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INVARIANT REDUCTION OF THE TWO-BODY PROBLEM WITH CENTRAL INTERACTION ON SIMPLY CONNECTED SPACES OF CONSTANT SECTIONAL CURVATURE

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Abstract. The problem of two classical particles with central interaction on simply connected spaces of a constant curvature is considered from the invariant point of view. The Hamiltonian reduction method is used for exluding a movement of the system as a whole.

1. Introduction

Everybody knows that the two body problem with central interaction in Euclidean space of an arbitrary dimension is reduced to the problem of one body in a central potential. On the other hand there exist spaces of a constant sectional curvature, which possess as wide isometry group as the Euclidean space of the same dimension and are homogeneous and isotropic. In this connection the following question arises: What is the most effective way of using their isometry group for the simplifying of the two-body problem on spaces of the constant sectional curvature?

Detail analysis shows that for the last problem there is no an analog of the Galilei transformation and a naturally defined center of mass doesn't move along a geodesic even in the case without interaction. So, for simplifying this problem we can use only the isometry group. It had been shown [1] that the exluding of a movement of the center of mass for n-particle system in Euclidean space can be carried out by the Marsden–Weinstein reduction method. Obviously the result is the same as after the using for the same purpose the Galilei transformation.

The Hamiltonian reduction had been used for the classical two-body problem in the spaces of a constant sectional curvature [2]. This reduction was based

there on the explicit direct very cumbersome calculations carried out with the help of a computer analytical calculation system. There was shown that in the general case the reduced dynamic system has the two degrees of freedom and there were found the canonically conjugated coordinates and the expressions of the reduced Hamiltonian function in these coordinates.

In this paper more geometric procedure of this reduction will be presented.

First of all we should mention that the classical two-body problem on the sphere and the hyperbolic space reaches it's maximal generality for 3-dimensional spaces. Indeed, for every moment of time in spaces of a greater dimension we can choose a constant curvature subspace of the dimension three such that it's tangent bundle contains both particles with their velocities vectors. Moreover, starting from this moment particles will stay in this subspace.

From the other hand it is clear that the two body problem on constant curvature surfaces is the special case of the movement on the three dimensional constant curvature space.

Below we will consider the two-body movement on the three dimensional constant curvature spaces.

2. Basic Notations

Let $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$ is supplied with the metric:

$$\mathfrak{g}_s = \left(4R^2 \sum_{i=1}^{3} dx_i^2\right) \Big/ \left(1 + \sum_{i=1}^{3} x_i^2\right)^2 ,$$

where $x_i, \quad i = 1, 2, 3$ are cartesian coordinates on $\mathbb{R}^3$, R is the radius of a curvature. The distance between two points we shall denote as $\rho^s(\cdot, \cdot)$. The connected component of a unit of the isometry group for this space is the 6 dimensional group $SO(4)$ with Lie algebra $so(4)$. The following Killing vector fields correspond to the left action of the isometry group:

$$X_1 = x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} , \quad X_2 = x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3} , \quad X_3 = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} ,$$

$$X_4 = \frac{1}{2}\left(1 + x_1^2 - x_2^2 - x_3^2\right) \frac{\partial}{\partial x_1} + x_1 \left(x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}\right) ,$$

$$X_5 = \frac{1}{2}\left(1 + x_2^2 - x_1^2 - x_3^2\right) \frac{\partial}{\partial x_2} + x_2 \left(x_1 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_3}\right) ,$$

$$X_6 = \frac{1}{2}\left(1 + x_3^2 - x_1^2 - x_2^2\right) \frac{\partial}{\partial x_3} + x_3 \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right) .$$

For the hyperbolic space $\mathbb{H}^3$ we take the Poincaré model in the unit ball $D^3 \subset \mathbb{R}^3$ with the metric:

$$\mathfrak{g}_h = \left(4R^2 \sum_{i=1}^{3} \mathrm{d}x_i^2\right) \Big/ \left(1 - \sum_{i=1}^{3} x_i^2\right)^2 , \qquad \sum_{i=1}^{3} x_i^2 < 1 .$$

The distance between two points of $\mathbb{H}^3$ we shall denote as $\rho^h(\cdot\,,\cdot)$. The connected component of a unit of the isometry group for this space is the 6 dimensional group $SO(1,3)$ with Lie algebra $so(1,3)$. The Killing vector fields are:

$$Z_1 = x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} , \quad Z_2 = x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3} , \quad Z_3 = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} ,$$

$$Y_4 = \frac{1}{2}\left(1 - x_1^2 + x_2^2 + x_3^2\right) \frac{\partial}{\partial x_1} - x_1 \left(x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}\right) ,$$

$$Y_5 = \frac{1}{2}\left(1 - x_2^2 + x_1^2 + x_3^2\right) \frac{\partial}{\partial x_2} - x_2 \left(x_1 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_3}\right) ,$$

$$Y_6 = \frac{1}{2}\left(1 - x_3^2 + x_1^2 + x_2^2\right) \frac{\partial}{\partial x_3} - x_3 \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right) .$$

Below we'll identify Killing vector fields and elements of the corresponding Lie algebra.

3. Representation of the Phase Space and Hamiltonian Functions

The configuration spaces of the two-body problem on the spaces $\mathbb{S}^3$ and $\mathbb{H}^3$ are $Q_s = (\mathbb{S}^3 \times \mathbb{S}^3)\backslash\{\text{diag}\}$ and $Q_h = (\mathbb{H}^3 \times \mathbb{H}^3)\backslash\{\text{diag}\}$ respectively.

The distance between particles $\rho^{s,h}(1,2)$ is the unique invariant of the isometry group. So, the corresponding degree of freedom is not reducable and we'll try to "separate" it as full as possible. Let introduce a foliation of the configuration space $Q_{s,h}$ by folios $\rho = \text{const}$.

The Hamiltonian functions of our systems $H_{s,h} = H_0^{s,h} + U(\rho^{s,h})$ are the Legendre transformations of the natural Lagrange functions:

$$\mathcal{L}_{s,h} = \frac{m_1}{2}\left(\frac{\mathrm{d}s_1}{\mathrm{d}t}\right)^2 + \frac{m_2}{2}\left(\frac{\mathrm{d}s_2}{\mathrm{d}t}\right)^2 + U(\rho^{s,h}) ,$$

where $\mathrm{d}s_i/\,\mathrm{d}t,\ i = 1,2$ is the velocity of the i-th particle on the $Q_{s,h}$ and $H_0^{s,h}$ are the Hamiltonian functions of the free particles.

3.1. The Case of the Sphere $\mathbb{S}^3$

Let $r = \tan(\rho^s(1,2)/2R)$. We denote a folio $r(\rho) = r_0 = \text{const}$ as F_{r_0}. The leaves F_r, $0 < r < \infty$ are homogeneous Riemannian manifolds of the group $SO(4)$ with the stationary subgroup $K = SO(2)$. Choosing a point $\mathbf{x}_0 \in F_r$ we can identify F_r with the factor space $SO(4)/SO(2)$ by the formula $\mathbf{x} = gK\mathbf{x}_0$, where gK is a left coset of the group $SO(4)$. Up to a set of a vanishing measure we have: $Q_s = \mathbb{R}_+ \times (SO(4)/SO(2))$, where $\mathbb{R}_+ = (0, \infty)$.

For any Lie group Γ with an algebra γ there exists a natural diffeomorphism between the space $T^*\Gamma$ and $\Gamma \times T_e^*\Gamma = \Gamma \times \gamma^*$.

The following theorem is valid [3]:

Theorem 1. Let $\mathbf{x}_0 = (x_1^{(1)}, 0, 0, x_1^{(2)}, 0, 0) \in \mathbb{S}^3 \times \mathbb{S}^3$. There exists a function $\tilde{H}^s$ on the manifold

$$T^*\tilde{Q}_s = T^*\left(\mathbb{R}_+ \times SO(4)\right) = T^*\mathbb{R}_+ \times SO(4) \times so^*(4)$$

such that it's natural projection on the space T^*Q_s equals to H^s. This function has the form

$$\begin{aligned}\tilde{H}_s = &\frac{(1+r^2)^2}{8mR^2}p_r^2 + \frac{1}{a}p_4^2 + A_s\left(p_2^2 + p_3^2\right) + C_s\left(p_5^2 + p_6^2\right)\\ &+ \frac{1}{2}B_s\left(p_3p_5 - p_2p_6\right) + U(r)\,,\end{aligned}$$

where p_r is an impulse conjugated to the coordinate r, p_i, $1 \le i \le 6$ are coordinates on $so^*(4)$, corresponding to the dual basis of the basis X_i, $1 \le i \le 6$ and A_s, B_s, C_s are the following functions:

$$A_s(r) = \frac{1}{2R^2}\left(\frac{(1+r^2)^2}{8mr^2} + \frac{1-r^4}{8mr^2}\cos\zeta + \frac{1+r^2}{4m_1m_2r}(m_1 - m_2)\sin\zeta\right),$$

$$B_s(r) = \frac{1}{2R^2}\left(\frac{(m_2 - m_1)}{m_1m_2r}(1+r^2)\cos\zeta + \frac{1-r^4}{2mr^2}\sin\zeta\right),$$

$$C_s(r) = \frac{1}{2R^2}\left(\frac{(1+r^2)^2}{8mr^2} - \frac{1-r^4}{8mr^2}\cos\zeta - \frac{1+r^2}{4m_1m_2r}(m_1 - m_2)\sin\zeta\right),$$

$$\zeta = 2\frac{m_1 - m_2}{m_1 + m_2}\arctan r\,, \qquad m = \frac{m_1m_2}{m_1 + m_2}\,.$$

3.2. The Case of the Hyperbolic Space $\mathbb{H}^3$

The main formulae in this case can be obtained from formulae of the previouse case by the formal transformation: $x_j \to ix_j, r \to ir, R \to iR, j = 1,2,3$, ($i$ is

the imaginary unit). Now $F_r \leftrightarrow \tanh\left(\rho^h(1,2)/2R\right) = r = \text{const}$, $0 < r < 1$, $F_r \cong SO(1,3)/SO(2)$.

Theorem 2. Let $\mathbf{x}_0 = \left(x_1^{(1)}, 0, 0, x_1^{(2)}, 0, 0\right) \in F_r$. There exists a function $\tilde{H}^h$ on the manifold

$$T^*\tilde{Q}_h = T^*\left(\mathbb{I} \times SO(1,3)\right) = T^*\mathbb{I} \times SO(1,3) \times so^*(1,3)$$

such that it's natural projection on the space T^*Q_h equals to H^h, $\mathbb{I} = (0,1)$. This function has the form:

$$\begin{aligned}\tilde{H}_h = {} & \frac{(1-r^2)^2}{8mR^2}p_r^2 + \frac{1}{a}p_4^2 + A_h\left(p_2^2 + p_3^2\right) - C_h\left(p_5^2 + p_6^2\right) \\ & + \frac{1}{2}B_h\left(p_3p_5 - p_2p_6\right) + U(r)\,,\end{aligned}$$

where p_r is the momentum conjugated to the coordinate r, p_i, $1 \le i \le 6$ are coordinates on $so^*(1,3)$, corresponding to the dual basis of the basis Z_i, Y_i, $1 \le i \le 3$ and A_h, B_h, C_h are the following functions:

$$A_h(r) = \frac{1}{2R^2}\left(\frac{(1-r^2)^2}{8mr^2} + \frac{1-r^4}{8mr^2}\cosh\zeta - \frac{1-r^2}{4m_1m_2r}(m_1-m_2)\sinh\zeta\right),$$

$$B_h(r) = \frac{1}{2R^2}\left(\frac{(m_2-m_1)}{m_1m_2r}(1-r^2)\cosh\zeta + \frac{1-r^4}{2mr^2}\sinh\zeta\right),$$

$$C_h(r) = \frac{1}{2R^2}\left(\frac{(1-r^2)^2}{8mr^2} - \frac{1-r^4}{8mr^2}\cosh\zeta + \frac{1-r^2}{4m_1m_2r}(m_1-m_2)\sinh\zeta\right),$$

$$\zeta = 2\frac{m_1-m_2}{m_1+m_2}\operatorname{arctanh} r.$$

4. One Result from the Field of the Hamiltonian Reduction

Let Γ be a Lie group with an algebra $\mathfrak{g}$, Γ_0 is a subgroup of Γ with an algebra $\mathfrak{g}_0 \subset \mathfrak{g}$, acting on Γ from the right. Let $M = T^*\Gamma_1$ be a cotangent bundle of a homogeneous space $\Gamma_1 = \Gamma/\Gamma_0$ equiped with a standart symplectic structure. A standart left action of the group Γ on M is Poisson. Let $\Phi : M \to \mathfrak{g}^*$ be a corresponding momentum map, and H be a Γ-invariant function on M. Let consider the Marsden–Weinstein reduction method in connection with Hamiltonian dynamic system with the function H on M. It is well known that for $\Gamma_0 = \{e\}$ a reduced phase space is symplectomorphic to an orbit of the group Γ in the coadjoint representation equiped with the Kirillov form (up to a sign). The following construction generalize this one.

Let O_{β_0} be an orbit of the coadjoint action of the group Γ on $\mathfrak{g}^*$, containing a point $\beta_0 \in \mathfrak{g}^*$, and

$$O'_{\beta_0} := \{\beta \in O_{\beta_0};\ \beta_{|\mathfrak{g}_0} = 0\}\ .$$

It is obvious that $\mathrm{Ad}^*_{\Gamma_0} O'_{\beta_0} = O'_{\beta_0}$. Let $\tilde{O}_{\beta_0} = O'_{\beta_0} \Big/ \mathrm{Ad}^*_{\Gamma_0}$ and $\pi : O'_{\beta_0} \to \tilde{O}_{\beta_0}$ be a canonical projecture. Let ω be a restriction of the Kirillov form on O'_{β_0}. This means that if elements $X, Y \in T_\beta O'_{\beta_0}$, $\beta \in O'_{\beta_0}$ have the form

$$X = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*_{\exp(tX')}\beta, \qquad Y = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*_{\exp(tY')}\beta, \qquad X', Y' \in \mathfrak{g}$$

then $\omega(X, Y) = \beta([X', Y'])$. Because X' corresponds to a vector tangent to Q'_{β_0}, it holds $\mathrm{Ad}^*_{\exp(tX')}\beta\Big|_{\mathfrak{g}_0} = 0$, so

$$\beta([X', Y'_0]) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*_{\exp(tX')}\beta(Y'_0) = 0$$

for any $Y'_0 \in \mathfrak{g}_0$. It means that a 2-form $\tilde{\omega}$ is well-defined on $T\tilde{O}_{\beta_0}$ by the formula:

$$\tilde{\omega}(\tilde{X}, \tilde{Y}) = \omega(d\pi^{-1}\tilde{X}, d\pi^{-1}\tilde{Y})$$

for $\tilde{X} \in T_{\pi\beta}\tilde{O}_{\beta_0}$, $\tilde{Y} \in T_{\pi\beta}\tilde{O}_{\beta_0}$.

Theorem 3. The reduced phase space $\tilde{M}_{\beta_0}$, which corresponds to the value β_0 of the momentum map, is symplectomorphic to the symplectic space $\left(\tilde{O}_{\beta_0},\ \tilde{\omega}\right)$.

Proof: Let us consider the point x which is in a set $M_{\beta_0} := \Phi^{-1}(\beta_0) \subset M$ being an orbit $O_{x'}$ of some point $x' = (\gamma, p) \in T^*\Gamma$, $\gamma \in \Gamma$, $p \in T^*_\gamma\Gamma$ under the right action of Γ_0 on $T^*\Gamma$. In order to avoid too cumbersome notations we preserve for the right (left) action:

$$(\gamma, p) \to (\gamma_1\gamma, L^*_{\gamma_1^{-1}}p) \quad \text{and} \quad (\gamma, p) \to (\gamma\gamma_1, R^*_{\gamma_1^{-1}}p)$$

of an element $\gamma_1 \in \Gamma$ on $T^*\Gamma$ the notation L_{γ_1} and R_{γ_1}, respectively. According to the definition of the momentum map if $X = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} L_{\exp(tX')}\gamma$, $X' \in \mathfrak{g}$, $X \in T_\gamma\Gamma$ then $p(X) = \beta_0(X')$, i. e. $p = R^*_{\gamma^{-1}}\beta_0$. If $X' \in \mathrm{Ad}_\gamma\mathfrak{g}_0$ then $X \in d\pi_1\left(T_{x'}O_{x'}\right)$, where $\pi_1 :\ T^*\Gamma \to \Gamma$ is a standard projection, and $p(X) = 0$. So $\mathrm{Ad}^*_\gamma\beta_0|_{\mathfrak{g}_0} = 0$. Let denote

$$O = \left\{x' = (\gamma, p) \in T^*\Gamma;\ \mathrm{Ad}^*_\gamma\beta_{0|\mathfrak{g}_0} = 0, \quad p = R^*_{\gamma^{-1}}\beta_0\right\}.$$

Let $\tau: \quad O \to \mathfrak{g}^* = T_e^*\Gamma$ be a map, acting according to the formula $\tau(\gamma, p) = L_\gamma^* p$. The following diagram is commutative [4]:

$$\begin{array}{ccc} T^*\Gamma & \xrightarrow{L_{\gamma^{-1}}} & T^*\Gamma \\ \downarrow \Phi & & \downarrow \Phi \\ \mathfrak{g}^* & \xrightarrow{Ad^*_\gamma} & \mathfrak{g}^* \end{array}$$

Therefore an orbit of a stationary subgroup Γ_{β_0} on $T^*\Gamma$ is mapped by τ into one point. According to the definition of the set O it holds $\tau(O) = O'_{\beta_0}$. The element (γ, p) is mapped into $\mathrm{Ad}^*_\gamma \beta_0$, so the element $R_{\gamma_0}(\gamma, p)$ is mapped into $\mathrm{Ad}^*_{\gamma\gamma_0}\beta_0 = \mathrm{Ad}^*_{\gamma_0} \circ \mathrm{Ad}^*_\gamma \beta_0$. Consequently, orbits of the right action of the group Γ_0 on O are transformed into orbits of the coadjoint action of Γ_0 on O'_{β_0}. This yields a diffeomorphism:

$$\phi: \ \tilde{M}_{\beta_0} = \Gamma_{\beta_0} \backslash M_{\beta_0} = \Gamma_{\beta_0} \backslash (O/\Gamma_0) \to O'_{\beta_0}/\mathrm{Ad}^*_{\Gamma_0} = \tilde{O}_{\beta_0}.$$

It remains to prove that the symplectic form $\hat{\omega}$ on $\tilde{M}_{\beta_0}$ is transformed into the form $-\tilde{\omega}$ under the action of the map ϕ. But this statement follows from it's validity for the case $\Gamma_0 = \{e\}$, the possibility to represent tangent vectors of $\tilde{M}_{\beta_0}$ via tangent vectors of O and the commutativity of the following diagram for any $\gamma_0 \in \Gamma_0$:

$$\begin{array}{ccc} O & \xrightarrow{R_{\gamma_0}} & O \\ \downarrow \tau & & \downarrow \tau \\ O'_{\beta_0} & \xrightarrow{R_{\gamma_0}} & O'_{\beta_0} \end{array}$$

The form $\hat{\omega}$ is symplectic, so we get:

Corollary 1. The form $\tilde{\omega}$ is symplectic on $\tilde{O}_{\beta_0}$, i. e. it is nondegenerate and closed.

5. Reduction of the Two-body System on the Sphere $\mathbb{S}^3$

Let now $M = T^*Q_s$ equiped with the standart symplectic structure. According to the Sect. 3.1 we can represent M as

$$M = T^*\mathbb{R}_+ \times T^*\left(SO(4)/SO(2)\right)$$

up to a set of zero measure, corresponding to the value $r = \infty$. The symmetry group $SO(4)$ acts only on the second term of this product and the construction of the Sect. 4 can be generalized easily on this case. After the reduction we obtain instead of M the space:

$$\tilde{M}_{\beta_0} = T^*\mathbb{R}_+ \times \tilde{Q}_{\beta_0}$$

where $\tilde{Q}_{\beta_0}$ is constructed as in the Sect. 4 for the case $\Gamma = SO(4)$, $\Gamma_0 = SO(2)$. We will introduce another basis in the Lie algebra $so(4)$:

$$L_1 = \frac{1}{2}(X_1 + X_4)\,, \quad L_2 = \frac{1}{2}(X_2 + X_5)\,, \quad L_3 = \frac{1}{2}(X_3 + X_6)\,,$$

$$G_1 = \frac{1}{2}(X_1 - X_4)\,, \quad G_2 = \frac{1}{2}(X_2 - X_5)\,, \quad G_3 = \frac{1}{2}(X_3 - X_6)\,.$$

The commutator relations in so(4) for this basis are as follows:

$$[L_i, L_j] = \sum_{k=1}^{3} \varepsilon_{ijk} L_k\,, \quad [G_i, G_j] = \sum_{k=1}^{3} \varepsilon_{ijk} G_k\,, \quad [L_i, G_j] = 0\,, \quad i, j = 1, 2, 3,$$

where ε_{ijk} is the completely antisymmetric tensor, $\varepsilon_{123} = 1$. This basis corresponds to the expansion $so(4) = so(3) \oplus so(3)$. Let $u_i, v_i,\ 1 = 1, 2, 3$ are the coordinates of an arbitrary element $\mathbf{p}$ in the space $so^*(4)$ with respect to the dual basis:

$$\mathbf{p} = \sum_{i=1}^{3} \left(u_i L^i + v_i G^i\right).$$

After substitutions $p_i = u_i + v_i,\ p_{3+i} = u_i - v_i,\ i = 1, 2, 3$ we obtain the function $\tilde{H}_s$ in the following form:

$$\begin{aligned}\tilde{H}_s = {} & \frac{(1+r^2)^2}{8mR^2} p_r^2 + \frac{1}{a}\left(u_1 - v_1\right)^2 + A_s\left(\left(u_2 + v_2\right)^2 + \left(u_3 + v_3\right)^2\right) \\ & + C_s\left(\left(u_2 - v_2\right)^2 + \left(u_3 - v_3\right)^2\right) + B_s\left(u_2 v_3 - v_2 u_3\right) + U(r)\,.\end{aligned}$$

Let us now construct the conjugated coordinates on $\tilde{O}_{\beta_0}$. The stationary subgroup $SO(2)$ of the point $\mathbf{x}_0 = (x_1^{(1)}, 0, 0, x_1^{(2)}, 0, 0)$ is generated by the element X_1. It is well-known that a nondegenerated orbit of the coadjoint action of the group $SO(3)$ is a sphere and the Kirillov form on this orbit is it's area. The same orbit for the group $SO(4)$ is therefore a direct product of its two spheres. So the orbit O_{β_0} can be specified as a set of elements in the space $so^*(4)$ with coordinates $u_i,\ v_i,\ i = 1, 2, 3$, which satisfy the following conditions:

$$u_1^2 + u_2^2 + u_3^2 = \mu^2\,, \qquad v_1^2 + v_2^2 + v_3^2 = \nu^2\,, \tag{1}$$

where μ, ν are arbitrary nonnegative real numbers. The subset $O'_{\beta_0} \subset O_{\beta_0}$ consists of those elements of O_{β_0} that are anulated by the element X_1. Therefore we must add the condition $p_1 = u_1 + v_1 = 0$ to the equations (1) to describe the set O'_{β_0}.

Firstly, let consider the case $\mu, \nu > 0$. Let u, ψ, χ be coordinates on O'_{β_0} that are introduced by following equations:

$$u_1 = -v_1 = u, \qquad u_2 = \sqrt{\mu^2 - u^2}\sin\psi\,, \qquad u_3 = \sqrt{\mu^2 - u^2}\cos\psi\,,$$
$$v_2 = \sqrt{\nu^2 - u^2}\sin\chi\,, \qquad v_3 = \sqrt{\nu^2 - u^2}\cos\chi\,,$$
$$-\min\{\mu,\nu\} \le u \le \min\{\mu,\nu\}.$$

The restriction of the Kirillov form from O_{β_0} to O'_{β_0} is as follows:

$$\omega = \frac{1}{\mu^2}\left(u_1\,\mathrm{d}u_2\wedge\mathrm{d}u_3 + u_2\,\mathrm{d}u_3\wedge\mathrm{d}u_1 + u_3\,\mathrm{d}u_1\wedge\mathrm{d}u_2\right)$$
$$+\frac{1}{\nu^2}\left(v_1\,\mathrm{d}v_2\wedge\mathrm{d}v_3 + v_2\,\mathrm{d}v_3\wedge v_1 + v_3\,\mathrm{d}v_1\wedge\mathrm{d}v_2\right) = \mathrm{d}u\wedge\mathrm{d}(\psi-\chi)\,.$$

The coadjoint action of the one-parametric group, corresponding to the element X_1, on the orbit O'_{β_0} is described by the formulae

$$u \to u, \qquad \psi \to \psi + \xi, \qquad \chi \to \chi + \xi,\ 0 \le \xi < 2\pi$$

Therefore $\phi = \psi - \chi,\ p_\phi = u$ are canonically conjugated coordinates on $\tilde{O}_{\beta_0}$. In fact $\tilde{O}_{\beta_0}$ is diffeomorphic to $\mathbb{S}^2$. The coordinate system p_ϕ, ϕ is singular at the points $p_\phi = \pm\min\{\mu,\nu\}$. The reduced Hamiltonian $\hat{H}_s$ has the following form:

$$\hat{H}_s = \frac{(1+r^2)^2}{8mR^2}p_r^2 + \frac{4p_\phi^2}{a} + A_s\left(\mu^2+\nu^2-2p_\phi^2+2\sqrt{\mu^2-p_\phi^2}\sqrt{\nu^2-p_\phi^2}\cos\phi\right)$$
$$+C_s\left(\mu^2+\nu^2-2p_\phi^2-2\sqrt{\mu^2-p_\phi^2}\sqrt{\nu^2-p_\phi^2}\cos\phi\right)$$
$$+B_s\sqrt{\mu^2-p_\phi^2}\sqrt{\nu^2-p_\phi^2}\sin\phi + U(r)\,.$$

In the case $\mu = 0,\ \nu > 0$ (or $\nu = 0, \mu > 0$) we obtain for O'_{β_0} the conditions $u_1 = u_2 = u_3 = v_1 = 0$, and so $O'_{\beta_0} = \mathbb{S}^1$ and $\tilde{O}_{\beta_0} = \mathrm{pt}$. Therefore the reduced phase space in this case is $T^*\mathbb{R}_+$ with the Hamiltonian:

$$\hat{H}_s = \frac{(1+r^2)^2}{8mR^2}\left(p_r^2 + \frac{\nu^2}{r^2}\right).$$

In the case $\mu = \nu = 0$ we obtain

$$\tilde{O}_{\beta_0} = O'_{\beta_0} = \mathrm{pt}\,, \qquad M = T^*\mathbb{R}_+\,, \qquad \hat{H}_s = \frac{(1+r^2)^2}{8mR^2}p_r^2\,.$$

6. Two-body System on the Hyperbolic Space $\mathbb{H}^3$

The Lie algebra $so(1,3)$ is a simple one so we can not represent an orbit of the coadjoint action of $SO(1,3)$ as a direct product like in Sect. 5. However the dynamic systems on the sphere $\mathbb{S}^3$ and the space $\mathbb{H}^3$ are connected by the formal replacement (see above). This motivate the following construction.

Let $Z^1, Z^2, Z^3, Y^1, Y^2, Y^3$ be the basis in $so^*(1,3)$ dual to the basis $Z_1, Z_2, Z_3, Y_1, Y_2, Y_3$. Let

$$\mathbf{p} = p_1 Z^1 + p_2 Z^2 + p_3 Z^3 + p_4 Y^1 + p_5 Y^2 + p_6 Y^3$$

be an arbitrary element of $so^*(1,3)$. Then it can be varified by direct calculations that the following formulae give the invariants of the coadjoint action of $SO(1,3)$:

$$I_1 = p_1^2 + p_2^2 + p_3^2 - p_4^2 - p_5^2 - p_6^2, \qquad I_2 = p_1 p_4 + p_2 p_5 + p_3 p_6\,.$$

Let O_{β_0} be an orbit of the coadjoint action of $SO(1,3)$ given by equations: $I_1 = \mu, I_2 = \nu$, $\mu, \nu \in \mathbb{R}$. The stationary subgroup of the point $\mathbf{x}_0 = \left(x_1^{(1)}, 0, 0, x_1^{(2)}, 0, 0\right) \in F_r$ is generated by the element Z_1. The coadjoint action of this subgroup are simultaneous rotations in the planes (p_2, p_3) and (p_5, p_6). The submanifold O'_{β_0} is given by the equations $I_1 = \mu$, $I_2 = \nu$, $p_1 = 0$. The following formulae give the coordinates p_4, ψ, χ on it:

$$p_2 = u\cosh\psi\cos\chi + v\sinh\psi\sin\chi\,, \quad p_3 = v\sinh\psi\cos\chi - u\cosh\psi\sin\chi\,,$$
$$p_5 = v\cosh\psi\cos\chi - u\sinh\psi\sin\chi\,, \quad p_6 = -u\sinh\psi\cos\chi - v\cosh\psi\sin\chi\,,$$

where $p_4, \psi \in \mathbb{R}$, $\chi \in \mathbb{R} \mod 2\pi$ and u, v can be found by the equations

$$u^2 - v^2 = \mu + p_4^2\,, \qquad uv = \nu\,.$$

The two solutions of the last equations differ from each other by a sign so we have to choose only one of them. The action of the stationary group $SO(2)$ corresponds to the rotation $\chi \to \chi + \xi$. The reduced phase space $\tilde{O}_{\beta_0}$ is obtained from O'_{β_0} by "forgetting" about the coordinate χ. The space $\tilde{O}_{\beta_0}$ is diffeomorphic to $\mathbb{R}^2$.

In order to find the canonically conjugated coordinates on $\tilde{O}_{\beta_0}$ we will use the degenerated Poisson bracket on $so^*(1,3)$ that corresponds to the Kirillov symplectic form. This bracket was constructed in [5] for any Lie algebra $\mathfrak{g}$ as follows.

Let $\{e_i\}_{i=1}^n$ be a basis of the algebra $\mathfrak{g}$, $[e_i, e_j] = c_{ij}^k e_k$, and $\{x_i\}_{i=1}^n$ be coordinates on $\mathfrak{g}^*$, corresponding to a dual basis $\{e^i\}_{i=1}^n$. Let f_1, f_2 be arbitrary

smooth functions on $\mathfrak{g}^*$. Then their Poisson bracket have the following form:

$$\{f_1, f_2\} = -\sum_{i,j,k=1}^{n} c_{ij}^k x_k \frac{\partial f_1}{\partial x_i} \frac{\partial f_2}{\partial x_j} .$$

The restriction of this bracket to an orbit of the coadjoint action is nondegenarated. Returning to our case and using the following formulae

$$\psi = \frac{1}{4} \ln \left(\frac{(p_2 - p_6)^2 + (p_5 + p_3)^2}{(p_2 + p_6)^2 + (p_5 - p_3)^2} \right)$$

$$\chi = \frac{1}{2} \left(\arctan \left(\frac{p_5 - p_3}{p_2 + p_6} \right) - \arctan \left(\frac{p_5 + p_3}{p_2 - p_6} \right) \right)$$

$$[Z_i, Z_j] = \sum_{k=1}^{3} \varepsilon_{ijk} Z_k , \quad [Y_i, Y_j] = -\sum_{k=1}^{3} \varepsilon_{ijk} Z_k , \quad [Z_i, Y_j] = \sum_{k=1}^{3} \varepsilon_{ijk} Y_k$$

we obtain by direct calculations the following relations:

$$\{p_4, \psi\} = 1, \quad \{p_4, \chi\} = 0, \quad \{\psi, \chi\} = 0 .$$

Therefore the symplectic form on $\tilde{O}_{\beta_0}$ is given by the formula: $\mathrm{d}p_4 \wedge \mathrm{d}\psi$. From the previous formulae we obtain:

$$p_2^2 + p_3^2 = \frac{1}{2} \left(\mu + p_4^2 + \sqrt{\left(\mu + p_4^2\right)^2 + 4\nu^2} \cosh 2\psi \right) ,$$

$$p_5^2 + p_6^2 = \frac{1}{2} \left(-\mu - p_4^2 + \sqrt{\left(\mu + p_4^2\right)^2 + 4\nu^2} \cosh 2\psi \right) ,$$

$$p_3 p_5 - p_2 p_6 = \frac{1}{2} \sqrt{\left(\mu + p_4^2\right)^2 + 4\nu^2} \sinh 2\psi .$$

Introducing new canonically conjugated coordinates $p_\phi = p_4/2$, and $\phi = 2\psi$ we obtain the final form of the reduced Hamiltonian function:

$$\hat{H}_h = \frac{(1 - r^2)^2}{8mR^2} p_r^2 + \frac{4p_\phi^2}{a} + A_h \left(\frac{\mu}{2} + 2p_\phi^2 + 2\sqrt{\left(\frac{\mu}{4} + p_\phi^2 \right)^2 + \frac{\nu^2}{4}} \cosh \phi \right)$$

$$+ C_h \left(\frac{\mu}{2} + 2p_\phi^2 - 2\sqrt{\left(\frac{\mu}{4} + p_\phi^2 \right)^2 + \frac{\nu^2}{4}} \cosh \phi \right)$$

$$+ B_h \sqrt{\left(\frac{\mu}{4} + p_\phi^2 \right)^2 + \frac{\nu^2}{4}} \sinh \phi + U(r) .$$

7. Conclusion

The so derived forms of reduced two-body classical systems on $\mathbb{S}^3$ and $\mathbb{H}^3$ can be used for proving the absence of particle's collision for some potentials [2]. In this work we have cleared up also the geometric meaning of the reduction procedure.

A similar analysis for the quantum two-body problem is also possible [2, 6].

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