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INTEGRABILITY OF CONTACT SCHWARZIAN DERIVATIVES AND ITS LINEARIZATION

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Abstract. We define the contact Schwarzian derivatives $s_{[ij,k]}(\phi)$ for a contact transformation $\phi : \mathbb{K}^3 \to \mathbb{K}^3$. Using the contact Schwarzian derivatives as coefficients, we give a system of linear differential equations such that the solutions give the contact transformation.

1. Introduction

For a contact transformation $\phi : (x, y, z) \mapsto (X, Y, Z)$, we define the contact Schwarzian derivatives $s_{[ij,k]}(\phi)$. A system of non-linear differential equations for a quadruple of functions is given as the condition that the quadruple is the Schwarzian derivatives of a contact transformation. Using a quadruple of functions on $\mathbb{K}^3$ as coefficients, we give a system of linear differential equations. The integrable condition of the linear system is just equal to the non-linear system. We call the linear system a linearization of the integrability condition of the contact Schwarzian derivatives. If the linear system is integrable, the solutions give the contact transformation whose contact Schwarzian derivatives are the coefficient functions. Details will appear in a joint paper with Tetsuya Ozawa [2].

2. Contact Schwarzian Derivative

On the affine 3-space $\mathbb{K}^3$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with the usual coordinate (x, y, z), we give the contact form $\alpha = dy - zdx$.

$$v_1 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}, \quad v_2 = \frac{\partial}{\partial z}, \quad v_3 = \frac{\partial}{\partial y}, \quad v_4 = v_2v_1 + v_1v_2 .$$

Notice that the vector fields v_1, v_2 and v_3 form a Heisenberg Lie algebra:

$$v_3 = [v_2, v_1]\,, \qquad \text{and} \qquad [v_3, v_1] = [v_3, v_2] = 0\,,$$

and that the vector fields v_1 and v_2 span the contact distribution; $\alpha(v_1) = \alpha(v_2) = 0$.

A local diffeomorphism ϕ is said to be a *contact transformation*, if it satisfies $\phi^*(\alpha) = \rho\alpha$ for some nonvanishing function ρ. For a contact transformation $\phi : (x, y, z) \mapsto (X, Y, Z)$, we define the contact Schwarzian derivatives for $i, j, k = 1, 2$ as follows:

$$s_{[ij,k]}(\phi) = v_i v_j(X) v_k(Z) - v_i v_j(Z) v_k(X)\,,$$

and

$$S_{\{ijk\}}(\phi) = \frac{1}{3\Delta(\phi)}\left(s_{[ij,k]}(\phi) + s_{[jk,i]}(\phi) + s_{[ki,j]}(\phi),\right)$$

where $\Delta(\phi) = v_1(X)v_2(Z) - v_1(Z)v_2(X)$. We call the functions $S_{\{ijk\}}(\phi)$ the *contact Schwarzian derivatives* of the contact transformation ϕ. We denote the quadruple of functions by

$$S(\phi) = (S_{\{111\}}(\phi), S_{\{112\}}(\phi), S_{\{122\}}(\phi), S_{\{222\}}(\phi))\,.$$

A diffeomorphism $\phi : \mathbb{K}^3 \to \mathbb{K}^3$ with $\phi(x, y, z) = (X, Y, Z)$ induces the prolongation $\tilde{\phi} : \mathbb{K}^4 \to \mathbb{K}^4$ given by

$$\tilde{\phi}(x, y, z, w) = \left(X, Y, Z, W = \frac{Z_x + zZ_y + wZ_z}{X_x + zX_y + wX_x}\right)\,.$$

By using this prolongation, we get the following.

Proposition 2.1. *The inverse ϕ^{-1} of a contact diffeomorphism $\phi : \mathbb{K}^3 \to \mathbb{K}^3$ maps the differential equation $Y''' = 0$ to*

$$y''' = S_{\{112\}}(\phi) + 3S_{\{111\}}(\phi)y'' + 3S_{\{222\}}(\phi)(y'')^2 + S_{\{122\}}(\phi)(y'')^3\,.$$

By [4], the condition that $y''' = f(x, y, y', y'')$ is mapped to $y''' = 0$ by a contact transformation is the vanishing of two curvatures A and $\mathbf{b}$. We obtain that $\mathbf{b} = 0$ is equivalent to $\partial^4 f/\partial(y'')^4 = 0$. Let us consider

$$y''' = P + 3Qy' + 3R(y'')^2 + S(y'')^3\,,$$

where $P = P(x, y, y')$, $Q = Q(x, y, y')$, $R = R(x, y, y')$, $S = S(x, y, y')$. Then the condition $A = 0$ together with $\mathbf{b} = 0$ is equal to the following system (IC) of nonlinear differential equations

$$\begin{aligned} v_3(P) &= 2(v_1 - 2Q)M_{11} + 4PM_4 \\ 3v_3(Q) &= 2(v_2 - 4R)M_{11} + 4(v_1 + Q)M_4 + 4PM_{22} \end{aligned}$$

$$\begin{aligned} 3v_3(R) &= 2(v_1 + 4Q)M_{22} + 4(v_2 - R)M_4 - 4SM_{11} \\ v_3(S) &= 2(v_2 + 2R)M_{22} - 4SM_4\,, \end{aligned}$$

where we put

$$\begin{aligned} M_{11} &= -\frac{1}{4}(v_1(Q) - v_2(P) - 2Q^2 + 2PR) \\ M_4 &= -\frac{1}{4}(v_1(R) - v_2(Q) - QR + PS) \\ M_{22} &= -\frac{1}{4}(v_1(S) - v_2(R) - 2R^2 + 2QS)\,. \end{aligned}$$

Thus we obtain that

Theorem 2.1. *Four functions P, Q, R, S on $\mathbb{K}^3$ are the Schwarzian derivarives of a contact transformation $\phi : \mathbb{K}^3 \to \mathbb{K}^3$;*

$$(P, Q, R, S) = S(\phi),$$

if and only if the system of the nonlinear differential equations (IC) *is satisfied.*

We seek a system of linear differential equations whose integrability equation is equvalent to (IC) and its solutions give the contact transformation. We call such linear system the linearization of (IC)

3. Fundamental System

Here is the linear differential system (Sp);

$$\begin{aligned} {v_1}^2(\vartheta) &= Qv_1(\vartheta) - Pv_2(\vartheta) + M_{11}\vartheta \\ v_4(\vartheta) &= 2(Rv_1(\vartheta) - Qv_2(\vartheta) + M_4\vartheta) \\ {v_2}^2(\vartheta) &= Sv_1(\vartheta) - Rv_2(\vartheta) + M_{22}\vartheta \end{aligned}$$

with unknown function ϑ.

Theorem 3.1. *The necessary and sufficient condition for the linear PDE system* (Sp) *to have 4-dimensional solution space is equvalent to the nonlinear PDE system* (IC).

Proposition 3.1. *For any two solutions α and β of the PDE system* (Sp)*, the function $I(\alpha, \beta)$ defined by*

$$I(\alpha, \beta) = \frac{1}{2}\alpha v_3(\beta) - \frac{1}{2}v_3(\alpha)\beta + v_1(\alpha)v_2(\beta) - v_2(\alpha)v_1(\beta)$$

is constant on (x, y, z). Moreover this skew product $I(\alpha, \beta)$ is non-degenerate, and thus it defines a symplectic structure on the solution space $\mathcal{S}(P, Q, R, S)$ of (Sp)*, provided the dimension of $\mathcal{S}(P, Q, R, S)$ is equal to 4.*

We have the following theorem:

Theorem 3.2. *If a map $\phi : (x, y, z) \mapsto (X, Y, Z)$ is contact, then there exists a symplectic basis $\{\vartheta, \xi, \zeta, \eta\}$ of the solution space $\mathcal{S}(S(\phi))$ of the PDE system* (Sp) *such that ϕ is given by*

$$(x, y, z) \mapsto \left(\frac{\xi}{\vartheta}, \frac{1}{2}\Big(\frac{\eta}{\vartheta} + \frac{\xi\zeta}{\vartheta^2} \Big), \frac{\zeta}{\vartheta} \right).$$

Conversely, given a symplectic basis $\{\vartheta, \xi, \zeta, \eta\}$ of the solution space $\mathcal{S}(P, Q, R, S)$ of (Sp)*, the map ϕ is a contact transformation whose contact Schwarzian derivatives are equal to*

$$S(\phi) = (P, Q, R, S).$$

Here a linear basis $\{\xi_0, \xi_1, \xi_2, \xi_3\}$ is symplectic if

$$\Big(I(\xi_i, \xi_j) \Big)_{i,j=0,3} = cJ\,, \qquad \text{where} \qquad J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and c is a nonzero constant.

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