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DIRAC AND SEIBERG–WITTEN MONOPOLES

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Abstract. Dirac magnetic monopoles, which may or may not exist in nature, seem to exist everywhere in mathematics. They are in one-to-one correspondence with the natural connections on principal $U(1)$-bundles over S^2 and, moreover, appear as solutions to the field equations of $SU(2)$ Yang–Mills–Higgs theory on $\mathbb{R}^3$ as well as Seiberg–Witten theory and its non-Abelian generalization on Minkowski spacetime. This talk will present an informal survey of the situation.

1. Classical Dirac Monopoles

We begin with the source-free Maxwell equations written in complex form as

$$\nabla \cdot (\vec{E} + \mathrm{i}\vec{B}) = 0, \qquad \frac{\partial}{\partial t}(\vec{E} + \mathrm{i}\vec{B}) + \mathrm{i}\nabla \times (\vec{E} + \mathrm{i}\vec{B}) = \vec{0}\,. \tag{1.1}$$

These equations have a great many well-known symmetries. They are, for example, Lorentz invariant, gauge invariant and conformally invariant, but they also possess what might be called a "duality symmetry". Specifically, if $\vec{E}+\mathrm{i}\vec{B}$ is a solution to (1.1), then so is $\mathrm{e}^{\mathrm{i}\varphi}(\vec{E} + \mathrm{i}\vec{B})$ for any complex number $\mathrm{e}^{\mathrm{i}\varphi}$ of modulus one. When $\varphi = \pi/2$ this reduces to the familiar fact that the substitutions $\vec{B} \to \vec{E}$ and $\vec{E} \to -\vec{B}$ carry one solution into another.

This last symmetry is lost, of course, if one includes charge densities and currents in Maxwell's equations, but Dirac [3] realized that it could be reinstated by including also (hypothetical) magnetic charges and currents. For this he introduced the magnetic analogue of a Coulomb field defined, on $\mathbb{R}^3 \backslash \{0\}$, by

$$\vec{E} = \vec{0}\,, \qquad \vec{B} = \frac{n/2}{\rho^2}\,\hat{e}_\rho\,, \tag{1.2}$$

where ρ is the standard radial spherical coordinate, $\hat{e}_\rho$ is the unit radial vector field and we have written the charge as $n/2$, n an integer, to conform with the Dirac quantization condition (see page 7 of [9]). This certainly satisfies (1.1) and the usual recipe expresses the field as a 2-form F on $\mathbb{R}^3\backslash\{0\}$:

$$F = \frac{n/2}{\rho^3}(x\,\mathrm{d}y \wedge\,\mathrm{d}z - y\,\mathrm{d}x \wedge\,\mathrm{d}z + z\,\mathrm{d}x \wedge\,\mathrm{d}y)\,. \tag{1.3}$$

In spherical coordinates (ρ, ϕ, θ) this becomes

$$F = \frac{n}{2}\sin\phi\,\mathrm{d}\phi \wedge\,\mathrm{d}\theta\,. \tag{1.4}$$

which, being independent of ρ, may be viewed as a 2-form on the unit sphere S^2.

It is easy to see that the monopole field 2-form F is not exact on $\mathbb{R}^3\backslash\{0\}$, i. e., there does not exist a potential 1-form A defined on all of $\mathbb{R}^3\backslash\{0\}$ for which $F = \mathrm{d}A$, since the existence of such a global potential would contradict Stokes' Theorem (see pages 2–3 of [9]). Dirac knew this, of course, and he also knew, although he did not phrase the matter in these terms, that by deleting from $\mathbb{R}^3$ not only the origin, but also some ray extending from the origin to infinity (a so-called "Dirac string") one obtains a subspace of $\mathbb{R}^3$ whose second de Rham cohomology is trivial.

$$H^2_{\text{de Rham}}(\mathbb{R}^3 - \text{Dirac String}) = 0\,. \tag{1.5}$$

On such a set every 2-form is exact and therefore so is F. For example, on $U_S = \mathbb{R}^3 - \{(0,0,z) : z \geq 0\}$, the 1-form

$$A_S = \frac{n/2}{\rho(\rho - z)}(y\,\mathrm{d}x - x\,\mathrm{d}y) = -\frac{n}{2}(1 + \cos\phi)\,\mathrm{d}\theta \tag{1.6}$$

satisfies $\mathrm{d}A_s = F_{|U_S}$, while on $U_N = \mathbb{R}^3 - \{(0,0,z) : z \leq 0\}$,

$$A_N = -\frac{n/2}{\rho(\rho + z)}(y\,\mathrm{d}x - x\,\mathrm{d}y) = \frac{n}{2}(1 - \cos\phi)\,\mathrm{d}\theta \tag{1.7}$$

satisfies $\mathrm{d}A_N = F_{|U_N}$. Since $\mathbb{R}^3\backslash\{0\} = U_S \cup U_N$ we have covered the domain of F by two open sets on which F has a potential 1-form. Note that, on $U_S \cap U_N$,

$$A_N = A_S + n\,\mathrm{d}\theta\,. \tag{1.8}$$

For convenience, we would now like to regard all of these as forms on the 2-sphere S^2 and introduce a "Lie algebra factor" of $-i$. Thus, we define

$$\mathcal{F} = -\mathrm{i}F = -\mathrm{i}\frac{n}{2}\sin\phi\,\mathrm{d}\phi \wedge \mathrm{d}\theta \quad \text{on } S^2\,, \tag{1.9}$$

$$\mathcal{A}_S = -\mathrm{i}A_S = \frac{n}{2}\mathrm{i}\,(1+\cos\phi)\;\mathrm{d}\theta \quad \text{on } U_S = S^2 - (0,0,1)\,, \tag{1.10}$$

$$\mathcal{A}_N = -\mathrm{i}A_N = -\frac{n}{2}\mathrm{i}\,(1-\cos\phi)\;\mathrm{d}\theta \quad \text{on } U_N = S^2 - (0,0,-1)\,. \tag{1.11}$$

2. Hopf Bundles

We will think of the 3-sphere S^3 as the 1-point compactification of $\mathbb{R}^3$, or as the subspace of $\mathbb{C}^2 = \mathbb{R}^4$ consisting of all (z^1, z^2) such that $|z^1|^2 + |z^2|^2 = 1$. Define an action of $U(1) \cong S^1$ on S^3 by $(z^1, z^2)\,.\,a = (z^1 a, z^2 a)$ for all $(z^1, z^2) \in S^3$ and $a \in S^1$. The orbits are copies of S^1 and the orbit space is $\mathbb{CP}^1$. Choosing a specific diffeomorphism of $\mathbb{CP}^1$ onto S^2 one obtains the complex Hopf bundle

$$U(1) \hookrightarrow S^3 \xrightarrow{\mathcal{P}} S^2\,, \tag{2.1}$$

where

$$\mathcal{P}(z^1,\, z^2) = \Big(2\,\mathrm{Re}(\bar{z}^1\, z^2)\,,\; -2\,\mathrm{Im}(\bar{z}^1\, z^2)\,,\; |z^1|^2 - |z^2|^2\Big)\,. \tag{2.2}$$

This is a principal $U(1)$-bundle over S^2 which trivializes over $U_S = S^2 - (0,0,1)$ and $U_N = S^2 - (0,0,-1)$. Specifically, the maps $\Psi_S : \mathcal{P}^{-1}(U_S) \to U_S \times U(1)$ and $\Psi_N : \mathcal{P}^{-1}(U_N) \to U_N \times U(1)$ given by

$$\Psi_S(z^1, z^2) = (\mathcal{P}(z^1, z^2), z^2/|z^2|) \tag{2.3}$$

and

$$\Psi_N(z^1\,,\, z^2) = (\mathcal{P}(z^1,\, z^2)\,,\, z^1/|z^1|) \tag{2.4}$$

are local trivializations. The corresponding transition function

$$g_{SN} : U_N \cap U_S \to U(1)\,,$$

is given by

$$g_{SN}(x) = \frac{z^2/|z^2|}{z^1/|z^1|} \tag{2.5}$$

where (z^1, z^2) is any point in $\mathcal{P}^{-1}(x)$. Expressed in terms of spherical coordinates (ϕ, θ) on S^2 this becomes

$$g_{SN}(\phi, \theta) = \mathrm{e}^{-\mathrm{i}\theta}\,. \tag{2.6}$$

The complex Hopf bundle admits a natural connection given by a $u(1) = \mathrm{Im}\,\mathbb{C}$-valued 1-form ω_1 on S^3 that is the restriction to S^3 of the 1-form

$$\begin{aligned}\bar{z}^1\,\mathrm{d}z^1 + \bar{z}^2\,\mathrm{d}z^2 &= (x^1\,\mathrm{d}x^1 + x^2\,\mathrm{d}x^2 + x^3\,\mathrm{d}x^3 + x^4\,\mathrm{d}x^4) \\ &\quad + \mathrm{i}(-x^2\,\mathrm{d}x^1 + x^1\,\mathrm{d}x^2 - x^4\,\mathrm{d}x^3 + x^3\,\mathrm{d}x^4)\end{aligned} \tag{2.7}$$

on $\mathbb{C}^2 = \mathbb{R}^4$ (here $z^1 = x^1 + \mathrm{i}x^2$ and $z^2 = x^3 + \mathrm{i}x^4$). One sees easy that the real part is zero on S^3 so, suppressing the inclusion $S^3 \hookrightarrow \mathbb{C}^2$, we may write

$$\begin{aligned}\omega_1 &= \mathrm{i}\,\mathrm{Im}(\bar{z}^1\,\mathrm{d}z^1 + \bar{z}^2\,\mathrm{d}z^2) \\ &= \mathrm{i}(-x^2\,\mathrm{d}x^1 + x^1\,\mathrm{d}x^2 - x^4\,\mathrm{d}x^3 + x^3\,\mathrm{d}x^4)\,.\end{aligned} \tag{2.8}$$

A simple computation shows that the horizontal subspace $\mathrm{Hor}_p(S^3) = \ker\omega_1(p)$ at each $p \in S^3$ corresponding to ω_1 is just that part of the $\mathbb{R}^4$-orthogonal complement of the tangent space to the orbit of p that lies in $T_p(S^3)$. Moreover, if $s_S : U_S \to \mathcal{P}^{-1}(U_S)$ and $s_N : U_N \to \mathcal{P}^{-1}(U_N)$ are the cross-sections corresponding to our chosen trivializations (i. e. $s_S(x) = \Psi_S^{-1}(x,1)$ and $s_N(x) = \Psi_N^{-1}(x,1)$), then one easily verifies that

$$s_S^*\omega_1 = \frac{\mathrm{i}}{2}(1+\cos\phi)\,\mathrm{d}\theta \quad \text{on} \quad U_S \tag{2.9}$$

and,

$$s_N^*\omega_1 = -\frac{\mathrm{i}}{2}(1-\cos\phi)\,\mathrm{d}\theta \quad \text{on} \quad U_N\,. \tag{2.10}$$

These are called the gauge potentials corresponding to the connection ω_1 and the given trivializations (gauges) are nothing other than the local potentials for the Dirac monopole of minimum positive strength.

This rather remarkable coincidence is actually just the beginning of the story. Recall that the $U(1)$-bundles over S^2 are characterized up to equivalence by the first Chern class and that this class can be computed from any connection on the bundle. Given such a bundle $U(1) \hookrightarrow P \to S^2$, the integral

$$\int_{S^2} c_1(P) \tag{2.11}$$

of the first Chern class over S^2 is an integer which also characterizes the bundle. One can show [10] that, for each $n \in \mathbb{Z}$, the $U(1)$-bundle $U(1) \hookrightarrow P_n \to S^2$ with

$$\int_{S^2} c_1(P_n) = n \tag{2.12}$$

admits a connection ω_n whose gauge potentials $s_S^*\omega_n$ and $s_N^*\omega_n$ are the potential one-forms for a Dirac monopole of strength $\frac{n}{2}$.

Thus, Dirac monopoles arise in at least one unexpected context. To set the stage for another we recall that the Hopf bundle $U(1) \hookrightarrow S^3 \to \mathbb{CP}^1$ has a natural quaternionic analogue obtained by replacing the complex numbers $\mathbb{C}$ by the quaternions $\mathbb{H}$. Specifically, we think of S^7 as the subspace of $\mathbb{H}^2$ consisting of those (q^1, q^2) with $|q^1|^2 + |q^2|^2 = 1$ and define on it an action by $Sp(1)$ (the group of unit quaternions) as follows: for $(q^1, q^2) \in S^7$ and $a \in Sp(1)$, $(q^1, q^2) \,.\, a = (q^1 a, q^2 a)$. The orbits are copies of $Sp(1) \cong S^3$ and the orbit space is $\mathbb{HP}^1 \cong S^4$. The quotient map gives a principal bundle

$$Sp(1) \hookrightarrow S^7 \to \mathbb{HP}^1 \cong S^4 \tag{2.13}$$

and this bundle admits a natural connection given by the $sp(1) \cong \operatorname{Im}\mathbb{H}$-valued 1-form

$$\omega = \operatorname{Im}(\bar{q}^1\, \mathrm{d}q^1 + \bar{q}^2\, \mathrm{d}q^2) \tag{2.14}$$

(again, we suppress the restriction to $S^7 \subseteq \mathbb{H}^2$). Remarkably, pulling back this connection by a natural cross-section $s : \mathcal{U} \to \mathcal{P}^{-1}(\mathcal{U})$ on S^4 minus a point once again gives a gauge potential that arose independently in physics:

$$\mathcal{A} = s^*\omega = \operatorname{Im}\Big(\frac{\bar{q}}{1 + |q|^2}\, \mathrm{d}q\Big)\,. \tag{2.15}$$

This is the famous BPST instanton first described in [2]. Its significance for physics arises in the following way. If $\Omega = \mathrm{d}\omega + \omega \wedge \omega$ is the curvature of the connection ω, then the pullback

$$\mathcal{F} = s^*\Omega = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \tag{2.16}$$

is called the field strength of $\mathcal{A}$ in gauge s. Identifying $\mathcal{U}$ with $\mathbb{R}^4$ via a stereographic projection one can think of $\mathcal{F}$ as defined on $\mathbb{R}^4$ and then $\mathcal{F}$ is anti-self-dual, i. e. satisfies $*\mathcal{F} = -\mathcal{F}$, where $*$ is the Hodge dual determined by the standard metric and orientation of $\mathbb{R}^4$, and finite action, i. e. satisfies

$$\int_{\mathbb{R}^4} -2\,\operatorname{trace}(\mathcal{F} \wedge *\mathcal{F}) \;<\; \infty\,. \tag{2.17}$$

Such anti-self-dual potentials also satisfy the Yang–Mills equations on $\mathbb{R}^4$ and the search for finite energy solutions to these was the physical motivation in [2].

Remark: Although we have a different story to tell here it is worth mentioning briefly the role this discovery played in the emergence of Donaldson theory. The anti-self-dual equations are conformally invariant and, by a judicious choice of conformal diffeomorphisms one can produce a finite action anti-self-dual potential $\mathcal{A}_{\lambda,n}$ for each $\lambda > 0$ and $n \in \mathbb{H} \cong \mathbb{R}^4$:

$$\mathcal{A}_{\lambda,n} = \mathrm{Im}\left(\frac{\bar{q} - \bar{n}}{\lambda^2 + |q-n|^2}\,\mathrm{d}q\right) \tag{2.18}$$

(thus, $\mathcal{A}$ in (2.15) corresponds to $\lambda = 1$, $n = 0$). The field strengths $\mathcal{F}_{\lambda,n}$ all satisfy

$$\frac{1}{8\pi^2}\int_{\mathbb{R}^4} -2\,\mathrm{trace}(\mathcal{F}_{\lambda,n} \wedge *\mathcal{F}_{\lambda,n}) = 1\,. \tag{2.19}$$

This is the second Chern number of the quaternionic Hopf bundle and indeed, each $\mathcal{A}_{\lambda,n}$ is a gauge potential for a connection $\omega_{\lambda,n}$ on $Sp(1) \hookrightarrow S^7 \to S^4$. A deep theorem of Atyah, Hitchin and Singer [1] implies that, up to gauge equivalence (i. e., an automorphism of the bundle), these are the only finite action anti-self-dual connections on the Hopf bundle. Thus, the moduli space of gauge equivalence classes of such connections is $(0, \infty) \times \mathbb{R}^4$. But this is conformally equivalent to the open 5-ball B^5. Now B^5 has a natural compactification with boundary S^4 (the base manifold of the Hopf bundle). Donaldson's first application of gauge-theoretic techniques to 4-manifold topology generalized this scenario to smooth 4-manifolds other than S^4. A more detailed outline is available in [9]; for the background and proof, see [8].

3. Yang–Mills–Higgs Theory on $\mathbb{R}^3$

Our interest in the anti-self-dual equations arises in the following way. Solutions to these equations on $\mathbb{R}^4$ that have finite energy are instantons. If one gives up the finite energy requirement but seeks instead solutions that are "static" in the sense that the potential $\mathcal{A}(x^1, x^2, x^3)$ does not depend on x^0, then dimensional reduction to $\mathbb{R}^3$ gives the following reformulation of the equations. The first three components of $\mathcal{A}$ give an $SU(2)$-potential $\hat{\mathcal{A}}$ on $\mathbb{R}^3$, while the fourth component ψ of $\mathcal{A}$ can be regarded as a matter field (Higgs field) coupled to $\hat{\mathcal{A}}$ by the anti-self-dual equations, which now take the form

$$\hat{\mathcal{F}} = -*\,\mathrm{d}^{\hat{\mathcal{A}}}\psi\,. \tag{3.1}$$

Here $\hat{\mathcal{F}}$ is the field strength of $\hat{\mathcal{A}}$ and the covariant derivative on the right-hand side is given by $\mathrm{d}^{\hat{\mathcal{A}}}\psi = \mathrm{d}\psi + [\hat{\mathcal{A}}, \psi]$. Equations (3.1) are called the

Bogomolny equations and they admit the following exact solution (the so called Hooft–Polyakov–Prasad–Sommerfeld monopole):

$$\begin{aligned} \psi^1 &= \psi^2 = 0\,, \qquad \psi^3 = \coth\rho - \frac{1}{\rho} \\ \hat{\mathcal{A}}^1 &= \frac{\rho}{\sinh\rho}(\sin\theta\,\mathrm{d}\phi + \cos\theta\sin\phi\,\mathrm{d}\theta) \\ \hat{\mathcal{A}}^2 &= -\frac{\rho}{\sinh\rho}(\cos\theta\,\mathrm{d}\phi - \sin\theta\sin\phi\,\mathrm{d}\theta) \\ \hat{\mathcal{A}}^3 &= (1-\cos\phi)\,\mathrm{d}\theta \end{aligned} \tag{3.2}$$

This solution is interesting for the following reason. Note first that it is globally defined and smooth on all of $\mathbb{R}^3$ (even at $\rho = 0$ where the component functions are actually real analytic). Furthermore, seen from a distance (i. e., in the limit as $\rho \to \infty$) the Higgs field approaches a constant field

$$\psi^1 = \psi^2 = 0\,, \qquad \psi^3 \to 1\,,$$

the first two components of the potential vanish

$$\hat{\mathcal{A}}^1 \to 0\,, \qquad \hat{\mathcal{A}}^2 \to 0$$

and the third component, which does not depend on ρ, is just the Dirac monopole of strength 2:

$$\hat{\mathcal{A}}^3 = \frac{2}{2}(1-\cos\phi)\,\mathrm{d}\theta\,. \tag{3.3}$$

Unlike the situation in classical electromagnetic theory, where monopoles are singular and must be inserted by hand, the equations of $SU(2)$ Yang–Mills–Higgs theory admit a smooth solution which, at large distances, behaves like a Dirac monopole.

4. Seiberg–Witten Equations on Flat Space

It is by now well-known that the Seiberg–Witten equations have usurped the role formerly played by the anti-self-dual equations in the application of gauge-theoretic techniques to 4-manifold topology (see Donaldson [5]). We will have nothing to say about this, but will instead describe some exact solutions to these equations on flat space in which, once again, Dirac monopoles put in an unexpected appearance. We begin by writing down the equations in their general form and then explain the meaning of the symbols locally, in coordinates. The basic ingredients consist of a connection A on a $U(1)$-bundle and a 2-component spinor field ψ.

$$\not{D}_A \psi = 0 \tag{4.1}$$

$$\rho^+(F_A) = (\psi \otimes \psi^*)_0 \tag{4.2}$$

For the moment we let R denote either Euclidean 4-space $\mathbb{R}^4$ or Minkowski spacetime $\mathbb{R}^{1,3}$. $\{e_0, e_1, e_2, e_3\}$, $\{e^0, e^1, e^2, e^3\}$, $e^0 \wedge e^1 \wedge e^2 \wedge e^3$, and (x_0, x_1, x_2, x_3) will denote the standard basis, dual basis, orientation and coordinates on R. We will write $A = A_\alpha \, \mathrm{d}x^\alpha$, $A_\alpha : R \to u(1) = \mathrm{Im}\,\mathbb{C}$, $\alpha = 0, 1, 2, 3$, for a $U(1)$-potential and

$$F_A = \mathrm{d}A = \sum_{\alpha<\beta} F_{\alpha\beta} \, \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta = \sum_{\alpha<\beta} (\partial_\alpha A_\beta - \partial_\beta A_\alpha) \, \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta$$

for its curvature (here $\partial_\alpha = \dfrac{\partial}{\partial x^\alpha}$). For any map

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} : \; R \to \mathbb{C}^2$$

we write $\psi \otimes \psi^*$ for the endomorphism of $\mathbb{C}^2$ defined by

$$\psi \otimes \psi^* = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (\bar{\psi}_1 \bar{\psi}_2) = \begin{pmatrix} |\psi_1|^2 & \psi_1 \bar{\psi}_2 \\ \bar{\psi}_1 \psi_2 & |\psi_2|^2 \end{pmatrix}.$$

The trace free part of $\psi \otimes \psi^*$ is

$$(\psi \otimes \psi^*)_0 = \begin{pmatrix} \frac{1}{2}(|\psi_1|^2 - |\psi_2|^2) & \psi_1 \bar{\psi}_2 \\ \bar{\psi}_1 \psi_2 & \frac{1}{2}(|\psi_2|^2 - |\psi_1|^2) \end{pmatrix}$$

which can be written in terms of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the basis quaternions $I = \mathrm{i}\sigma_3$, $J = \mathrm{i}\sigma_2$ and $K = \mathrm{i}\sigma_1$, as

$$(\psi \otimes \psi^*)_0 = \frac{1}{2}(\psi^* \sigma_1 \psi)\sigma_1 + \frac{1}{2}(\psi^* \sigma_2 \psi)\sigma_2 + \frac{1}{2}(\psi^* \sigma_3 \psi)\sigma_3 \tag{4.3}$$

and

$$(\psi \otimes \psi^*)_0 = -\frac{1}{2}(\psi^* I \psi) I - \frac{1}{2}(\psi^* J \psi) J - \frac{1}{2}(\psi^* K \psi) K\,. \tag{4.4}$$

To proceed further (i. e., to define $\not{D}_A$ and ρ^+ in (4.1) and (4.2)) we require information about the Clifford algebras of $\mathbb{R}^4$ and $\mathbb{R}^{1,3}$. Since these are different we consider each separately. On $\mathbb{R}^4$ the standard inner product is given by

$$\langle x,\, y\rangle_4 = x^0\, y^0 + x^1\, y^1 + x^2\, y^2 + x^3\, y^3 = \delta_{\alpha\beta} x^\alpha\, y^\beta$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. We construct a convenient matrix model of this inner product space as follows: Let $\mathbb{R}^4$ be the set of all 2×2 complex matrices of the form

$$\begin{aligned} X &= \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} = \begin{pmatrix} x^0 + \mathrm{i}x^1 & x^2 + \mathrm{i}x^3 \\ -x^2 + \mathrm{i}x^3 & x^0 - \mathrm{i}x^1 \end{pmatrix} \\ &= x^0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x^1 I + x^2 J + x^3 K \,. \end{aligned}$$

Note that $\det X = \langle x,\, x\rangle_4$. Thus, defining an inner product on $\mathbb{R}^4$ via polarization from the norm $||X||^2 = \det X$ we find that $\mathbb{R}^4$ is isomorphic (as an inner product space) to $\mathbb{R}^4$. Now, for each $x \in \mathbb{R}^4$ let

$$\tilde X = \bar X^T = x^0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - x^1 I - x^2 J - x^3 K$$

and define a map

$$T : \mathbb{R}^4 \to \mathbb{C}^{4\times 4}$$

by

$$T(x) = \begin{pmatrix} 0 & X \\ -\tilde X & 0 \end{pmatrix}.$$

T is clearly linear and injective so we may identify $\mathbb{R}^4$ with the subspace $T(\mathbb{R}^4)$ of $\mathbb{C}^{4\times 4}$. A basis for this copy of $\mathbb{R}^4$ is then given by

$$\gamma_\alpha = T(e_\alpha)\,, \quad \alpha = 0,\, 1,\, 2,\, 3\,. \tag{4.5}$$

Performing the matrix multiplications shows that

$$\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = -2\delta_{\alpha\beta}\mathbb{I}\,, \qquad \alpha, \beta = 0,\, 1,\, 2,\, 3\,, \tag{4.6}$$

where $\mathbb{I}$ is the 4×4 identity matrix. The Clifford algebra $Cl(\mathbb{R}^4)$ of $\mathbb{R}^4$ is the <u>real</u> subalgebra of $\mathbb{C}^{4\times 4}$ generated by $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$. It is a simple matter to write out a basis for this algebra. Using σ_0 for the 2×2 identity matrix one such is

$$\mathbb{I} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix},$$

$$\gamma_0 = \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

$$
\begin{aligned}
\gamma_2 &= \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}, \\
\gamma_0\gamma_1 &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma_0\gamma_2 = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}, \\
\gamma_0\gamma_3 &= \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix}, \quad \gamma_1\gamma_2 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \\
\gamma_1\gamma_3 &= \begin{pmatrix} -J & 0 \\ 0 & -J \end{pmatrix}, \quad \gamma_2\gamma_3 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \\
\gamma_0\gamma_1\gamma_2 &= \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}, \quad \gamma_0\gamma_1\gamma_3 = \begin{pmatrix} 0 & -J \\ J & 0 \end{pmatrix}, \\
\gamma_0\gamma_2\gamma_3 &= \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & -\sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}, \\
\gamma_0\gamma_1\gamma_2\gamma_3 &= \begin{pmatrix} -\sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}.
\end{aligned}
\tag{4.7}
$$

The dimension of $Cl(\mathbb{R}^4)$ over $\mathbb{R}$ is therefore 16. The complexified Clifford algebra $Cl(\mathbb{R}^4) \otimes \mathbb{C}$ (same basis, but complex scalars) is therefore a subspace of $\mathbb{C}^{4\times 4}$ of complex dimension 16, i. e., it is all of $\mathbb{C}^{4\times 4}$. Viewing $\mathbb{C}^{4\times 4}$ as $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^4)$ we find that $\mathbb{R}$, $Cl(\mathbb{R}^4)$ and $Cl(\mathbb{R}^4) \otimes \mathbb{C}$ all act on $\mathbb{C}^4$ as linear transformations. This action is called Clifford multiplication and will be denoted with a dot "$\cdot$". We will write

$$\mathbb{C}^4 = W^+ \oplus W^-,$$

where W^+ is the set of elements of the form

$$\begin{pmatrix} z_1 \\ z_2 \\ 0 \\ 0 \end{pmatrix}$$

while W^- consists of those elements of the form

$$\begin{pmatrix} 0 \\ 0 \\ z_3 \\ z_4 \end{pmatrix}.$$

Note that Clifford multiplication by even elements of the Clifford algebra (i. e., those in the span of $\mathbb{I}$, $\gamma_0\gamma_1$, $\gamma_0\gamma_2$, $\gamma_0\gamma_3$, $\gamma_1\gamma_2$, $\gamma_1\gamma_3$, $\gamma_2\gamma_3$, $\gamma_0\gamma_1\gamma_2\gamma_3$ preserve $W^{\pm}$, while odd elements interchange W^+ and W^-.

A map $\Psi : \mathbb{R}^4 \to \mathbb{C}^4$ will be called a 4-component spinor field on $\mathbb{R}^4$. A positive 2-component spinor field is a map $\psi\colon \mathbb{R}^4 \to W^+$ and a negative 2-component spinor field is a map $\phi\colon \mathbb{R}^4 \to W^-$. We will generally abuse the notation slightly and write, for example,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

for

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\Psi = \begin{pmatrix} \psi \\ \phi \end{pmatrix},$$

etc. Associated with the $U(1)$-potential $A = A_\alpha \, \mathrm{d}x^\alpha$ we introduce the covariant differential

$$\nabla\Psi = (\nabla_\alpha \Psi)\, \mathrm{d}x^\alpha = (\partial_\alpha + A_\alpha)\Psi \, \mathrm{d}x^\alpha. \tag{4.8}$$

For each tangent vector v one therefore has the covariant derivative

$$\nabla_v \Psi = \left((\partial_\alpha + A_\alpha)\Psi \, \mathrm{d}x^\alpha\right)(v) = (\partial_\alpha + A_\alpha)\Psi v^\alpha \,. \tag{4.9}$$

Now we introduce what we will refer to as the "physicist's Dirac operator" on spinors by

$$\begin{aligned}\Psi \to \sum_{\alpha=0}^{3} e_\alpha \cdot \nabla_{e_\alpha}\Psi &= \sum_{\alpha=0}^{3} \gamma_\alpha \nabla_\alpha \Psi \\ &= \begin{pmatrix} \nabla_0\phi + I\nabla_1\phi + J\nabla_2\phi + K\nabla_3\phi \\ -\nabla_0\psi + I\nabla_1\psi + J\nabla_2\psi + K\nabla_3\psi \end{pmatrix}.\end{aligned}$$

Notice that this operator carries a positive spinor to a negative spinor (and vice versa) and it is this operator that interests us. We define the (mathematician's) Dirac operator $\not{D}_A$ by

$$\not{D}_A \, \psi = -\nabla_0\psi + I\nabla_1\psi + J\nabla_2\psi + K\nabla_3\psi \tag{4.10}$$

for every positive 2-component spinor field ψ. The result $\not{D}_A \, \psi$ is a negative 2-component spinor and the first Seiberg–Witten equation (4.1) requires that this be zero, i. e., that

$$\nabla_0\psi = I\nabla_1\psi + J\nabla_2\psi + K\nabla_3\psi \,. \tag{4.11}$$

Written out in detail this is

$$\begin{pmatrix} -(\partial_0 + A_0) + \mathrm{i}(\partial_1 + A_1) & (\partial_2 + A_2) + \mathrm{i}(\partial_3 + A_3) \\ -(\partial_2 + A_2) + \mathrm{i}(\partial_3 + A_3) & -(\partial_0 + A_0) - \mathrm{i}(\partial_1 + A_1) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.12}$$

To understand the second Seiberg–Witten equation (4.2) we must describe a natural action of complex-valued 2-forms on $\mathbb{C}^4$. Thus, we define

$$\rho\colon \Lambda^2(\mathbb{R}^4) \otimes \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^4)$$

by

$$\rho(F) = \rho(\sum_{\alpha<\beta} F_{\alpha\beta} e^\alpha \wedge e^\beta) = \sum_{\alpha<\beta} F_{\alpha\beta} T(e_\alpha) T(e_\beta)$$
$$= \sum_{\alpha<\beta} F_{\alpha\beta}\gamma_\alpha\gamma_\beta = \begin{pmatrix} \begin{array}{l} (F_{01}+F_{23})I \\ +(F_{02}+F_{31})J \\ +(F_{03}+F_{12})K \end{array} & 0 \\ 0 & \begin{array}{l} -(F_{01}-F_{23})I \\ -(F_{02}+F_{13})J \\ -(F_{03}-F_{12})K \end{array} \end{pmatrix}$$

Being diagonal, $\rho(F)$ preserves $W^\pm$ so we may define

$$\rho^\pm(F) = \rho(F)|W^\pm \,.$$

In particular,

$$\rho^+(F) = (F_{01}+F_{23})I + (F_{02}+F_{31})J + (F_{03}+F_{12})K \,. \tag{4.13}$$

Using (4.13) and (4.4) we write the second Seiberg–Witten equation (4.2) as

$$\begin{aligned} F_{01} + F_{23} &= -\frac{1}{2}(\psi^* I \psi) \\ F_{02} + F_{31} &= -\frac{1}{2}(\psi^* J \psi) \\ F_{03} + F_{12} &= -\frac{1}{2}(\psi^* K \psi) \end{aligned} \tag{4.14}$$

or, in still more detail,

$$\begin{aligned} (\partial_0 A_1 - \partial_1 A_0) + (\partial_2 A_3 - \partial_3 A_2) &= -\frac{\mathrm{i}}{2}(|\psi_1|^2 - |\psi_2|^2) \\ (\partial_0 A_2 - \partial_2 A_0) + (\partial_3 A_1 - \partial_1 A_3) &= -\mathrm{i}\,\mathrm{Im}(\bar{\psi}_1\psi_2) \\ (\partial_0 A_3 - \partial_3 A_0) + (\partial_1 A_2 - \partial_2 A_1) &= -\mathrm{i}\,\mathrm{Re}(\bar{\psi}_1\psi_2) \end{aligned} \tag{4.15}$$

Now that we have the Seiberg–Witten equation on $\mathbb{R}^4$ in hand we sketch the proof of a result of Witten which seems to suggest that they have no physically interesting solutions.

Theorem (Witten): *Suppose* $A \in \Lambda^1(\mathbb{R}^4, \operatorname{Im}\mathbb{C})$ *and* $\psi \in C^\infty(\mathbb{R}^4, \mathbb{C}^2)$ *satisfy (4.11) and (4.14). Then* $\psi \in L^2(\mathbb{R}^4)$ *implies* $\psi \equiv 0$*.*

Proof: (sketch) Let $\Delta = -\sum_{\alpha=0}^{3} \frac{\partial^2}{(\partial x^\alpha)^2}$ be the usual Laplacian on $\mathbb{R}^4$. A computation, using (4.11) and (4.14) and the Weitzenböck formula shows that

$$\Delta||\psi||^2 = -\sum_{\alpha=0}^{3} ||\nabla_\alpha \psi||^2 - |\psi^* I \psi|^2 - |\psi^* J \psi|^2 - |\psi^* K \psi|^2 .$$

But then $\Delta||\psi||^2 \leq 0$ on $\mathbb{R}^4$ so $||\psi||^2$ is subharmonic on $\mathbb{R}^4$ and so satisfies a mean value property on $\mathbb{R}^4$. Specifically, for any $r > 0$ and any $x \in \mathbb{R}^4$,

$$||\psi(x)||^2 \leq \frac{2}{\pi^2 r^4} \int_{\bar{B}_r(x)} ||\psi||^2,$$

where $\bar{B}_r(x)$ is the closed ball of radius r about x. Thus, $\psi \in L^2(\mathbb{R}^4)$ implies

$$||\psi(x)||^2 \leq \frac{k}{r^4}$$

for some constant k and any $r > 0$. It follows that $||\psi(x)|| = 0$ for every $x \in \mathbb{R}^4$ so $||\psi|| \equiv 0$. $\square$

To see that there are, nevertheless, physically interesting (albeit non-L^2) solutions to the Seiberg–Witten equations on flat space we must extend the equations from Euclidean $\mathbb{R}^4$ to Minkowski spacetime $\mathbb{R}^{1,3}$. The entire discussion is exactly the same on $\mathbb{R}^{1,3}$ except for the Clifford algebra, i. e., the γ-matrices used to define $\not{D}_A$ and ρ^+. We sketch the construction of $Cl(\mathbb{R}^{1,3})$.
The inner product on $\mathbb{R}^{1,3}$ is given by

$$\langle x, y \rangle_{1,3} = x^0\, y^0 - x^1\, y^1 - x^2\, y^2 - x^3\, y^3 = \eta_{\alpha\beta} x^\alpha\, y^\beta$$

where

$$\eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Let $\mathcal{R}^{1,3}$ consist of all 2×2 complex matrices X of the form

$$X = x^0 \sigma_0 + x^1 \sigma_1 + x^2 \sigma_2 + x^3 \sigma_3 = \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i} x^2 \\ x^1 + \mathrm{i} x^2 & x^0 - x^3 \end{pmatrix}.$$

Then $\det X = \langle x, x \rangle_{1,3}$ so, introducing an inner product on $\mathbb{R}^{1,3}$ via polarization from the norm $||X||^2 = \det X$ we find that $\mathcal{R}^{1,3}$ is isomorphic (as an inner product space) to $\mathbb{R}^{1,3}$. For each $X = x^\alpha \sigma_\alpha \in \mathbb{R}^{1,3}$ we let

$$\tilde{X} = x^0 \sigma_0 - x^1 \sigma_1 - x^2 \sigma_2 - x^3 \sigma_3$$

and define a map $T : \mathbb{R}^{1,3} \to \mathbb{C}^{4\times 4}$ by

$$T(x) = \begin{pmatrix} 0 & \tilde{X} \\ -\tilde{X} & 0 \end{pmatrix}.$$

Then T is linear and injective so we may identify $\mathbb{R}^{1,3}$ with the subspace $T(\mathbb{R}^{1,3})$ of $\mathbb{C}^{4\times 4}$. A basis for this copy of $\mathbb{R}^{1,3}$ is

$$\gamma_\alpha = T(e_\alpha), \quad \alpha = 0, 1, 2, 3 \tag{4.16}$$

and these satisfy

$$\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = -2\eta_{\alpha\beta}\mathbb{I}, \qquad \alpha, \beta = 0, 1, 2, 3. \tag{4.17}$$

The Clifford algebra of $\mathbb{R}^{1,3}$ is the real subalgebra $Cl(\mathbb{R}^{1,3})$ of $\mathbb{C}^{4\times 4}$ generated by these γ-matrices.

Proceeding in exactly the same way as for $\mathbb{R}^4$, but with these γ - matrices, gives the following Seiberg–Witten equations on $\mathbb{R}^{1,3}$:

$$\nabla_0 \psi = \sigma_1 \nabla_1 \psi + \sigma_2 \nabla_2 \psi + \sigma_3 \nabla_3 \psi \tag{4.18}$$

$$\begin{aligned} F_{01} + \mathrm{i}F_{23} &= \frac{1}{2}(\psi^* \sigma_1 \psi) \\ F_{02} + \mathrm{i}F_{31} &= \frac{1}{2}(\psi^* \sigma_2 \psi) \\ F_{03} + \mathrm{i}F_{12} &= \frac{1}{2}(\psi^* \sigma_3 \psi) \end{aligned} \tag{4.19}$$

Witten's Theorem is still true on $\mathbb{R}^{1,3}$ and the proof is virtually the same. Nevertheless, Peter Freund [7] has pointed out that equations (4.18) and (4.19) have the following interesting solution. To ease comparison with earlier formulas we will write $x^0 = t$, $x^1 = x$, $x^2 = y$ and $x^3 = z$. Then, on $\mathbb{R}^{1,3} - \{(t, 0, 0, z);\ -\infty < t < \infty,\ z \geq 0\}$, a solution $A = A_0\,\mathrm{d}t + A_1\,\mathrm{d}x + A_2\,\mathrm{d}y + A_3\,\mathrm{d}z$, $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ to (4.18) and (4.19) is given by

$$A_0 = A_3 = 0\,, \quad A_1 = \frac{-y\mathrm{i}}{2\rho(\rho - z)}\,, \quad A_2 = \frac{x\mathrm{i}}{2\rho(\rho - z)}$$
$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \frac{1}{\rho\sqrt{2\rho(\rho - z)}}\begin{pmatrix}x - y\mathrm{i}\\ \rho - z\end{pmatrix}\,. \tag{4.20}$$

The potential A therefore once again represents a Dirac monopole of minimal positive strength.

Remark: Equation (4.18) is the so-called Weyl–Dirac equation so the spinor field Ψ can be thought of as a massless spin $\frac{1}{2}$ field coupled to the Dirac monopole A. Moreover,

$$\frac{1}{2}(\psi^*\sigma_1\psi) = \mathrm{Re}(\bar{\psi}_1\psi_2) = \frac{1}{2}\frac{x}{\rho^3}$$
$$\frac{1}{2}(\psi^*\sigma_2\psi) = \mathrm{Im}(\bar{\psi}_1\psi_2) = \frac{1}{2}\frac{y}{\rho^3}$$
$$\frac{1}{2}(\psi^*\sigma_3\psi) = \frac{1}{2}(|\psi_1|^2 - |\psi_2|^2) = \frac{1}{2}\frac{z}{\rho^3}$$

so the curvature equations essentially say that ψ determines a Coulomb field.

5. $SU(2)$ Generalization of the Seiberg–Witten Equations on $\mathbb{R}^{1,3}$

The Seiberg–Witten equations admit natural generalizations to other gauge groups. We will briefly describe the generalization for $SU(2)$ in order to write down some recently discovered monopole solutions. As a basis for the Lie algebra $su(2)$ of $SU(2)$ we take $\{T_1, T_2, T_3\}$, where, $T_a = \frac{\mathrm{i}}{2}\sigma_a\,,\ a = 1, 2, 3.$ Then the structure constants are the Levi-Civita symbols:

$$[T_a\,,\,T_b] = \sum_{a=1}^{3}\varepsilon_{abc}T_c\,. \tag{5.1}$$

A gauge potential on $\mathbb{R}^{1,3}$ can then be written

$$A = A^aT_a = A_\alpha\,\mathrm{d}x^\alpha = (A^a_\alpha T_a)\,\mathrm{d}x^\alpha$$

and the corresponding field strength is

$$F_A = \mathrm{d}A + \frac{1}{2}A\wedge A = F^aT_a = \frac{1}{2}F_{\alpha\beta}\,\mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\,,$$

where

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + \frac{1}{2}[A_\alpha\, ,\, A_\beta], \quad \alpha, \beta = 0, 1, 2, 3\,.$$

A 4-component spinor field takes the form

$$\Psi = \Psi^a T_a = \begin{pmatrix} \psi^a \\ \phi^a \end{pmatrix} T_a$$

and its covariant differential is

$$\nabla\Psi = \mathrm{d}\Psi + \frac{1}{2}[A\, ,\, \Psi]\,.$$

Thus, for example,

$$\begin{aligned}
\nabla_{e_\alpha}\Psi &= (\nabla_{e_\alpha}\Psi)^1 T_1 + (\nabla_{e_\alpha}\Psi)^2 T_2 + (\nabla_{e_\alpha}\Psi)^3 T_3 \\
&= (\partial_\alpha \Psi^1 - \frac{1}{2}A^2_\alpha \Psi^3 + \frac{1}{2}A^3_\alpha \Psi^2)T_1 \\
&\quad + (\partial_\alpha \Psi^2 - \frac{1}{2}A^3_\alpha \Psi^1 + \frac{1}{2}A^1_\alpha \Psi^3)T_2 \\
&\quad + (\partial_\alpha \Psi^3 - \frac{1}{2}A^1_\alpha \Psi^2 + \frac{1}{2}A^2_\alpha \Psi^1)T_3\,.
\end{aligned}$$

From this we build the physicist's Dirac operator:

$$\begin{aligned}
\Psi = \begin{pmatrix} \psi \\ \phi \end{pmatrix} \to \sum_{\alpha=0}^{3} e_\alpha . \nabla_{e_\alpha}\Psi &= (\sum_{\alpha=0}^{3} \gamma_\alpha (\nabla_{e_\alpha}\Psi)^a)T_a \\
&= \begin{pmatrix} (\not{D}_A^* \phi)^a \\ (\not{D}_A \psi)^a \end{pmatrix} T_a \\
&= \begin{pmatrix} \not{D}_A^* \phi \\ \not{D}_A \psi \end{pmatrix}.
\end{aligned}$$

The first (generalized) Seiberg–Witten equation is

$$\not{D}_A \Psi = 0\,, \tag{5.2}$$

i. e.,

$$(\not{D}_A \Psi)^1 = (\not{D}_A \Psi)^2 = (\not{D}_A \Psi)^3 = 0\,. \tag{5.3}$$

As an illustration we write out $(\not{D}_A \Psi)^1 = 0$ explicitly,

$$\begin{aligned}
&-\sigma_0(\partial_0 \Psi^1 - \frac{1}{2}A^2_0 \Psi^3 + \frac{1}{2}A^3_0 \Psi^2) + \sigma_1(\partial_1 \Psi^1 - \frac{1}{2}A^2_1 \Psi^3 + \frac{1}{2}A^3_1 \Psi^2) \\
&+\sigma_2(\partial_2 \Psi^1 - \frac{1}{2}A^2_2 \Psi^3 + \frac{1}{2}A^3_2 \Psi^2) + \sigma_3(\partial_3 \Psi^1 - \frac{1}{2}A^2_3 \Psi^3 + \frac{1}{2}A^3_3 \Psi^2) = 0.
\end{aligned} \tag{5.4}$$

The map ρ^+ is defined componentwise in the Lie algebra, i. e., if $F = F^a T_a$ is an $su(2)$-valued 2-form on $\mathbb{R}^{1,3}$ we take

$$\rho^+(F) = \rho^+(F^a) T_a \,. \tag{5.5}$$

The natural definition of $\Psi \otimes \Psi^*$ is

$$\Psi \otimes \Psi^* = (\Psi^a T_a) \otimes ((\Psi^b)^* T_b) = \Psi^a (\Psi^b)^* [T_a \,,\, T_b], \tag{5.6}$$

and this, with (5.1), gives

$$\begin{aligned} \Psi \otimes \Psi^* = {} & (\Psi^2(\Psi^3)^* - \Psi^3(\Psi^2)^*)\, T_1 + (\Psi^3(\Psi^1)^* \\ & - \Psi^1(\Psi^3)^*)\, T_2 + (\Psi^1(\Psi^2)^* - \Psi^2(\Psi^1)^*)\, T_3 \,. \end{aligned} \tag{5.7}$$

One computes the tracefree part componentwise so

$$\begin{aligned} (\Psi \otimes \Psi^*)_0 &= (\Psi^2(\Psi^3)^* - \Psi^3(\Psi^2)^*)_0 T_1 + \cdots \\ &= \{((\Psi^3)^* \sigma_1 \Psi^2 - (\Psi^2)^* \sigma_1 \Psi^3)\sigma_1 \\ &\qquad + ((\Psi^3)^* \sigma_2 \Psi^2 - (\Psi^2)^* \sigma_2 \Psi^3)\sigma_2 \\ &\qquad + ((\Psi^3)^* \sigma_3 \Psi^2 - (\Psi^2)^* \sigma_3 \Psi^3)\sigma_3\} T_1 + \cdots \end{aligned} \tag{5.8}$$

The second Seiberg–Witten equation is the formally identical to the $U(1)$ equation $\rho^+(F_A) = (\Psi \otimes \Psi^*)_0$, but is, of course, rather more complicated. We will write out explicitly both Seiberg–Witten equations in the very special case of interest to us, i. e., $A^1 = A^2 = 0$, $A^3_0 = A^3_0 = 0$, $\Psi^3 = 0$, and both A and Ψ independent of x^0:

$$\begin{aligned} \partial_1 \Psi^1_2 + \frac{1}{2} A^3_1 \Psi^2_2 - \mathrm{i}(\partial_2 \Psi^1_2 + \frac{1}{2} A^3_2 \Psi^2_2) + \partial_3 \Psi^1_1 = 0 \\ \partial_1 \Psi^1_1 + \frac{1}{2} A^3_1 \Psi^2_1 + \mathrm{i}(\partial_2 \Psi^1_1 + \frac{1}{2} A^3_2 \Psi^2_1) - \partial_3 \Psi^1_2 = 0 \\ \partial_1 \Psi^2_2 - \frac{1}{2} A^3_1 \Psi^1_2 - \mathrm{i}(\partial_2 \Psi^2_2 - \frac{1}{2} A^3_2 \Psi^1_2) + \partial_3 \Psi^2_1 = 0 \\ \partial_1 \Psi^1_2 - \frac{1}{2} A^3_1 \Psi^1_1 + \mathrm{i}(\partial_2 \Psi^2_1 - \frac{1}{2} A^3_2 \Psi^1_1) - \partial_3 \Psi^2_2 = 0 \end{aligned} \tag{5.9}$$

$$\begin{aligned} -\partial_3 A^3_2 &= \mathrm{Im}(\bar\Psi^2_1 \Psi^1_2 + \bar\Psi^2_2 \Psi^1_1) \\ \partial_3 A^3_1 &= \mathrm{Re}(\bar\Psi^2_2 \Psi^1_1 - \bar\Psi^2_1 \Psi^1_2) \\ \partial_1 A^3_2 - \partial_2 A^3_1 &= \mathrm{Im}(\bar\Psi^2_1 \Psi^1_1 - \bar\Psi^2_2 \Psi^1_2) \end{aligned} \tag{5.10}$$

Dereli and Tekmen [6] found a solution to these equations which we record below (reverting again to $x^1 = x$, $x^2 = y$ and $x^3 = z$):

$$A^1 = A^2 = 0 \qquad A^3 = -(1+\cos\phi)\,\mathrm{d}\theta = \frac{1}{\rho(\rho - z)}(y\,\mathrm{d}x - x\,\mathrm{d}y)$$

$$\Psi^1 = \frac{1}{\sqrt{2}}\frac{1}{\rho}(\xi + \eta), \quad \Psi^2 = \frac{1}{\sqrt{2}}\frac{1}{\rho}(-\xi + \eta), \qquad \Psi^3 = 0$$

$$\xi = \frac{1}{\sqrt{2\rho(\rho - z)}}\begin{pmatrix} x - y\mathrm{i} \\ \rho - z \end{pmatrix} \tag{5.11}$$

$$\eta = \frac{1}{\sqrt{2\rho(\rho - z)}}\begin{pmatrix} \rho - z \\ -(x + y\mathrm{i}) \end{pmatrix}$$

Thus,

$$A = \left(\frac{-y\mathrm{i}}{\rho(\rho - z)}\,\mathrm{d}x + \frac{x\mathrm{i}}{\rho(\rho - z)}\,\mathrm{d}y\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.12}$$

and again we find ourselves face-to face with a monopole.

There are analogous generalizations of the Seiberg–Witten equations for any $SU(n)$ and all of these admit such Abelian Dirac monopole solutions. The $SU(2)$ Seiberg–Witten equations also admit a non-Abelian monopole solution (without string singularities) that can be obtained from the Abelian solution by a singular gauge transformation:

$$\begin{aligned} A^1 &= \frac{1}{\rho^2}\,(z\,\mathrm{d}y - y\,\mathrm{d}z) \\ A^2 &= \frac{1}{\rho^2}\,(x\,\mathrm{d}z - z\,\mathrm{d}x) \\ A^3 &= \frac{1}{\rho^2}\,(y\,\mathrm{d}x - x\,\mathrm{d}y) \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} \Psi^1 &= \frac{\sqrt{3}}{2\rho}(a\xi + \bar{a}\eta) \\ \Psi^2 &= \frac{\sqrt{3}}{2\rho}(b\xi + \bar{b}\eta) \\ \Psi^3 &= \frac{\sqrt{3}}{2\rho}(c\xi + \bar{c}\eta) \end{aligned} \tag{5.14}$$

where

$$
\begin{aligned}
a &= \frac{\rho - z}{2\rho} - \frac{(x + y\mathrm{i})^2}{2\rho(\rho - z)} \\
b &= \mathrm{i}\,\frac{\rho - z}{2\rho} + \mathrm{i}\,\frac{(x + y\mathrm{i})^2}{2\rho(\rho - z)} \\
c &= \frac{x + y\mathrm{i}}{\rho}\,.
\end{aligned}
\tag{5.15}
$$

For more details we refer to Dereli and Tekmen [6].

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