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CONSTANT CURVATURE SPACELIKE HYPERSURFACES IN THE LORENTZ–MINKOWSKI SPACE

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Abstract. In this paper we will report on some of our recent results about compact spacelike hypersurfaces with spherical boundary in the Lorentz–Minkowski space $\mathbb{L}^{n+1}$. In particular we will prove that the only compact spacelike hypersurfaces in $\mathbb{L}^{n+1}$ with constant mean curvature and spherical boundary are the hyperplanar balls and the hyperbolic caps. As for the case of the scalar curvature, we will prove that the only compact spacelike hypersurfaces in $\mathbb{L}^{n+1}$ with nonzero constant scalar curvature and spherical boundary are the hyperbolic caps. Our approach is based on the use of several integral formulas, among them there are a volume formula and a flux formula.

1. Introduction and Statement of the Main Results

The study of spacelike hypersurfaces in the Lorentz–Minkowski space has been of increasing interest in recent years from both physical and mathematical points of view. From a physical point of view, that interest is motivated by the role that spacelike hypersurfaces in Lorentzian spacetimes play in different problems of general relativity. For instance, Lichnerowicz [10] showed that maximal hypersurfaces (that is, zero mean curvature spacelike hypersurfaces) are convenient as initial data for solving the Cauchy problem of the Einstein equations. Other reasons justifying their importance in general relativity can be found in [4, 7, 11] and [12], and references therein.

On the other hand, their mathematical interest is also motivated by the fact that spacelike hypersurfaces in the Lorentz–Minkowski space exhibit nice Bernstein-type properties. Let us recall that the Bernstein problem for maximal hypersurfaces in the Lorentz–Minkowski space $\mathbb{L}^{n+1}$ was introduced by Calabi [5], who proposed the study of the maximal hypersurface equation in

$\mathbb{L}^{n+1}$. For $n \leq 4$, Calabi found that the only entire solutions to that equation are affine functions. Later on, Cheng and Yau [6] extended this result to the general n-dimensional case and gave its parametric version, proving that the only complete maximal hypersurfaces in $\mathbb{L}^{n+1}$ are the spacelike hyperplanes. As for the case of nonzero constant mean curvature, some other authors such us Treibergs [13], Hano and Nomizu [8], and Ishihara and Hara [9] constructed many nonlinear examples of complete spacelike hypersurfaces with nonzero constant mean curvature.

In this paper we will report on some of our recent results about the study of compact spacelike hypersurfaces with spherical boundary in $\mathbb{L}^{n+1}$. In particular we will prove the following Bernstein-type result (Theorem in [2]).

Theorem 1. *The only compact spacelike hypersurfaces in the Lorentz–Minkowski space with constant mean curvature and spherical boundary are the hyperplanar balls and the hyperbolic caps.*

The corresponding problem for hypersurfaces in Euclidean space concerning hyperplanar balls and hyperspherical caps remains open even in the two-dimensional case. Some partial results have recently been obtained by different authors, but it is still unkown if planar discs and spherical caps are the only embedded compact surfaces with circular boundary. The first named author jointly with López and Palmer [1] recently showed that the only stable constant mean curvature discs which are bounded by a circle are flat discs and spherical caps.

As for the case of the scalar curvature, we will prove the following uniqueness result (Theorem in [3]).

Theorem 2. *The only compact spacelike hypersurfaces in the Lorentz–Minkowski space with nonzero constant scalar curvature and spherical boundary are the hyperbolic caps.*

Our approach is based on the use of several integral formulas, among them we would like to emphasize a volume formula (Lemma 5) and a flux formula (Lemma 7). We also derive some further consequences for the general case where the boundary is an $(n-1)$-dimensional hyperplanar closed submanifold embedded in $\mathbb{L}^{n+1}$.

2. Preliminaries

Let $\mathbb{L}^{n+1}$ denote the $(n+1)$-dimensional Lorentz–Minkowski space, that is, the real vector space $\mathbb{R}^{n+1}$ endowed with the Lorentzian metric

$$\langle =, (\rangle dx_1)^2 + \cdots + (dx_n)^2 - (dx_{n+1})^2 \,,$$

where $(x_1, \ldots, x_{n+1})$ are the canonical coordinates in $\mathbb{R}^{n+1}$. A smooth immersion $\psi\colon M \to \mathbb{L}^{n+1}$ of an n-dimensional connected manifold M is said to be a *spacelike hypersurface* if the induced metric via ψ is a Riemannian metric on M, which, as usual, is also denoted by $\langle , \rangle$.

Let us first remark that every spacelike hypersurface in $\mathbb{L}^{n+1}$ is orientable. Actually, observe that $(0, \ldots, 0, 1)$ is a unit timelike vector field globally defined on $\mathbb{L}^{n+1}$, which determines a time-orientation on $\mathbb{L}^{n+1}$. Thus, there exists a unique timelike unit normal field N on M which is future-directed in $\mathbb{L}^{n+1}$, and hence we may assume that M is oriented by N. This future-directed normal field N can be regarded as a map $N\colon M \to \mathbb{H}^n$, where $\mathbb{H}^n$ denotes the n-dimensional hyperbolic space, that is

$$\mathbb{H}^n = \{x \in \mathbb{L}^{n+1}\colon \langle x, x\rangle = -1, \quad x_{n+1} \geq 1\}.$$

We will refer to N as the *future-directed Gauss map* of the hypersurface M. The image $N(M) \subset \mathbb{H}^n$ will be called the *hyperbolic image* of M.

A second basic remark about spacelike hypersurfaces in $\mathbb{L}^{n+1}$ is that they cannot be closed (compact without boundary). In other words, every compact spacelike hypersurface M in $\mathbb{L}^{n+1}$ necessarily has nonempty boundary. This follows easily from the fact that if $\vec{a} \in \mathbb{L}^{n+1}$ is a fixed arbitrary vector, then the gradient of the height function $\langle \vec{a}, \psi\rangle$ on M satisfies

$$\nabla\langle \vec{a}, \psi\rangle = \vec{a}^{\top} = \vec{a} + \langle \vec{a}, N\rangle N,$$

where $\vec{a}^{\top} \in \mathcal{X}(M)$ is tangent to M, and

$$|\nabla\langle \vec{a}, \psi\rangle|^2 = \langle \vec{a}, \vec{a}\rangle + \langle \vec{a}, N\rangle^2 \geq \langle \vec{a}, \vec{a}\rangle.$$

In particular, if $\vec{a}$ is chosen to be spacelike then the corresponding height function has no critical points in M, so that M necessarily has nonempty boundary ∂M. As usual, if Σ is an $(n-1)$-dimensional closed submanifold in $\mathbb{L}^{n+1}$, a spacelike hypersurface $\psi\colon M \to \mathbb{L}^{n+1}$ is said to be a *hypersurface with boundary* Σ if the restriction of ψ to ∂M is a diffeomorphism onto Σ.

The orientation of M induces a natural orientation on ∂M as follows: a basis $\{v_1, \ldots, v_{n-1}\}$ for $T_p(\partial M)$ is positively oriented if and only if $\{u, v_1, \ldots, v_{n-1}\}$ is a positively oriented basis for T_pM, whenever $u \in T_pM$ is *outward pointing*. We will denote by ν the *outward pointing unit conormal vector* along ∂M.

In what follows, we will assume that $\psi\colon M \to \mathbb{L}^{n+1}$ is a compact spacelike hypersurface bounded by an $(n-1)$-dimensional closed submanifold $\Sigma = \psi(\partial M)$, and we will assume that Σ is contained in a hyperplane Π of $\mathbb{L}^{n+1}$. We will refer to it saying that M has *hyperplanar boundary*. In that case, it is not difficult to see that the hyperplane Π is necessarily spacelike. We may assume without loss of generality that Π passes through the origin and $\Pi = \vec{a}^{\perp}$, for a unit future-directed timelike vector $\vec{a} \in \mathbb{L}^{n+1}$.

3. Basic Formulas and Examples

In order to set up the notation to be used later, let us denote by ∇^{o} the flat Levi-Civita connection of $\mathbb{L}^{n+1}$ and by ∇ the Levi-Civita connection of M. Then the Gauss and Weingarten formulas for M in $\mathbb{L}^{n+1}$ are written respectively as

$$\nabla^{\mathrm{o}}_X Y = \nabla_X Y - \langle AX, Y\rangle N\,,$$

and

$$A(X) = -\nabla^{\mathrm{o}}_X N\,,$$

for all tangent vector fields $X, Y \in \mathcal{X}(M)$. Here $A : \mathcal{X}(M) \to \mathcal{X}(M)$ defines the *shape operator* of M with respect to the future-directed Gauss map N. Associated to the shape operator of M there is the *mean curvature* of the hypersurface, which is the main extrinsic curvature of M, and it is given by

$$H = -\frac{1}{n}\,\mathrm{trace}(A) = -\frac{1}{n}\sum_{i=1}^{n}\kappa_i\,,$$

where $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of the hypersurface. On the other hand, the (intrinsic) *Ricci curvature* of M is written in terms of the shape operator A as follows

$$\mathrm{Ric}(X, Y) = nH\langle AX, Y\rangle + \langle AX, AY\rangle\,,$$

so that the *scalar curvature* S of the hypersurface is given by

$$S = \mathrm{trace}(\mathrm{Ric}) = \mathrm{trace}(A^2) - n^2H^2 = \sum_{i=1}^{n}\kappa_i^2 - \left(\sum_{i=1}^{n}\kappa_i\right)^2. \qquad (1)$$

Before proving the main results, we will briefly study two particularly simple and illustrative examples of compact spacelike hypersurfaces in $\mathbb{L}^{n+1}$ with spherical boundary.

Example 3. (Hyperplanar balls) *Let $\vec{a} \in \mathbb{L}^{n+1}$ be a unit future-directed timelike vector. For each $r > 0$, let $\mathbb{B}_r$ be the hyperplanar ball given by*

$$\mathbb{B}_r = \{x \in \mathbb{L}^{n+1}\,;\ \langle \vec{a}, x\rangle = 0\,,\ \langle x, x\rangle \le r^2\}\,,$$

which is a compact embedded spacelike hypersurface in $\mathbb{L}^{n+1}$ bounded by an $(n-1)$-sphere of radius r,

$$\partial\mathbb{B}_r = \{x \in \mathbb{B}_r\,;\ \langle x, x\rangle = r^2\} = \mathbb{S}^{n-1}(r)\,.$$

The future-directed Gauss map of $\mathbb{B}_r$ *is the constant map* $N(p) = \vec{a}$, $p \in \mathbb{B}_r$, *so that its hyperbolic image reduces to the single point* $\vec{a}$, *and* $\mathbb{B}_r$ *has zero mean curvature* $H = 0$ *and zero scalar curvature* $S = 0$.

Example 4. (Hyperbolic caps) *Let* $\vec{a} \in \mathbb{L}^{n+1}$ *be a unit future-directed timelike vector. For each* $r, R > 0$, *let us consider* $\mathbb{H}_{r,R}$ *the hyperbolic cap given by*

$$\mathbb{H}_{r,R} = \{x \in \mathbb{L}^{n+1}\,;\, \langle x, x\rangle = -R^2, -\sqrt{r^2 + R^2} \leq \langle \vec{a}, x\rangle \leq -R\}\,,$$

which is a compact embedded spacelike hypersurface in $\mathbb{L}^{n+1}$ *bounded by an* $(n-1)$*-sphere of radius* r,

$$\partial\mathbb{H}_{r,R} = \{x \in \mathbb{H}_{r,R}\,;\, \langle \vec{a}, x\rangle = -\sqrt{r^2 + R^2}\} = \mathbb{S}^{n-1}(r)\,.$$

Is future-directed Gauss map is given by $N(p) = -p/R$, $p \in \mathbb{H}_{r,R}$, *and its hyperbolic image is the geodesic ball in* $\mathbb{H}^n$ *centered at* $\vec{a}$ *of radius* $\operatorname{arccosh}\sqrt{1 + r^2/R^2}$. $\mathbb{H}_{r,R}$ *is an example of a spacelike hypersurface with nonzero constant mean curvature* $H = 1/R$ *and negative constant scalar curvature* $S = -n(n-1)/R$.

4. An Integral Formula

In this section we will state an integral formula for the n-dimensional volume of Ω, the domain in Π bounded by Σ. We refer the reader to Lemma 1 in [3] for a detailed proof.

Lemma 5. (Volume formula) *Let* $\psi\colon M \to \mathbb{L}^{n+1}$ *be a compact spacelike hypersurface with hyperplanar boundary* $\Sigma = \psi(\partial M)$, *and assume that* Σ *is contained in a hyperplane* $\Pi = \vec{a}^{\perp}$, $\vec{a}$ *being the unit future-directed timelike vector normal to* Π. *Then the* n*-dimensional volume of* Ω, *the domain in* Π *bounded by* Σ, *is given by*

$$\operatorname{vol}(\Omega) = -\int_M \langle \vec{a}, N\rangle \mathrm{d}v\,, \tag{2}$$

where $\mathrm{d}v$ *is the* n*-dimensional volume element of* M *with respect to the induced metric and the chosen orientation.*

The integral formula (2) will be essential for deriving our main results. Moreover, a first interesting application of this formula is the following result on the volume of the hypersurface M (Theorem 2 in [3]).

Proposition 6. *Let $\psi\colon M \to \mathbb{L}^{n+1}$ be a compact spacelike hypersurface with hyperplanar boundary $\Sigma = \psi(\partial M)$, and assume that Σ is contained in a hyperplane Π of $\mathbb{L}^{n+1}$. Let $\vec{a} \in \mathbb{H}^n$ be the unit future-directed timelike vector in $\mathbb{L}^{n+1}$ such that $\Pi = \vec{a}^{\perp}$. Then the n-dimensional volume of M satisfies the following inequalities,*

$$\frac{\mathrm{vol}(\Omega)}{\cosh(\varrho)} \leq \mathrm{vol}(M) \leq \mathrm{vol}(\Omega), \tag{3}$$

where Ω is the n-dimensional domain in Π bounded by Σ and $\varrho \geq 0$ is the radius of a geodesic ball in $\mathbb{H}^n$ centered at $\vec{a}$ and containing the hyperbolic image of M. Moreover, each equality holds if and only if M is the hyperplanar domain Ω.

Hyperbolic caps in example 4 show that inequalities (3) are sharp. Actually, choosing $r = 1$ the hyperbolic cap $\mathbb{H}_{1,R}$ is bounded by an $(n-1)$-sphere of radius one and its volume is given by

$$\mathrm{vol}(\mathbb{H}_{1,R}) = \omega_{n-1} R^n \int_0^{\arctan(1/R)} \frac{\sin^{n-1}(\theta)}{\cos^n(\theta)}\,\mathrm{d}\theta\,,$$

where ω_{n-1} is the $(n-1)$-dimensional volume of $\mathbb{S}^{n-1}$. From here it follows that the volume of $\mathbb{H}_{1,R}$ is an increasing function on R with

$$\lim_{R\to 0} \mathrm{vol}(\mathbb{H}_{1,R}) = 0 \tag{4}$$

and

$$\lim_{R\to\infty} \mathrm{vol}(\mathbb{H}_{1,R}) = \frac{\omega_{n-1}}{n} = \mathrm{vol}(\mathbb{B}_1)\,, \tag{5}$$

where $\mathbb{B}_1 = \Omega$ is the unitary n-dimensional round ball.

Equation (5) means that the second inequality in (3) is sharp. On the other hand, equation (4) implies that it is not possible to find a positive lower bound for the volume of all compact spacelike hypersurfaces with a fixed boundary. This justifies the necessity of including some additional geometric quantity (in our case, the radius of the geodesic ball containing the hyperbolic image of the hypersurface) in order to find an appropriate lower estimation for the volume of the hypersurfaces. Moreover, the first inequality in (3) is also sharp because

$$\lim_{R\to\infty} \cosh(\varrho_R)\mathrm{vol}(\mathbb{H}_{1,R}) = \mathrm{vol}(\mathbb{B}_1),$$

where $\varrho_R = \mathrm{arccosh}\sqrt{1 + 1/R^2}$ is the radius of the geodesic ball in $\mathbb{H}^n$ centered at $\vec{a}$ and containing the hyperbolic image of $\mathbb{H}_{1,R}$.

5. Proof of Theorem 1

The proof of our first main result is based on the two following results, whose proofs can be found in [2] (Proposition 2 and Proposition 3 in [2]).

Lemma 7. (Flux formula) *Let $\psi\colon M \to \mathbb{L}^{n+1}$ be a compact spacelike hypersurface with hyperplanar boundary $\Sigma = \psi(\partial M)$, and assume that Σ is contained in a hyperplane Π of $\mathbb{L}^{n+1}$. Let $\vec{a}$ be the unit future-directed timelike vector in $\mathbb{L}^{n+1}$ such that $\Pi = \vec{a}^{\perp}$. If the mean curvature H is constant, then the flux is given by*

$$\oint_{\partial M} \langle \nu, \vec{a} \rangle \mathrm{d}s = -nH \ \mathrm{vol}(\Omega), \tag{6}$$

where ν is the outward pointing unit conormal and $\mathrm{d}s$ is the induced $(n-1)$-dimensional volume element on ∂M. In particular, the flux does not depend on the hypersurface, but only on the value of H and Σ.

Proposition 8. (An integral inequality) *Let $\psi\colon M \to \mathbb{L}^{n+1}$ be a compact spacelike hypersurface with hyperplanar boundary $\Sigma = \psi(\partial M)$, and assume that Σ is contained in a hyperplane Π of $\mathbb{L}^{n+1}$. Let $\vec{a}$ be the unit future-directed timelike vector in $\mathbb{L}^{n+1}$ such that $\Pi = \vec{a}^{\perp}$. If the mean curvature H is constant, then*

$$-\oint_{\partial M} H_{\Sigma} \langle \nu, \vec{a} \rangle^2 \mathrm{d}s \le nH^2 \, \mathrm{vol}(\Omega)\,, \tag{7}$$

where H_{Σ} stands for the mean curvature of Σ in Π with respect to the outward pointing unitary normal. Moreover, the equality holds if and only if M is totally umbilical.

Proof: Since the boundary $\Sigma = \psi(\partial M)$ is a round sphere $\mathbb{S}^{n-1}(r)$ of radius $r > 0$, then $H_{\Sigma} = -1/r$, and $\mathrm{vol}(\Omega) = r\omega_{n-1}(r)/n$, where $\omega_{n-1}(r)$ stands for the $(n-1)$-dimensional volume of $\mathbb{S}^{n-1}(r)$. From inequality (7) we know that

$$\oint_{\partial M} \langle \nu, \vec{a} \rangle^2 \mathrm{d}s \le H^2 r^2 \omega_{n-1}(r)\,, \tag{8}$$

with equality if and only if M is either a hyperplanar ball or a hyperbolic cap. On the other hand, from flux formula (6) we also know that

$$\oint_{\partial M} \langle \nu, \vec{a} \rangle \mathrm{d}s = -Hr\omega_{n-1}(r)\,,$$

and the Cauchy–Schwarz inequality yields

$$\oint_{\partial M} \langle \nu, \vec{a} \rangle^2 \mathrm{d}s \geq \frac{\left(\oint_{\partial M} \langle \nu, \vec{a} \rangle \mathrm{d}s\right)^2}{\oint_{\partial M} \mathrm{d}s} = H^2 r^2 \omega_{n-1}(r)\,.$$

Therefore, we have the equality in (8) and M must be either a hyperplanar ball (when $H = 0$) or a hyperbolic cap (when $H \neq 0$).

6. Proof of Theorem 2

The proof of our second main result is based on the following integral formula, whose proof can be found in [3] (Proposition 4 in [3]).

Proposition 9. (An integral formula) *Let $\psi\colon M \to \mathbb{L}^{n+1}$ be a compact spacelike hypersurface with hyperplanar boundary $\Sigma = \psi(\partial M)$, and assume that Σ is contained in a hyperplane Π of $\mathbb{L}^{n+1}$. Let $\vec{a}$ be the unit future-directed timelike vector in $\mathbb{L}^{n+1}$ such that $\Pi = \vec{a}^{\perp}$. If the scalar curvature S is constant, then*

$$(n-1) \oint_{\partial M} H_{\Sigma} \langle \nu, \vec{a} \rangle^2 \mathrm{d}s = S \;\; \mathrm{vol}(\Omega)\,, \tag{9}$$

where H_{Σ} stands for the mean curvature of Σ in Π with respect to the outward pointing unitary normal.

Proof: Let us assume that the boundary $\Sigma = \psi(\partial M)$ is a round sphere $\mathbb{S}^{n-1}(r)$ of radius $r > 0$. In that case, $H_{\Sigma} = -1/r$, $\mathrm{v}ol(\Omega) = r\omega_{n-1}(r)/n$, and the integral formula (9) becomes

$$\oint_{\partial M} \langle \nu, \vec{a} \rangle^2 \mathrm{d}s = \frac{-S}{n(n-1)} r^2 \omega_{n-1}(r)\,.$$

In particular, $S < 0$ and by the Cauchy–Schwarz inequality we obtain that

$$\left| \oint_{\partial M} \langle \nu, \vec{a} \rangle \mathrm{d}s \right| \leq \frac{\sqrt{-S}}{\sqrt{n(n-1)}} r\omega_{n-1}(r)\,. \tag{10}$$

On the other hand, from equation (1) we know that

$$S = \sum_{i=1}^{n} \kappa_i^2 - \left(\sum_{i=1}^{n} \kappa_i \right)^2 = \mathrm{trace}(A^2) - [\mathrm{trace}(A)]^2\,. \tag{11}$$

The Cauchy–Schwarz inequality applied to the vectors $(\kappa_1, \ldots, \kappa_n)$ and $(1, \ldots, 1) \in \mathbb{R}^n$ yields

$$\mathrm{trace}(A^2) - \frac{1}{n}\left[\mathrm{trace}(A)\right]^2 = \sum_{i=1}^{n} \kappa_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} \kappa_i\right)^2 \geq 0\,,$$

the equality holding only at umbilical points, which jointly with (11) implies that

$$H^2 \geq \frac{-S}{n(n-1)} > 0\,,$$

the equality holding only at umbilical points. In particular, the mean curvature H does not vanish on M and

$$n|H|(-\langle \vec{a}, N\rangle) \geq \frac{n\sqrt{-S}}{\sqrt{n(n-1)}}(-\langle \vec{a}, N\rangle)\,, \tag{12}$$

with equality if and only if M is a hyperbolic cap. Integrating (12) on M, and using our volume formula in Lemma 5, we have that

$$\left|\oint_{\partial M} \langle \nu, \vec{a}\rangle \mathrm{d}s\right| = n \int_M |H|(-\langle \vec{a}, N\rangle)\mathrm{d}v \geq \frac{\sqrt{-S}}{\sqrt{n(n-1)}} r\omega_{n-1}(r)\,,$$

with equality if and only if M is a hyperbolic cap. Therefore, by equation (10) we have the equality above and the result.

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