\beta_x \end{pmatrix}. \quad (103)$$

This can be rewritten in the following matrix form

$$\begin{pmatrix} \mathbf{c} \\ \mathbf{c}^* \end{pmatrix} = i \begin{pmatrix} 0 & -v & -u & 0 \\ -u & 0 & v & 0 \\ v^* & 0 & u^* & 0 \\ 0 & u^* & -v^* & 0 \end{pmatrix} \frac{d}{dx} \begin{pmatrix} \beta \\ \beta^* \\ \frac{\alpha - \delta}{2} \\ \alpha + \delta \end{pmatrix}. \quad (104)$$

One can write relation (99) in a more detailed way by substituting (104) in it as follows

$$\begin{pmatrix} u \\ v \\ u^* \\ v^* \end{pmatrix}_\tau = \mathcal{A} \begin{pmatrix} \beta \\ \beta^* \\ \frac{\alpha-\delta}{2} \\ \frac{\alpha+\delta}{2} \end{pmatrix}, \quad \mathcal{A} := i \frac{d}{dx} \begin{pmatrix} 0 & -v & -u & 0 \\ -u & 0 & v & 0 \\ v^* & 0 & u^* & 0 \\ 0 & u^* & -v^* & 0 \end{pmatrix} \frac{d}{dx}. \quad (105)$$

What remains is to find the function J involved in the expressions for α, β etc. For that purpose we make use of (102). After substituting the explicit expressions needed one derives a linear differential equation for J

$$\begin{aligned} 2J_x + i(u^*u_t + v^*v_t)_x - i[(uu_t^* + v^*v_t)(|v|^2 - |u|^2)_x \\ - (uv_t^* - v^*u_t)(u^*v)_x + (u^*v_t - v^*u_t)(uv^*)_x] = 0. \end{aligned} \quad (106)$$

But one can verify that

$$\begin{aligned} (uu_t^* + v^*v_t)(|v|^2 - |u|^2)_x - (uv_t^* - v^*u_t)(u^*v)_x + (u^*v_t - v^*u_t)(uv^*)_x \\ = u_t u_x^* + v_t v_x^* - u_t^* u_x - v_t^* v_x \end{aligned} \quad (107)$$

and therefore J can be written in the form

$$J = \frac{i}{2}(uu_t^* + vv_t^*) + \frac{i}{2}\partial_x^{-1}(u_t u_x^* + v_t v_x^* - u_t^* u_x - v_t^* v_x) \quad (108)$$

where $\partial_x^{-1} := \int_{\pm\infty}^x dy$. Taking into account the explicit expression (108) for the function J one could easily derive

$$\begin{pmatrix} \beta \\ \beta^* \\ \frac{\alpha-\delta}{2} \\ \frac{\alpha+\delta}{2} \end{pmatrix} = (\mathcal{B}_{\text{loc}} + \mathcal{B}_{\text{nonl}}) \begin{pmatrix} u \\ v \\ u^* \\ v^* \end{pmatrix}_t \quad (109)$$

where \mathcal{B}_{loc} and $\mathcal{B}_{\text{nonl}}$ are a local and a nonlocal part, namely

$$\begin{aligned} \mathcal{B}_{\text{loc}} = \frac{i}{4} \begin{pmatrix} 0 & -4u^* & 2v(3|v|^2 - 1) & -6u^*v^2 \\ 2v^*(1 - 3|v|^2) & 6u(v^*)^2 & 0 & 4u \\ -4u^* & 0 & 3u(|v|^2 - |u|^2) & -v(1 + 6|u|^2) \\ -u^* & -v^* & 0 & 0 \end{pmatrix} \\ \mathcal{B}_{\text{nonl}} = -\frac{3i}{4} \begin{pmatrix} 2u^*v \\ 2uv^* \\ |u|^2 - |v|^2 \\ -1/3 \end{pmatrix} \partial_x^{-1} [(u_x^*, v_x^*, -u_x, -v_x)]. \end{aligned} \quad (110)$$

The recursion operator maps the vector $(\mathbf{u}_t, \mathbf{u}_t^*)^T$ into $(\mathbf{u}_\tau, \mathbf{u}_\tau^*)^T$. Hence combining (105), (109) and (110) one obtains the recursion operator \mathcal{R}

$$\mathcal{R} = \mathcal{A} (\mathcal{B}_{\text{loc}} + \mathcal{B}_{\text{nonl}}). \quad (111)$$

4.2. Recursion Operators Through the Recursion Relations

Consider a general flow Lax pair

$$L := i\partial_x + \lambda L_1, \quad A := i\partial_t + \sum_{k=1}^N \lambda^k A_k \quad (112)$$

where L and A are subject to the action of the reductions (50) and (52). Then the matrix L_1 is given by (54) again and the coefficients of A are hermitian matrices to fulfill

$$J_1 A_{2q-1} J_1 = -A_{2q-1} \in \mathfrak{g}^{(1)}, \quad J_1 A_{2q} J_1 = A_{2q} \in \mathfrak{g}^{(0)}.$$

The compatibility condition $[L, A] = 0$ can be written in a more detailed way as follows

$$\lambda^{N+1} : [L_1, A_N] = 0 \quad (113)$$

...

$$\lambda^k : i\partial_x A_k + [L_1, A_{k-1}] = 0, \quad k = 2, \dots, N \quad (114)$$

...

$$\lambda : \partial_x A_1 - \partial_t L_1 = 0. \quad (115)$$

It directly follows from (113) that the highest order term is a second order polynomial of L_1 , see Remark 1. Since $L_1 \in \mathfrak{g}^{(1)}$ and $L_2 \in \mathfrak{g}^{(0)}$ we have two types of choices for A_N

$$\begin{aligned} \text{a)} \quad & A_N^{\text{a}} = \frac{3}{2} f_{2p} L_2, & \text{for } N = 2p \\ \text{b)} \quad & A_N^{\text{b}} = \frac{1}{2} f_{2p+1} L_1, & \text{for } N = 2p + 1 \end{aligned} \quad (116)$$

where L_2 is given in equation (58).

Each element A_k obeys a splitting (see Remark 1),

$$A_{2q-1} = A_{2q-1}^\perp + \frac{1}{2} f_{2q-1} L_1, \quad A_{2q} = A_{2q}^\perp + \frac{3}{2} f_{2q} L_2 \quad (117)$$

into two mutually orthogonal parts A_{2q-1}^\perp (respectively A_{2q}^\perp) and $f_{2q-1} L_1/2$ (respectively $3f_{2q} L_2/2$). Substituting (117) into (114) for $k = N$ and performing the

splitting of orthogonal parts one convinces himself that $f_N = c_N = \text{const}$ and

$$\begin{aligned} \text{a)} \quad A_{2p-1}^\perp &= -\frac{3i}{2}c_{2p} \text{ad}_{L_1}^{-1} L_{2,x} \\ \text{b)} \quad A_{2p}^\perp &= -\frac{i}{2}c_{2p+1} \text{ad}_{L_1}^{-1} L_{1,x}. \end{aligned} \quad (118)$$

Quite analogously after placing the splitting (117) in (114) we obtain

$$\begin{aligned} \frac{i}{2}f_{2q-1,x}L_1 + \frac{i}{2}f_{2q-1}L_{1,x} + i \left(A_{2q-1}^\perp \right)_x + [L_1, A_{2q-2}^\perp] &= 0 \\ if_{2q,x}L_2 + if_{2q}L_{2,x} + i \left(A_{2q}^\perp \right)_x + [L_1, A_{2q-1}^\perp] &= 0. \end{aligned} \quad (119)$$

We would like to stress on the fact that $(A_k^\perp)_x \neq (A_{k,x})^\perp$. After taking the Killing form with L_1 (respectively L_2) in the left hand side of (119) and performing an elementary integration one can express the coefficients f_k through A_k^\perp as follows

$$\begin{aligned} f_{2q-1} &= c_{2q-1} - \frac{1}{2}\partial_x^{-1} \left\langle \left(A_{2q-1}^\perp \right)_x, L_1 \right\rangle \\ f_{2q} &= c_{2q} - \frac{3}{2}\partial_x^{-1} \left\langle \left(A_{2q}^\perp \right)_x, L_2 \right\rangle \end{aligned} \quad (120)$$

where c_k are constants of integration.

Remark 3. It is well known that ∂_x^{-1} is determined up to an additional constant. We have two natural ways to fix it up

$$\partial_x^{-1}Z(x) \equiv \int_{\pm\infty}^x dy Z(y) + z_\pm, \quad z_\pm = \lim_{x \rightarrow \pm\infty} Z(x). \quad (121)$$

Therefore strictly speaking each of the recursion operators Λ_k introduced below is one of the two: Λ_k^+ or Λ_k^- depending on the choice of the lower limit of the integral in equation (121). We note also that if the NLEE (126) are local in terms of L_1 and L_2 then their explicit form will be the same for both types of recursion operators.

Substituting (120) into (119) we get

$$\begin{aligned} A_{2q}^\perp &= \Lambda_1^\pm \left(A_{2q+1}^\perp \right) - \frac{i}{2}c_{2q+1} \text{ad}_{L_1}^{-1} L_{1,x} \\ A_{2q-1}^\perp &= \Lambda_2^\pm \left(A_{2q}^\perp \right) - \frac{3i}{2}c_{2q} \text{ad}_{L_1}^{-1} L_{2,x} \end{aligned} \quad (122)$$

where the integro-differential operators $\Lambda_{1,2}$ read

$$\begin{aligned} \Lambda_1^\pm X &= -i \text{ad}_{L_1}^{-1} \left(\frac{\partial X}{\partial x} - \frac{1}{2}L_{1,x} \partial_{x,\pm}^{-1} \left\langle L_1, \frac{\partial X}{\partial y} \right\rangle \right) \\ \Lambda_2^\pm Y &= -i \text{ad}_{L_1}^{-1} \left(\frac{\partial Y}{\partial x} - \frac{3}{2}L_{2,x} \partial_{x,\pm}^{-1} \left\langle L_2, \frac{\partial Y}{\partial y} \right\rangle \right). \end{aligned} \quad (123)$$

Finally we consider equation (115). Splitting again $A_1 = A_1^\perp + f_1 L_1/2$ we find

$$f_1 = c_1 - \frac{1}{2} \partial_x^{-1} \langle (A_1^\perp)_x, L_1 \rangle \quad (124)$$

and for the corresponding NLEE we get

$$i \operatorname{ad}_{L_1}^{-1} \partial_t L_1 + \Lambda_1 A_1^\perp - \frac{i}{2} c_1 \mathcal{L}_1 = 0. \quad (125)$$

Next we have to solve the recurrent relations (122) for each of the cases. Skipping the details we get the following hierarchies of NLEE

$$\begin{aligned} \text{a) } i \operatorname{ad}_{L_1}^{-1} \partial_t L_1 - \frac{3i}{2} \sum_{q=1}^p c_{2q} (\Lambda_1 \Lambda_2)^{q-1} \Lambda_1 \mathcal{L}_2 - \frac{i}{2} \sum_{q=0}^{p-1} c_{2q+1} (\Lambda_1 \Lambda_2)^q \mathcal{L}_1 &= 0 \\ \text{b) } i \operatorname{ad}_{L_1}^{-1} \partial_t L_1 - \frac{3i}{2} \sum_{q=1}^p c_{2q} (\Lambda_1 \Lambda_2)^{q-1} \Lambda_1 \mathcal{L}_2 - \frac{i}{2} \sum_{q=0}^p c_{2q+1} (\Lambda_1 \Lambda_2)^q \mathcal{L}_1 &= 0. \end{aligned} \quad (126)$$

We have also used the notations

$$\mathcal{L}_1 = \operatorname{ad}_{L_1}^{-1} L_{1,x}, \quad \mathcal{L}_2 = \operatorname{ad}_{L_1}^{-1} L_{2,x}. \quad (127)$$

The explicit expressions for \mathcal{L}_1 and \mathcal{L}_2 are given by equations (174) and (179) which can be found in the Appendix.

Acting in analogy with Subsection 3.2, equation (75) we derive the dispersion laws for these equations

$$f(\lambda) = \lim_{x \rightarrow \infty} \psi_{0,\pm}^{-1} \sum_{k=1}^N \lambda^k A_k(x) \psi_{0,\pm} = \lim_{x \rightarrow \infty} \psi_{0,\pm}^{-1} \sum_{k=1}^N \lambda^k f_k A_{k,\text{ass}} \psi_{0,\pm} \quad (128)$$

where $A_{2k+1,\text{ass}} = L_1/2$ and $A_{2k,\text{ass}} = 3L_2/2$. The result is

$$\begin{aligned} \text{a) } f(\lambda) &= \frac{1}{2} \sum_{q=0}^{p-1} c_{2q+1} \lambda^{2q+1} K_1 + \frac{3}{2} \sum_{q=1}^p c_{2q} \lambda^{2q} K_2 \\ \text{b) } f(\lambda) &= \frac{1}{2} \sum_{q=0}^p c_{2q+1} \lambda^{2q+1} K_1 + \frac{3}{2} \sum_{q=1}^p c_{2q} \lambda^{2q} K_2 \end{aligned} \quad (129)$$

$$K_1 = \operatorname{diag}(1, 0, -1), \quad K_2 = \operatorname{diag}(-1/3, 2/3, -1/3).$$

One can convince himself through a direct computation that equation (126a) contains as a special case (63) in the case a) with $N = 2$, $p = 1$, $c_2 = -1$ and $c_1 = \alpha$.

5. Wronskian Relations and ‘Squared Solutions’ of L

Wronskian relations [2,3] provide an important tool for analyzing the relevant class of NLEE and the mapping $\mathfrak{F} : \mathfrak{M} \rightarrow \mathfrak{T}$, where \mathfrak{M} is the set of allowed potentials of L (in our case L_1) and \mathfrak{T} is the minimal set of scattering data.

In deriving them we will need along with the first equation in (48) also two other related equations

$$i \frac{\partial \hat{\chi}}{\partial x} - \lambda \hat{\chi}(x, \lambda) L_1(x) = 0, \quad \hat{\chi}(x, \lambda) \equiv \chi^{-1}(x, \lambda) \quad (130)$$

$$i \frac{\partial \delta \chi}{\partial x} + \lambda L_1(x) \delta \chi(x, \lambda) + \lambda \delta L_1(x) \chi(x, \lambda) = 0 \quad (131)$$

where the variation of $\chi(x, \lambda)$ is due to the variation $\delta L_1(x)$.

The first type of Wronskian relations interrelates the asymptotics of FAS with L_1 and its powers as shown in the examples below

$$i (\hat{\chi} J_0 \chi(x, \lambda) - J_0) |_{-\infty}^{\infty} = \lambda \int_{-\infty}^{\infty} dx \hat{\chi} [L_1, J_0] \chi(x, \lambda) \quad (132)$$

$$\hat{\chi} L_1(x) \chi(x, \lambda) |_{-\infty}^{\infty} = \int_{-\infty}^{\infty} dx \hat{\chi} L_{1,x} \chi(x, \lambda) \quad (133)$$

$$\hat{\chi} A_2(x) \chi(x, \lambda) |_{-\infty}^{\infty} = \int_{-\infty}^{\infty} dx \hat{\chi} A_{2,x} \chi(x, \lambda). \quad (134)$$

A second class of Wronskian relations connects the variation $\delta \chi(x, \lambda)$ with the variation of L_1 , i.e.,

$$\hat{\chi} \delta \chi(x, \lambda) |_{-\infty}^{\infty} = i \lambda \int_{-\infty}^{\infty} dx \hat{\chi} \delta L_1 \chi(x, \lambda). \quad (135)$$

In the left hand sides of the Wronskian relations are involved the scattering data and its variation while the right hand sides can be viewed as Fourier type integrals. To make this obvious let us take the Killing form of the Wronskian relation (132) with a Cartan-Weyl generator E_α and use the invariance of the Killing form

$$i \langle \hat{\chi} J_0 \chi(x, \lambda) - J_0, E_\alpha \rangle |_{-\infty}^{\infty} = \lambda \int_{-\infty}^{\infty} dx \langle [L_1, J_0], e_\alpha(x, \lambda) \rangle. \quad (136)$$

The quantity $e_\alpha(x, \lambda) = \chi E_\alpha \hat{\chi}(x, \lambda)$ introduced above is called a ‘squared solution’. Due to the fact that we have two FAS $\chi^+(x, \lambda)$ and $\chi^-(x, \lambda)$ we obtain two types of ‘squared solutions’ $e_\alpha^\pm(x, \lambda)$. Similarly, taking the Killing form in (133)

and (134) we find

$$\begin{aligned}
\langle \hat{\chi} L_1 \chi(x, \lambda), E_\alpha \rangle|_{-\infty}^{\infty} &= \int_{-\infty}^{\infty} dx \langle L_{1,x}, e_\alpha(x, \lambda) \rangle \\
\langle \hat{\chi} A_2 \chi(x, \lambda), E_\alpha \rangle|_{-\infty}^{\infty} &= \int_{-\infty}^{\infty} dx \langle A_{2,x}, e_\alpha(x, \lambda) \rangle \\
\langle \hat{\chi} \delta \chi(x, \lambda), E_\alpha \rangle|_{-\infty}^{\infty} &= i\lambda \int_{-\infty}^{\infty} dx \langle \delta L_1, e_\alpha(x, \lambda) \rangle.
\end{aligned} \tag{137}$$

We are interested more specifically in variations that are due to the time evolution of $L_1(x)$, i.e.,

$$\delta L_1(x) = L_1(x, t + \delta t) - L_1(x, t) \simeq \delta t \frac{\partial L_1}{\partial t}.$$

Therefore up to first order terms of δt we obtain

$$\langle \hat{\chi} \chi_t(x, \lambda), E_\alpha \rangle|_{x=-\infty}^{\infty} = i\lambda \int_{-\infty}^{\infty} dx \langle L_{1,t}, e_\alpha(x, \lambda) \rangle. \tag{138}$$

Now we can explain why the Wronskian relations are important for analyzing the mapping $\mathfrak{F} : \mathfrak{M}_{L_1} \rightarrow \mathfrak{T}$. Indeed, taking $\chi(x, \lambda)$ to be a fundamental analytic solution of L we can express the left hand sides of (137) through the Gauss factors S^\pm , T^\mp and D^\pm (respectively through the Gauss factors and their variations). The right hand side of (137) can be interpreted as a Fourier-like transformation of the potential $L_1(x)$ (respectively of the variation $\delta L_1(x)$). As a natural generalization of the usual exponents there appear the ‘squared solutions’. The ‘squared solutions’ are analytic functions of λ . This fact underlies the proof of their completeness in the space of allowed potentials, see e.g. [5, 6, 9, 11].

5.1. The Skew-Scalar Product

It is obvious, that only some of the matrix elements of the squared solutions $e_\alpha(x, \lambda)$ contribute to the right-hand sides of the Wronskian relations. To make this more clear we will use the \mathbb{Z}_2 -grading of the Lie algebra (see (53) and (54)) which hint that we should use the splitting

$$e_\alpha(x, \lambda) = H_\alpha(x, \lambda) + K_\alpha(x, \lambda), \quad H_\alpha(x, \lambda) \in \mathfrak{g}^0, \quad K_\alpha(x, \lambda) \in \mathfrak{g}^1. \tag{139}$$

In addition each components $H_\alpha(x, \lambda)$ and $K_\alpha(x, \lambda)$ can be split according to Remark 1 into

$$\begin{aligned}
H_\alpha(x, \lambda) &= H_\alpha^\perp(x, \lambda) + \frac{3}{2} h_\alpha L_2(x), & h_\alpha &= \langle L_2(x), H_\alpha(x, \lambda) \rangle \\
K_\alpha(x, \lambda) &= K_\alpha^\perp(x, \lambda) + \frac{1}{2} k_\alpha L_1(x), & k_\alpha &= \langle L_1(x), K_\alpha(x, \lambda) \rangle
\end{aligned} \tag{140}$$

where $H_\alpha^\perp(x, \lambda)$ and $K_\alpha^\perp(x, \lambda)$ do not commute with $L_1(x)$. It is not difficult to realize that only $H_\alpha^\perp(x, \lambda)$ and $K_\alpha^\perp(x, \lambda)$ contribute to the right-hand sides of the Wronskian relations.

In what follows we will make use also of the skew-scalar product

$$[[X, Y]] = \int_{-\infty}^{\infty} dy \langle X(y), [L_1(y), Y(y)] \rangle \quad (141)$$

where X and Y are matrix-valued functions that belong to \mathfrak{g}^1 , vanish for $x \rightarrow \pm\infty$ and such that $X = X^\perp$ and $Y = Y^\perp$, or equivalently

$$\langle L_1, X \rangle = 0, \quad \langle L_1, Y \rangle = 0. \quad (142)$$

We will denote the linear space of such functions by \mathfrak{M}_{L_1} . Note, that we can express the right hand sides of the Wronskian relations using the skew-scalar product

$$\begin{aligned} \langle \hat{\chi} L_1 \chi(x, \lambda), E_\alpha \rangle|_{x=-\infty}^{\infty} &= [[e_\alpha(x, \lambda), \text{ad}_{L_1}^{-1} L_{1,x}]] \\ \langle \hat{\chi} A_2 \chi(x, \lambda), E_\alpha \rangle|_{x=-\infty}^{\infty} &= [[e_\alpha(x, \lambda), \text{ad}_{L_1}^{-1} A_{2,x}]] \\ \langle \hat{\chi} \delta \chi(x, \lambda), E_\alpha \rangle|_{x=-\infty}^{\infty} &= i\lambda [[e_\alpha(x, \lambda), \text{ad}_{L_1}^{-1} \delta L_1]] \\ \langle \hat{\chi} \chi_t(x, \lambda), E_\alpha \rangle|_{x=-\infty}^{\infty} &= i\lambda [[e_\alpha(x, \lambda), \text{ad}_{L_1}^{-1} L_{1,t}]]. \end{aligned} \quad (143)$$

We should also point out that the skew-scalar product is non-degenerate on \mathfrak{M}_{L_1} .

5.2. The Mapping $\mathfrak{M}_{L_1} \rightarrow \mathfrak{T}_j$

Here we first calculate the left hand sides of (137)–(138). Inserting the asymptotic behavior of $\chi^\pm(x, \lambda)$

$$\begin{aligned} \lim_{x \rightarrow -\infty} \chi^\pm(x, \lambda) \hat{S}^\pm(\lambda) e^{-i\lambda K_1 x} \hat{\psi}_{0,-} &= \mathbb{1} \\ \lim_{x \rightarrow \infty} \chi^\pm(x, \lambda) \hat{D}^\pm(\lambda) \hat{T}^\mp(\lambda) e^{-i\lambda K_1 x} \hat{\psi}_{0,+} &= \mathbb{1}. \end{aligned} \quad (144)$$

We make use also of the basic properties of the Cartan-Weyl basis [16]. The results are

$$\begin{aligned} \tau_\alpha^{(k),\pm}(\lambda) &\equiv \langle E_{\mp\alpha}, \hat{S}^\pm K_k S^\pm(\lambda) \rangle = [[\text{ad}_{L_1}^{-1} L_{k,x}, e_{\mp\alpha}^\pm(x, \lambda)]] \\ \rho_\alpha^{(k),\pm}(\lambda) &\equiv \langle E_{\pm\alpha}, \hat{D}^\pm \hat{T}^\mp K_k T^\mp D^\pm(\lambda) \rangle = [[e_{\pm\alpha}^\pm(x, \lambda), \text{ad}_{L_1}^{-1} L_{k,x}]] \end{aligned} \quad (145)$$

where $k = 1, 2$ and the diagonal matrices K_k were introduced in equation (129). From the second type Wronskian relations we get

$$\begin{aligned} \delta \tau_\alpha^\pm(\lambda) &\equiv \langle E_{\mp\alpha}, \hat{S}^\pm \delta S^\pm(\lambda) \rangle = i\lambda [[\text{ad}_{L_1}^{-1} \delta L_1, e_{\mp\alpha}^\pm(x, \lambda)]] \\ \delta \rho_\alpha^\pm(\lambda) &\equiv \langle E_{\pm\alpha}, \hat{D}^\pm \hat{T}^\mp \delta(T^\mp D^\pm(\lambda)) \rangle = i\lambda [[e_{\pm\alpha}^\pm(x, \lambda), \text{ad}_{L_1}^{-1} \delta L_1]] \end{aligned} \quad (146)$$

$$\begin{aligned} \tau_{\alpha,t}^\pm(\lambda) &\equiv \langle E_{\mp\alpha}, \hat{S}^\pm S_t^\pm(\lambda) \rangle = i\lambda [[\text{ad}_{L_1}^{-1} L_{1,t}, e_{\mp\alpha}^\pm(x, \lambda)]] \\ \rho_{\alpha,t}^\pm(\lambda) &\equiv \langle E_{\pm\alpha}, \hat{D}^\pm \hat{T}^\mp T_t^\mp D^\pm(\lambda) \rangle = i\lambda [[e_{\pm\alpha}^\pm(x, \lambda), \text{ad}_{L_1}^{-1} L_{1,t}]]. \end{aligned} \quad (147)$$

These formulae can be used in the analysis of the mapping $\mathfrak{M}_{L_1} \rightarrow \mathfrak{T}_k$, $k = 1, 2$. Indeed $L_{1,x} \in \mathfrak{M}_{L_1}$, while the sets of coefficients

$$\begin{aligned} \mathfrak{T}_1 &\equiv \mathfrak{T}_1^{(1)} \cup \mathfrak{T}_1^{(2)}, & \mathfrak{T}_1^{(k)} &\equiv \left\{ \tau_\alpha^{(k),\pm}(\lambda); \lambda \in \mathbb{R} \quad \alpha > 0 \right\} \\ \mathfrak{T}_2 &\equiv \mathfrak{T}_2^{(1)} \cup \mathfrak{T}_2^{(2)}, & \mathfrak{T}_2^{(k)} &\equiv \left\{ \rho_\alpha^{(k),\pm}(\lambda); \lambda \in \mathbb{R} \quad \alpha > 0 \right\} \end{aligned} \quad (148)$$

are candidates for the minimal sets of scattering data of the Lax operator on the continuous spectrum. The ‘squared solutions’ can be viewed as generalized exponents and as a result (145) and (146) become analogous of the generalized Fourier transform.

Of course, the justifications of the above statements must be based on a proof of the completeness relation for the ‘squared solutions’. We will outline that proof elsewhere.

6. Recursion Operators – An Alternative Approach

We will derive in this section the recursion operators using an alternative definition. We are to introduce them as operators, whose eigenfunctions are the ‘squared solutions’, i.e.,

$$\Lambda_\pm K_\alpha^{\pm,\perp}(x, \lambda) = \lambda^2 K_\alpha^{\pm,\perp}(x, \lambda). \quad (149)$$

We start from equation

$$i\partial_x e_\alpha + \lambda[L_1(x), e_\alpha(x, \lambda)] = 0 \quad (150)$$

satisfied by each of the ‘squared solutions’. Then H_α and K_α introduced in (139) have to be solutions to the following system

$$\begin{aligned} i\partial_x H_\alpha + \lambda[L_1(x), K_\alpha(x, \lambda)] &= 0 \\ i\partial_x K_\alpha + \lambda[L_1(x), H_\alpha(x, \lambda)] &= 0. \end{aligned} \quad (151)$$

Now we insert the splittings (140), multiply the first (respectively the second) of the equation (140) by $L_2(x)$ (respectively by $L_1(x)$) and take the Killing form. This gives two equations

$$\begin{aligned} \left\langle L_2, \partial_x H_\alpha^\perp \right\rangle + \partial_x h_\alpha &= 0 \quad \Rightarrow \quad h_\alpha = h_{\alpha,0} - \partial_x^{-1} \left\langle L_2, \partial_x H_\alpha^\perp \right\rangle \\ \left\langle L_1, \partial_x K_\alpha^\perp \right\rangle + \partial_x k_\alpha &= 0 \quad \Rightarrow \quad k_\alpha = k_{\alpha,0} - \partial_x^{-1} \left\langle L_1, \partial_x K_\alpha^\perp \right\rangle \end{aligned} \quad (152)$$

where $h_{\alpha,0}$ and $k_{\alpha,0}$ are constants. On the other hand from (151) it follows the equations below hold true

$$\begin{aligned} i\partial_x H_\alpha^\perp + \frac{3i}{2} h_\alpha L_{2,x} &= -\lambda[L_1(x), K_\alpha^\perp] \\ i\partial_x K_\alpha^\perp + \frac{i}{2} k_\alpha L_{1,x} &= -\lambda[L_1(x), H_\alpha^\perp]. \end{aligned} \quad (153)$$

After substituting (152) in (153) and applying $\text{ad}_{L_1}^{-1}$ in both hand sides of the latter one gets equations, which involve H_α^\perp and K_α^\perp only

$$\Lambda_1 K_\alpha^\perp = \lambda H_\alpha^\perp + \frac{i}{2} k_{\alpha,0} \mathcal{L}_2, \quad \Lambda_2 H_\alpha^\perp = \lambda K_\alpha^\perp + \frac{3i}{2} h_{\alpha,0} \mathcal{L}_1 \quad (154)$$

where Λ_1 and Λ_2 are given by (123). Finally we obtain

$$\begin{aligned} \Lambda_2 \Lambda_1 K_\alpha^\perp &= \lambda^2 K_\alpha^\perp + \frac{3i}{2} \lambda h_{\alpha,0} \mathcal{L}_1 + \frac{i}{2} k_{\alpha,0} \Lambda_2 \mathcal{L}_2 \\ \Lambda_1 \Lambda_2 H_\alpha^\perp &= \lambda^2 H_\alpha^\perp + \frac{i}{2} \lambda k_{\alpha,0} \mathcal{L}_2 + \frac{3i}{2} h_{\alpha,0} \Lambda_1 \mathcal{L}_2. \end{aligned} \quad (155)$$

Note that the recursion operators Λ_k (125) may be factorized as follows

$$\Lambda_1 X = -i \text{ad}_{L_1}^{-1} \mathcal{J}_1 \frac{\partial X}{\partial x}, \quad \Lambda_2 Y = -i \text{ad}_{L_1}^{-1} \mathcal{J}_2 \frac{\partial Y}{\partial x} \quad (156)$$

where the integral operators \mathcal{J}_k are given by

$$\mathcal{J}_1 X = X - \frac{1}{2} L_{1,x} \partial_x^{-1} \langle L_1, X \rangle, \quad \mathcal{J}_2 Y = X - \frac{3}{2} L_{2,x} \partial_x^{-1} \langle L_2, X \rangle. \quad (157)$$

Note also that the operators \mathcal{J}_k can be inverted

$$\mathcal{J}_1^{-1} X = X + \frac{1}{2} L_{1,x} \partial_x^{-1} \langle L_1, X \rangle, \quad \mathcal{J}_2^{-1} Y = X + \frac{3}{2} L_{2,x} \partial_x^{-1} \langle L_2, X \rangle. \quad (158)$$

These facts allow us to derive also the inverse of the recursion operators

$$\Lambda_1^{-1} X = i \partial_x^{-1} \mathcal{J}_1^{-1} \text{ad}_{L_1} X, \quad \Lambda_2^{-1} X = i \partial_x^{-1} \mathcal{J}_2^{-1} \text{ad}_{L_1} Y. \quad (159)$$

It is appropriate here to remember that the constants $h_{\alpha,0}$ and $k_{\alpha,0}$ are determined by the asymptotic of the relevant ‘squared solution’ for $x \rightarrow \infty$ or for $x \rightarrow -\infty$, depending on the proper definition of the recursion operator, see Remark 3. For each of the recursion operators Λ_1^\pm (respectively Λ_2^\pm) there exist special choices of the roots α for which the constants $k_{\alpha,0}$ (respectively $h_{\alpha,0}$) vanish, namely

$$\begin{aligned} \Lambda_1^+ K_{\mp\alpha}^{\pm,\perp}(x, \lambda) &= \lambda H_{\mp\alpha}^{\pm,\perp}(x, \lambda), & \Lambda_1^- K_{\pm\alpha}^{\pm,\perp}(x, \lambda) &= \lambda H_{\pm\alpha}^{\pm,\perp}(x, \lambda) \\ \Lambda_2^+ H_{\mp\alpha}^{\pm,\perp}(x, \lambda) &= \lambda K_{\mp\alpha}^{\pm,\perp}(x, \lambda), & \Lambda_2^- K_{\pm\alpha}^{\pm,\perp}(x, \lambda) &= \lambda K_{\pm\alpha}^{\pm,\perp}(x, \lambda) \end{aligned} \quad (160)$$

for all positive roots $\alpha > 0$ and therefore

$$\begin{aligned} \Lambda_2^+ \Lambda_1^+ K_{\mp\alpha}^{\pm,\perp}(x, \lambda) &= \lambda^2 K_{\mp\alpha}^{\pm,\perp}(x, \lambda), & \Lambda_2^- \Lambda_1^- K_{\pm\alpha}^{\pm,\perp}(x, \lambda) &= \lambda^2 K_{\pm\alpha}^{\pm,\perp}(x, \lambda) \\ \Lambda_1^+ \Lambda_2^+ H_{\mp\alpha}^{\pm,\perp}(x, \lambda) &= \lambda^2 H_{\mp\alpha}^{\pm,\perp}(x, \lambda), & \Lambda_1^- \Lambda_2^- H_{\pm\alpha}^{\pm,\perp}(x, \lambda) &= \lambda^2 H_{\pm\alpha}^{\pm,\perp}(x, \lambda). \end{aligned} \quad (161)$$

7. The Class of NLEE and the Recursion Operators

In this last Section we briefly outline an alternative way for constructing the hierarchy of NLEE based on the Wronskian relations and the properties of the recursion operators. Let us start with a particular dispersion law which provides the following linear time evolution of the reflection coefficients

$$i\partial_t S^\pm + \frac{3}{2}c_{2k}\lambda^{2k}[K_2, S^\pm(\lambda)] = 0. \quad (162)$$

Comparing (143), (144), (145) and (147) we find

$$\begin{aligned} & \frac{1}{i} \left\langle E_{\mp\alpha}, i\hat{S}^\pm \partial_t S^\pm + \frac{3}{2}c_{2k}\lambda^{2k}\hat{S}^\pm K_2 S^\pm(\lambda) - \frac{3}{2}c_{2k}\lambda^{2k}K_2 \right\rangle \\ &= \lambda \left[[i \operatorname{ad}_{L_1}^{-1} L_{1,t}, e_{\mp\alpha}^\pm(x, \lambda)] \right] + \frac{3}{2}c_{2k}\lambda^{2k} [[\mathcal{L}_2, e_{\mp\alpha}^\pm]] \\ &= \lambda \left[[i \operatorname{ad}_{L_1}^{-1} L_{1,t}, K_{\mp\alpha}^{\pm, \perp}(x, \lambda)] \right] + \frac{3}{2}c_{2k}\lambda^{2k} [[\mathcal{L}_2, H_{\mp\alpha}^{\pm, \perp}]] \\ &= \lambda \left\{ \left[[i \operatorname{ad}_{L_1}^{-1} L_{1,t}, K_{\mp\alpha}^{\pm, \perp}(x, \lambda)] \right] + \frac{3}{2}c_{2k}\lambda^{2k-1} [[\mathcal{L}_2, H_{\mp\alpha}^{\pm, \perp}]] \right\} \quad (163) \\ &= \lambda \left\{ \left[[i \operatorname{ad}_{L_1}^{-1} L_{1,t}, K_{\mp\alpha}^{\pm, \perp}(x, \lambda)] \right] - \frac{3i}{2}c_{2k} [[\mathcal{L}_2, (\Lambda_1^+ \Lambda_2^+)^{k-1} \Lambda_1 * + K_{\mp\alpha}^{\pm, \perp}]] \right\} \\ &= \lambda \left\{ \left[[i \operatorname{ad}_{L_1}^{-1} L_{1,t}, K_{\mp\alpha}^{\pm, \perp}(x, \lambda)] \right] - \frac{3i}{2}c_{2k} [[\mathcal{L}_2, (\Lambda_1^+ \Lambda_2^+)^{k-1} \Lambda_1 * + K_{\mp\alpha}^{\pm, \perp}]] \right\} \\ &= \lambda \left\{ \left[[i \operatorname{ad}_{L_1}^{-1} L_{1,t} - \frac{3i}{2}c_{2k}(\Lambda_1^- \Lambda_2^-)^{k-1} \Lambda_1^- \mathcal{L}_2, K_{\mp\alpha}^{\pm, \perp}]] \right] \right\} = 0. \end{aligned}$$

In deriving the above relations we made use of several useful facts. First, from the basic properties of the Killing form, namely that $\langle X, Y \rangle = 0$ if $X \in \mathfrak{g}^{(1)}$ and $Y \in \mathfrak{g}^{(0)}$ there follows that

$$[[X_1, X_2]] = 0, \quad [[Y_1, Y_2]] = 0$$

for any pair of elements $X_1, X_2 \in \mathfrak{g}^{(1)}$ and $Y_1, Y_2 \in \mathfrak{g}^{(0)}$. This allowed us to identify $[[\mathcal{L}_2, e_{\mp\alpha}^\pm]] = [[\mathcal{L}_2, H_{\mp\alpha}^{\pm, \perp}]]$. Second, we made use of equations (160) and (161) and identified the factor λ^{2k} by the action of the recursion operators. Third, we used the adjoint properties of the recursion operators derived in Appendix B.

As a consequence of equation (163) we find that if $L_1(x, t)$ satisfies the NLEE

$$i \operatorname{ad}_{L_1}^{-1} L_{1,t} - \frac{3i}{2}c_{2k}(\Lambda_1^- \Lambda_2^-)^{k-1} \Lambda_1^- \mathcal{L}_2 = 0 \quad (164)$$

then the scattering data of the Lax operator must satisfy the linear evolution equation (162).

Obviously we can repeat the above arguments for any generic dispersion law of the form (129).

One can check that the result for the nonlinear part of the NLEE with polynomial dispersion laws are local, and therefore both pairs of recursion operators Λ_1^+ , Λ_1^+ and Λ_1^- , Λ_1^- lead to the same NLEE.

8. Conclusions

A system of coupled equations which generalize Heisenberg ferromagnet equations have been obtained. The system is associated with a polynomial bundle Lax operator L related to the symmetric space $SU(3)/S(U(1) \times U(2))$, see also [7]. The spectral properties of the operator L in the case of the simplest constant boundary condition (67) have been studied. The continuous spectrum of L fills up the real axis in the complex λ -plane and divides it into two regions: the upper half plane \mathbb{C}_+ and the lower half plane \mathbb{C}_- . Each region is an analyticity domain of a fundamental analytic solution to the auxiliary linear problem. The FAS can be constructed solutions of a set of integral equations, see (80), (81) and (84), (85) respectively. Wronskian relations for L_1 and its variation have been derived. Using the Wronskian relations one is able to construct ‘squared solutions’ and an integro-differential operator called recursion operator whose eigenfunctions they are. There exists another viewpoint on recursion operator — they generate hierarchy of symmetries of NLEEs. Thus one can derive the recursion operator of a NLEE from purely symmetry considerations.

Our results can be extended in several directions. Firstly one can consider operator L related to the generic Cartan symmetric space of the type **A.III** $\cong SU(n+k)/S(U(n) \times U(k))$ or more generally related to other types of symmetric spaces. This will allow us to treat multi-component generalizations of our NLEE.

The second direction of generalization concerns developing the theory in the case of a rational bundle L , namely

$$L = i\partial_x + \lambda L_1 + \frac{1}{\lambda} L_{-1}.$$

That modification is required when one imposes an additional \mathbb{Z}_2 reduction of the form $\lambda \rightarrow 1/\lambda$. This case is more complicated and much richer than the one we have exploited here, see [10]. It requires the construction of automorphic Lie algebras and studying their properties following the ideas of [17, 23, 26].

The third direction of our investigations involves the construction of the spectral decompositions for the recursion operators. More precisely, one can derive the completeness relation for the ‘squared solutions’ which will allow us to prove rigorously the equivalence of the equations (162) and (164). Next this completeness

relation can be used to derive on a common basis all fundamental properties of the NLEE, including their hierarchy of Hamiltonian structures, see [11].

Of course, using the dressing method [27, 28] one can construct explicitly the soliton solutions to the above equations. These generalizations will be considered elsewhere.

Acknowledgements

The authors acknowledges support from the Bulgarian Academy of Sciences and the Royal Society via joint research project ‘‘Reductions of Nonlinear Evolution Equations and Analytic Spectral Theory’’. One of us (T.V.) acknowledges partial support from the European Operational programm HRD with the Bulgarian Ministry of Education, contract BGO051PO001/07/3.3-02/53.

Appendix A. Derivation of $\text{ad}_{L_1}^{-1}$

Here we outline the derivation of the operator $\text{ad}_{L_1}^{-1}$ following the ideas in [13].

First we note that L_1 has as eigenvalues $\pm 1, 0$ and satisfies the characteristic equation

$$L_1^3 = L_1. \quad (165)$$

Therefore ad_{L_1} has eigenvalues $\pm 2, \pm 1, 0$ and satisfies the characteristic equation

$$(\text{ad}_{L_1}^2 - 4)(\text{ad}_{L_1}^2 - 1)\text{ad}_{L_1} = 0. \quad (166)$$

Projecting out the kernel of ad_{L_1} from (166) we get

$$\text{ad}_{L_1}^{-1} = \frac{1}{4} (5\text{ad}_{L_1} - \text{ad}_{L_1}^3). \quad (167)$$

Using (165) we get

$$\begin{aligned} \text{ad}_{L_1}^3 X &= [L_1, [L_1, [L_1, X]]] = [L_1^3, X] - \frac{3}{4}[L_1, L_1 X L_1] \\ &= [L_1, X - \frac{3}{4}L_1 X L_1]. \end{aligned} \quad (168)$$

If we choose X in the form

$$X = \begin{pmatrix} 0 & a & b \\ a^* & 0 & 0 \\ b^* & 0 & 0 \end{pmatrix} \quad (169)$$

then we have

$$L_1 X L_1 = \begin{pmatrix} 0 & uw & vw \\ u^* w^* & 0 & 0 \\ v^* w^* & 0 & 0 \end{pmatrix} \quad (170)$$

where $w = ua^* + vb^*$ and therefore

$$\begin{aligned} \text{ad}_{L_1}^{-1}X &= [L_1, X + \frac{3}{4}L_1XL_1] \\ &= \begin{pmatrix} w - w^* & 0 & 0 \\ 0 & u^*a - a^*u & u^*b - va^* \\ 0 & v^*a - ub^* & v^*b - b^*v \end{pmatrix} + \frac{3}{4}(w - w^*) \begin{pmatrix} -1 & 0 & 0 \\ 0 & |u|^2 & u^*v \\ 0 & v^*u & |v|^2 \end{pmatrix}. \end{aligned}$$

Then one can check that $\langle L_2, \text{ad}_{L_1}^{-1}X \rangle = 0$. In other words we can consider the condition

$$w + w^* = ua^* + vb^* + au^* + bv^* = 0 \quad (171)$$

as a constraint that $X = X^\perp$.

The operator $\text{ad}_{L_1}^{-1}$ is defined only on the image of ad_{L_1} , i.e., acting by $\text{ad}_{L_1}^{-1}$ on X we should recover its projection X^\perp

$$X^\perp = X - \frac{1}{2}L_1\langle L_1, X \rangle = X - \frac{1}{2}(w + w^*)L_1. \quad (172)$$

One can check that

$$[L_1, \text{ad}_{L_1}^{-1}X^\perp] = X^\perp. \quad (173)$$

In particular, choosing $X = L_{1,x}$ we find

$$\begin{aligned} \mathcal{L}_1 &\equiv \text{ad}_{L_1}^{-1}L_{1,x} \\ &= \begin{pmatrix} \frac{1}{2}w_0 & 0 & 0 \\ 0 & u^*u_x - u_x^*u + \frac{3}{2}w_0|u|^2 & u^*v_x - vu_x^* + \frac{3}{2}w_0u^*v \\ 0 & v^*u_x - uv_x^* + \frac{3}{2}w_0uv^* & v^*v_x - v_x^*v + \frac{3}{2}w_0|v|^2 \end{pmatrix} \end{aligned} \quad (174)$$

where $w_0 = uu_x^* + vv_x^*$.

Next we can choose

$$Y = \begin{pmatrix} -k - n & 0 & 0 \\ 0 & k & m \\ 0 & m^* & n \end{pmatrix} \quad (175)$$

and derive $\text{ad}_{L_1}^{-1}Y$. In what follows we will need also

$$\begin{aligned} L_1YL_1 &= \begin{pmatrix} W & 0 & 0 \\ 0 & -(k+n)|u|^2 & -(k+n)u^*v \\ 0 & -(k+n)uv^* & -(k+n)|v|^2 \end{pmatrix} \\ Y^\perp &= Y - (k|v|^2 + n|u|^2 - muv^* - m^*u^*v) L_2 \\ W &= k|u|^2 + n|v|^2 + vm^*u^* + umv^*. \end{aligned} \quad (176)$$

Then as above

$$\begin{aligned} \text{ad}_{L_1}^{-1}Y &= [L_1, Y + \frac{3}{4}L_1YL_1] \\ &= \frac{1}{4} \begin{pmatrix} 0 & u\alpha_0 + vm^* & v\alpha_1 + um \\ -u^*\alpha_0 - mv^* & 0 & 0 \\ -v^*\alpha_1 - u^*m^* & 0 & 0 \end{pmatrix} \\ \alpha_0 &= 5k + n - 3W, \quad \alpha_1 = k + 5n - 3W. \end{aligned} \quad (177)$$

Again we have the condition

$$k|v|^2 + n|u|^2 - muv^* - m^*u^*v = 0 \quad (178)$$

which ensures that $Y = Y^\perp$. In particular, choosing $Y = L_{2,x}$ we find

$$\begin{aligned} \mathcal{L}_2 &\equiv \text{ad}_{L_1}^{-1}L_{2,x} \\ &= \begin{pmatrix} 0 & -u|u|_x^2 - v(uv^*)_x & -v|v|_x^2 - u(v^*u)_x \\ u^*|u|_x^2 + v^*(u^*v)_x & 0 & 0 \\ v|v|_x^2 + u(u^*v)_x & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -u_x - u(uu_x^* + vv_x^*) & -v_x - v(uu_x^* + vv_x^*) \\ u_x^* + u^*(u^*u_x + v^*v_x) & 0 & 0 \\ v_x^* + v^*(u^*u_x + v^*v_x) & 0 & 0 \end{pmatrix} \end{aligned} \quad (179)$$

by comparing with the respective expressions for a and b in equation (62).

Appendix B. Derivation of the ‘Adjoint’ Recursion Operators

Here we derive the recursion operators, that are adjoint with respect to the skew-scalar product, i.e., we define $\Lambda_{1,2}^*$ by the relations

$$[[X_2, \Lambda_{1,\pm}X_1]] = [[\Lambda_{1,\pm}^*X_2, X_1]], \quad [[Y_2, \Lambda_{2,\pm}Y_1]] = [[\Lambda_{2,\pm}^*Y_2, Y_1]] \quad (180)$$

where $X_{1,2} = X_{1,2}^\perp \in \mathfrak{g}^1$ and $Y_{1,2} = Y_{1,2}^\perp \in \mathfrak{g}^0$.

In doing this we use several times integration by parts and the following properties of the operator $\text{ad}_{L_1}^{-1}$

$$\begin{aligned} [L_1, \text{ad}_{L_1}^{-1}X_{1,2}] &= X_{1,2}, & \langle X_1, \text{ad}_{L_1}^{-1}X_2 \rangle &= -\langle \text{ad}_{L_1}^{-1}X_1, X_2 \rangle \\ [L_1, \text{ad}_{L_1}^{-1}Y_{1,2}] &= Y_{1,2}, & \langle Y_1, \text{ad}_{L_1}^{-1}Y_2 \rangle &= -\langle \text{ad}_{L_1}^{-1}Y_1, Y_2 \rangle. \end{aligned} \quad (181)$$

The derivation of $\Lambda_{1,\pm}^*$ goes as follows

$$\begin{aligned}
 [[X_2, \Lambda_{1,\pm} X_1]] &= - \int_{-\infty}^{\infty} dx \left\langle X_2, \left[L_1, \text{iad}_{L_1}^{-1} \frac{\partial X_1}{\partial x} \right] \right\rangle \\
 &\quad + \frac{i}{2} \int_{-\infty}^{\infty} dx \left\langle X_2, \left[L_1, \text{iad}_{L_1}^{-1} L_{1,x} \right] \right\rangle \int_{\pm\infty}^x dy \left\langle L_1, \frac{\partial X_1}{\partial y} \right\rangle \\
 &= i \int_{-\infty}^{\infty} dx \left\langle \frac{\partial X_2}{\partial x}, X_1 \right\rangle - \frac{i}{2} \int_{-\infty}^{\infty} dx \langle X_2, L_{1,x} \rangle \int_{\pm\infty}^x dy \langle X_1, L_{1,y} \rangle \\
 &= -i \int_{-\infty}^{\infty} dx \left\langle \text{ad}_{L_1}^{-1} \frac{\partial X_2}{\partial x}, [L_1, X_1] \right\rangle \\
 &\quad - \frac{i}{2} \int_{-\infty}^{\infty} dx \left\langle \text{ad}_{L_1}^{-1} L_{1,x} \right\rangle \left(\int_{\pm\infty}^x dy \left\langle L_{1,y}, \frac{\partial X_2}{\partial y} \right\rangle \right) \\
 &= [[\Lambda_{1,\pm}^* X_2, X_1]].
 \end{aligned}$$

The other recursion operators are treated analogously. Thus we obtain

$$\Lambda_{1,\pm}^* = \Lambda_{1,\mp}, \quad \Lambda_{2,\pm}^* = \Lambda_{2,\mp}. \quad (182)$$

References

- [1] Ablowitz M., Kaup D., Newell A. and Segur H., *The Inverse Scattering Transform – Fourier Analysis for Nonlinear Problems*, Studies Appl. Math. **53** (1974) 249–315.
- [2] Calogero F. and Degasperis A., *Nonlinear Evolution Equations Solvable by the Inverse Spectral Transform I*, Nuovo Cimento B **32** (1976) 201–242.
- [3] Calogero F. and Degasperis A., *Nonlinear Evolution Equations Solvable by the Inverse Spectral Transform II*, Nuovo Cimento B **39** (1976) 1–54.
- [4] Drinfel'd V. and Sokolov V., *Lie Algebras and Equations of Korteweg-de Vries Type*, Sov. J. Math. **30** (1985) 1975–2036.
- [5] Gerdjikov V., *Generalised Fourier Transforms for the Soliton Equations. Gauge Covariant Formulation*, Inv. Problems **2** (1986) 51–74.
- [6] Gerdjikov V., *Algebraic and Analytic Aspects of N-wave Type Equations*, Contemp. Math. **301** (2002) 35–68 [nlin.SI/0206014](#).
- [7] Gerdjikov V., *On Spectral Theory of Lax Operators on Symmetric Spaces: Vanishing Versus Constant Boundary Conditions*, J. Geom. Symm. Phys. **15** (2009) 1–41.
- [8] Gerdjikov V., Grahovski G. and Kostov N., *On N-wave Type Systems and their Gauge Equivalent*, European J. Phys. B **29** (2002) 243–248.
- [9] Gerdjikov V. and Kulish P., *The Generating Operator for the $n \times n$ Linear System*, Physica D **3** (1981) 549–564.
- [10] Gerdjikov V., Mikhailov A. and Valchev T., *Reductions of Integrable Equations on A.III-type Symmetric Spaces*, J. Phys. A: Math & Theor. **43** (2010) 434015.

-
- [11] Gerdjikov V., Vilasi G. and Yanovski A., *Integrable Hamiltonian Hierarchies. Spectral and Geometric Methods*, Lecture Notes in Physics vol. **748**, Springer, Berlin, 2008.
- [12] Gerdjikov V. and Yanovski A., *Gauge Covariant Theory of the Generating Operator I. The Zakharov–Shabat System*, Commun. Math. Phys. **103** (1986) 549–568.
- [13] Gerdjikov V. and Yanovsky A., *Gauge Covariant Formulation of the Generating Operator. II. Systems on Homogeneous Spaces*, Phys. Lett. A **110** (1985) 53–58.
- [14] Golubchik I. and Sokolov V., *Multicomponent Generalization of the Hierarchy of the Landau-Lifshitz Equation*, Theor. Math. Phys. **124** (2000) 909–917.
- [15] Gürses M., Karasu A. and Sokolov V., *On Construction of Recursion Operators From Lax Representation*, J. Math. Phys. **40** (1999) 6473–6490.
- [16] Helgasson S., *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York, 1978.
- [17] Lombardo S. and Sanders J., *On the Classification of Automorphic Lie Algebras*, arXiv:0912.1697v1 [nlin.SI].
- [18] Mikhailov A., *On the Integrability of Two-dimensional Generalization of the Toda Lattice*, Lett. J. Exp. Theor. Phys. **30** (1979) 443–448.
- [19] Mikhailov A., *Reductions in Integrable Systems. The Reduction Groups*, Lett. J. Exp. Theor. Phys. **32** (1980) 187–192.
- [20] Mikhailov A., *The Reduction Problem and The Inverse Scattering Method*, Physica D **3** (1981) 73–117.
- [21] Mikhailov A., *The Landau-Lifschitz Equation and the Riemann Boundary Problem on a Torus*, Phys. Lett. A **92** (1982) 51.
- [22] Mikhailov A. and Lombardo S., *Reductions of Integrable Equations: Dihedral Group*, J. Phys. A: Math. & Gen. **37** (2004) 7727–7742.
- [23] Mikhailov A. and Lombardo S., *Reduction Groups and Automorphic Lie Algebras*, Commun. Math. Phys. **258** (2005) 179–202.
- [24] Mikhailov A., Olshanetski M. and Perelomov A., *Two-dimensional Generalized Toda Lattice*, Commun. Math. Phys. **79** (1981) 473–488.
- [25] Takhtadjan L. and Faddeev L., *The Hamiltonian Approach to Soliton Theory*, Springer, Berlin, 1987.
- [26] Wang J., *Lenard Scheme for Two-dimensional Periodic Volterra Chain*, J. Math. Phys. **50** (2009) 023506.
- [27] Zakharov V., Manakov S., Novikov S. and Pitaevskii L., *Theory of Solitons: The Inverse Scattering Method*, Plenum, New York, 1984.
- [28] Zakharov V. and Mikhailov A., *Relativistically Invariant Two-dimensional Models of Field Theory Which are Integrable by Means of the Inverse Scattering Problem Method*, Zh. Eksp. Teor. Fiz. **74** (1978) 1953–1973
Zakharov V. and Mikhailov A., *On the Integrability of Classical Spinor Models in Two-dimensional Space-time*, Comm. Math. Phys. **74** (1980) 21–40