Geometry, Integrability and Quantization
September 1–10, 1999, Varna, Bulgaria
Ivaïlo M. Mladenov and Gregory L. Naber, Editors
Coral Press, Sofia 2000, pp 163-174

DEFORMATIONS OF MINIMAL SURFACES

IVAÏLO M. MLADENOV and BORISLAV ANGELOV

Institute of Biophysics, Bulgarian Academy of Sciences,
Acad. G. Bonchev Street, Block 21, 1113 Sofia, Bulgaria

Abstract. Here we combine group-theoretical and differential-geometric techniques for considerations of minimal surface deformations in the ordinary three-dimensional space. This approach allows a consideration of a novel family of transformations generated by complex rotations. The resulting generalized deformations are compared with the well-known Bonnet and Goursat transformations and illustrated via Schwarz skew quadrilateral to provide a clarification of their geometrical origin.

1. Introduction

The most fundamental quantities of the theory of smooth surfaces in ordinary three-dimensional space are the Gaussian, respectively the mean curvature of the surface. The vanishing of the first quantity selects the class of the so-called flat surfaces while the vanishing of the later distinguishes the class of really remarkable surfaces known as “minimal”. The term “flat” is coined here because the prime example of a flat surface is the plane. The history of the minimal surfaces began more than two centuries ago with answering the following question (raised by Lagrange in connection with his studies of the variational problems): “What does the surface bounded by a given contour look like when it has the smallest surface area?”. This variational problem leads to a partial differential equation which turns out to be just the minimal surface equation (for more details cf. Darboux 1914; Osserman 1986; Karcher 1989; Jost 1994 and Oprea 1997).

In view of the fundamental observation in the minimal surface theory that every minimal surface belongs to one-parameter family of minimal surfaces, the so-called Bonnet family, the situation with the other quite interesting transformation discovered by Goursat is a little strange. While Bonnet transformation have found a great many applications, both in mathematics (Bonnet 1853; Große-Brauckmann & Wohlgemuth 1996) and other sciences (Andersson $et\,al.$ 1984; Seddon & Templer 1993; Charvolin & Sadoc 1996), Goursat transformation is less known and according to the authors knowledge no concrete investigation or application have been published (cf. Andersson $et\,al.$ 1988; Terrones & Mackay 1997 and references therein). This can be explained up to some extent by the fact that despite the family to which it belongs is still conformal the distances are not preserved in the process of deformations. Actually, as we shall see later on, the class of these transformations is much larger than the one which Goursat himself had in mind. It is the aim of the present paper to supply the missing part of the most general three-parameter family of homotopy transformations in the Goursat class, to provide illustrative examples and to clarify the geometrical manifestation of these parameters. The real importance of the possibility to control (locally!) the curvature of the surfaces is that it is responsible for interfacial morphology and eventually the morphogenesis as discussed in Klinowski $et\,al.$ 1996 and in more details by Hyde $et\,al.$ 1997.

2. Conformal Transformations in the Plane

For the purposes of the exposition to follow we will need the fundamental notion of Gauss map and stereographic projection. Let us recall that the Gauss map of the surface M equipped with the local coordinates σ, τ is a mapping from the surface M to the two-dimensional sphere S^2, denoted by $G : M \to S^2$ and given by $G(m) = N_m$, where N_m is the unit normal to M at m. In terms of the chosen parameterization, one may write $G(\mathbf{x}(\sigma, \tau)) = N(\sigma, \tau)$ and, for a small patch of M, think of $N(\sigma, \tau)$ as a parameterization of the sphere S^2. The stereographic projection from the north pole of the Riemann sphere

$$\xi^2 + \eta^2 + \zeta^2 = 1\,, \tag{2.1}$$

to the equatorial plane is given by

$$St(\xi, \eta, \zeta) = \left(\frac{\xi}{1-\zeta}, \frac{\eta}{1-\zeta}\right). \tag{2.2}$$

The following set of (generally complex) coordinates have been proven to be useful alternative of the Cartesian coordinates (ξ, η, ζ) on the sphere

$$\begin{aligned} u &= \frac{\xi + i\eta}{1-\zeta} = \frac{1+\zeta}{\xi - i\eta}\,, \\ -\frac{1}{v} &= \frac{\xi - i\eta}{1-\zeta} = \frac{1+\zeta}{\xi + i\eta}\,. \end{aligned} \tag{2.3}$$

These formulae can be easily inverted and give

$$\xi = \frac{1 - uv}{u - v}\,, \qquad \eta = \mathrm{i}\,\frac{1 + uv}{u - v}\,, \qquad \zeta = \frac{u + v}{u - v}\,. \tag{2.4}$$

Another calculation provides a proof that any rotational motion around the center of the sphere results in a linear fractional transformation

$$\tilde{u} = \frac{au + b}{cu + d}\,, \qquad \tilde{v} = \frac{av + b}{cv + d}\,, \tag{2.5}$$

of the complex plane with $ad - bc = 1$. It should be mentioned that the set of all such linear fractional transformations form the Lie group $SL(2, \mathbb{C})$ which consists of the 2×2 matrices with complex entries and determinant one, i. e.

$$SL(2, \mathbb{C}) := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix};\ \det g = ad - bc = 1\,,\ \ a,\, b,\, c,\, d \in \mathbb{C} \right\}. \tag{2.6}$$

A straightforward but tedious calculation gives that the linear fractional transformation (2.5) generated by the group element $g \in SL(2, \mathbb{C})$ produces the following (generally complex) rotation in $\mathbb{R}^3$

$$\Lambda(g) = \begin{pmatrix} (a^2 - b^2 - c^2 + d^2)/2 & \mathrm{i}(a^2 + b^2 - c^2 - d^2)/2 & cd - ab \\ \mathrm{i}(-a^2 + b^2 - c^2 + d^2)/2 & (a^2 + b^2 + c^2 + d^2)/2 & \mathrm{i}(ab + cd) \\ bd - ac & -\mathrm{i}(ac + bd) & ad + bc \end{pmatrix}. \tag{2.7}$$

The form of this matrix makes obvious the fact that the real rotations are carrying out by a special kind of $SL(2, \mathbb{C})$ group elements which generate the Lie subgroup $SU(2)$ inside $SL(2, \mathbb{C})$

$$SU(2) := \left\{ h = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix};\ \det h = |\alpha|^2 + |\beta|^2 = 1,\ \ \alpha,\, \beta \in \mathbb{C} \right\}. \tag{2.8}$$

More precisely, we have

$$\Lambda(h) = \begin{pmatrix} \Re(\alpha^2 - \beta^2) & -\Im(\alpha^2 + \beta^2) & -2\Re(\alpha\,\beta) \\ \Im(\alpha^2 - \beta^2) & \Re(\alpha^2 + \beta^2) & -2\,\Im(\alpha\,\beta) \\ 2\Re(\alpha\,\bar{\beta}) & -2\Im(\alpha\,\bar{\beta}) & \alpha\bar{\alpha} - \beta\bar{\beta} \end{pmatrix}. \tag{2.9}$$

The bar here means a complex conjugation while $\Re$ and $\Im$ denote the operations of taking the real, respectively the imaginary part of complex quantities.

From the group-theoretical point of view any matrix $g \in SL(2, \mathbb{C})$ can be represented in the form

$$g = kh\,, \qquad k \in K,\ h \in SU(2)\,, \tag{2.10}$$

where the set of 2×2 matrices K

$$K := \left\{ k = \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix}; \qquad \lambda \in \dot{\mathbb{R}} = \mathbb{R} \backslash \{0\},\, \mu \in \mathbb{C} \right\}, \tag{2.11}$$

form another subgroup of $SL(2, \mathbb{C})$. In explicit form the decomposition (2.10) can be rewritten as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \tag{2.12}$$

and therefore, for its fulfillment it is necessary and sufficient that

$$a = \lambda^{-1}\alpha - \mu\bar{\beta}, \qquad b = \lambda^{-1}\beta + \mu\bar{\alpha}, \qquad c = -\lambda\bar{\beta}, \qquad d = \lambda\bar{\alpha}. \tag{2.13}$$

Taking into account the defining condition

$$|\alpha|^2 + |\beta|^2 = 1, \tag{2.14}$$

we get

$$\alpha = \bar{d}\lambda^{-1}, \qquad \beta = -\bar{c}\lambda^{-1}, \qquad \lambda^2 = |c|^2 + |d|^2, \tag{2.15}$$

$$\mu = \begin{cases} (\lambda b + \lambda^{-1} c)/\bar{d} & \text{for } d \neq 0, \\ \lambda a/\bar{c} & \text{for } d = 0. \end{cases} \tag{2.16}$$

It is clear that α, β, λ and μ are defined unambiguously by the above relations, i. e. given a, b, c, d one can find them in an unique way. It is worth to point out also that the decomposition (2.10) makes obvious the fact that the linear fractional transformations (2.6) belong to the class of conformal mappings.

3. Complex Rotations and Minimal Surfaces

One can ask the question: what is the relation between minimal surfaces and the matrix groups listed above? The answer is as follows: all minimal surfaces in $\mathbb{R}^3$ can be identified with the real parts of the holomorphic null curves in $\mathbb{C}^3$. By definition $\Gamma \equiv (A(w), B(w), C(w))$ is such a curve if

$$[A'(w)]^2 + [B'(w)]^2 + [C'(w)]^2 = 0, \tag{3.1}$$

and

$$|A'(w)|^2 + |B'(w)|^2 + |C'(w)|^2 \neq 0. \tag{3.2}$$

Here $(A(w),\, B(w),\, C(w))$ are holomorphic functions of the complex variable $w = \sigma + \mathrm{i}\tau$ defined in terms of the conformal coordinates (σ, τ), and the respective minimal surface M is given by the formulae

$$x(w, \bar{w}) = \Re A(w), \qquad y(w, \bar{w}) = \Re B(w), \qquad z(w, \bar{w}) = \Re C(w). \tag{3.3}$$

By its very definition, the group of conformal motions in the three-dimensional complex vector space $\mathbb{C}^3$, i. e. the set of invertible linear homogeneous maps which preserve the quadric in the left hand side of (3.1) up to a multiplicative factor, are the most general transformations in the class of the minimal surfaces. The matrices in (2.7) belong obviously to this class and supply us directly with a six-dimensional family of such transformations. As we shall see later on, just three are essential and among them one can recognize the parameter in Goursat transformation.

An important step in passing from (3.1) to (3.3) is that we are able to describe explicitly all solutions of the former equation. They can be represented in the form

$$\begin{aligned} A'(w) &= (1-w^2)R(w)\,, \\ B'(w) &= \mathrm{i}(1+w^2)\,R(w), \\ C'(w) &= 2wR(w)\,, \end{aligned} \tag{3.4}$$

where $R(w)$ is an arbitrary holomorphic function. Therefore, we immediately obtain the Weierstrass–Enneper representation of the minimal surfaces in the form

$$\begin{aligned} x(\sigma,\,\tau) &= \Re\int (1-\tilde{w}^2)R(\tilde{w})\,\mathrm{d}\tilde{w} + x_0\,, \\ y(\sigma,\,\tau) &= \Re\int \mathrm{i}(1+\tilde{w}^2)R(\tilde{w})\,\mathrm{d}\tilde{w} + y_0\,, \\ z(\sigma,\,\tau) &= \Re\int 2\tilde{w}R(\tilde{w})\,\mathrm{d}\tilde{w} + z_0\,, \end{aligned} \tag{3.5}$$

where $\tilde{w} = \tilde{\sigma} + \mathrm{i}\tilde{\tau}$ and (x_0, y_0, z_0) are the integration constants.

If we exchange in the above formulae the Weierstrass function $R(w)$ with $\tilde{R}(w) = \mathrm{e}^{\mathrm{i}\theta}R(w)$ where θ is a real parameter and perform the integration we still have a minimal surface $\tilde{S}$ called associated to the initial surface S specified by (3.5). Traditionally, the transition

$$S \longrightarrow \tilde{S}(\theta)\,, \tag{3.6}$$

is called Bonnet transformation since the time he had introduced it in the differential geometry (Bonnet 1853), while θ is referred as the angle of association. The surface obtained for $\theta = \pi/2$ is called "adjoint" and will be denoted further on as S^*. It is interesting to point out that the Bonnet transformation does not change the Gaussian curvature as well. This can be seen by realizing that this transformation can be succinctly written into the form

$$(\tilde{x}(\theta),\,\tilde{y}(\theta),\,\tilde{z}(\theta)) = (x,\,y,\,z)\cos\theta + (x^*,\,y^*,\,z^*)\sin\theta\,, \tag{3.7}$$

which also makes obvious that Bonnet related points trace out ellipses in the space.

Going back to our considerations of the rotational motions induced by the linear fractional transformations of the complex plane, let us look closer to the decomposion (2.10).

We already know that the second factor on the right produces the $SO(3, \mathbb{R})$ group (2.9). Using the relativistic terminology we can say that we have decomposed the general Lorentz transformation into a real rotation and a "boost" transformation (cf. Crampin & Pirani 1986 for more details). This later can be represented as a complex rotation in the form

$$\Lambda(k) = \begin{pmatrix} (\lambda^{-2} - \mu^2 + \lambda^2)/2 & \mathrm{i}(\lambda^{-2} + \mu^2 - \lambda^2)/2 & -\mu\lambda^{-1} \\ -\mathrm{i}(\lambda^{-2} - \mu^2 - \lambda^2)/2 & (\lambda^{-2} + \mu^2 + \lambda^2)/2 & \mathrm{i}\mu\lambda^{-1} \\ \lambda\mu & -\mathrm{i}\lambda\mu & 1 \end{pmatrix}. \tag{3.8}$$

Besides, any element $k \in K$ can be factorized as follows:

$$k = \dot{r}n\,, \qquad \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \varrho^{-1} & 0 \\ 0 & \varrho \end{pmatrix} \begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix}, \quad \text{where } \varrho = \lambda \text{ and } \chi = \lambda\mu\,. \tag{3.9}$$

As a Lie group the first factor is isomorphic with the group $\dot{\mathbb{R}} = \mathbb{R}\backslash\{0\}$ of nonzero real numbers under multiplication and the second one with the group of complex numbers $\mathbb{C}$ under addition. Their images in the group of the (complex) rotations are respectively:

$$\Lambda(\dot{r}) = \begin{pmatrix} (\varrho^{-2} + \varrho^2)/2 & \mathrm{i}(\varrho^{-2} - \varrho^2)/2 & 0 \\ -\mathrm{i}(\varrho^{-2} - \varrho^2)/2 & (\varrho^{-2} + \varrho^2)/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.10}$$

and

$$\Lambda(n) = \begin{pmatrix} (2 - \chi^2)/2 & \mathrm{i}\chi^2/2 & -\chi \\ \mathrm{i}\chi^2/2 & (2 + \chi^2)/2 & \mathrm{i}\chi \\ \chi & -\mathrm{i}\chi & 1 \end{pmatrix}. \tag{3.11}$$

Let us note that the matrix $\Lambda(\dot{r})$ represents a complex rotation around the third coordinate axis while the complex rotation generated by $\Lambda(n)$ is around the complex axis defined by the vector $(\mathrm{i}, 1, 0)$. Summarizing, we can say that any rotation in three-dimensional complex vector space can be decomposed as a product of a real rotation followed by a complex rotation around a real axis and one more complex rotation around a complex axis. This should be compared with the analogical statement about the group $SO(3, \mathbb{C})$ in Goursat paper (cf. p. 139 in Goursat 1888). There the third rotation is recognized again as a *real* one and therefore neglected further on as inessential!

However we know that the non-trivial part of the deformations in which we are interested is generated by the whole group K. This group acts on the complex plane producing transformation of the form

$$\tilde{u} = Uu + V\,, \quad U = \lambda^{-2}\,, \quad V = \mu\lambda^{-1}\,, \qquad U \in \mathbb{R}^{+},\ V \in \mathbb{C}\,. \tag{3.12}$$

In his paper Goursat (1888), by the reasons explained above, had restricted himself just to the stretching from the origin in the complex plane

$$\tilde{u} = \kappa u\,, \qquad \kappa \in \mathbb{R}^{+}, \qquad \kappa \neq 1\,, \tag{3.13}$$

and determined the complex rotation corresponding to that part of the boost transformation in the form

$$\begin{aligned} \tilde{\xi} &= \frac{1+\kappa^2}{2\kappa}\xi - \mathrm{i}\frac{\kappa^2-1}{2\kappa}\eta\,, \\ \tilde{\eta} &= \mathrm{i}\frac{\kappa^2-1}{2\kappa}\xi + \frac{1+\kappa^2}{2\kappa}\eta\,, \\ \tilde{\zeta} &= \zeta\,. \end{aligned} \tag{3.14}$$

We immediately recognize in the above formulae our matrix (3.10) which can be obtained directly by setting here $\kappa = \varrho^2$. Properly speaking the generalization of the Goursat transformation is encoded in the matrix (3.11) which produces

$$\begin{aligned} x(w,\bar{w},\chi) = {} & \Re\Bigg[\frac{2-\chi^2}{2}\int(1-\tilde{w}^2)R(\tilde{w})\,\mathrm{d}\tilde{w} \\ & - \frac{\chi^2}{2}\int(1+\tilde{w}^2)R(\tilde{w})\,\mathrm{d}\tilde{w} - 2\chi\int\tilde{w}R(\tilde{w})\,\mathrm{d}\tilde{w}\Bigg]\,, \\ y(w,\bar{w},\chi) = {} & -\Im\Bigg[\frac{\chi^2}{2}\int(1-\tilde{w}^2)R(\tilde{w})\,\mathrm{d}\tilde{w} \\ & + \frac{2+\chi^2}{2}\int(1+\tilde{w}^2)R(\tilde{w})\,\mathrm{d}\tilde{w} + 2\chi\int\tilde{w}R(\tilde{w})\,\mathrm{d}\tilde{w}\Bigg]\,, \\ z(w,\bar{w},\chi) = {} & \Re\Bigg[\chi\int(1-\tilde{w}^2)R(\tilde{w})\,\mathrm{d}\tilde{w} \\ & + \chi\int(1+\tilde{w}^2)R(\tilde{w})\,\mathrm{d}\tilde{w} + 2\int\tilde{w}R(\tilde{w})\,\mathrm{d}\tilde{w}\Bigg] \end{aligned}$$

and after some simplifications

$$\begin{aligned} x(\sigma,\tau,\chi) &= \Re\int [1-(\tilde{w}+\chi)^2]R(\tilde{w})\,\mathrm{d}\tilde{w}\,, \\ y(\sigma,\tau,\chi) &= \Re\int \mathrm{i}[1+(\tilde{w}+\chi)^2]R(\tilde{w})\,\mathrm{d}\tilde{w}\,, \\ z(\sigma,\tau,\chi) &= \Re\int 2[\tilde{w}+\chi]R(\tilde{w})\,\mathrm{d}\tilde{w}\,. \end{aligned} \tag{3.15}$$

4. Examples and Comments

It is a trivial observation that the multiplication with a non-zero complex number ε of the null curve (3.4) produces a new minimal surface. The modulus of this complex number is responsible for dilation or contraction of the minimal surface in the space while its argument is already associated with the Bonnet transformation. It seems appropriate to refer to these transformations as "external" and to consider the proper complex rotations (2.7) as "internal" transformations. This situation hints also for their unification in the form

$$W(\varepsilon,g) = \varepsilon\Lambda(g) = |\varepsilon|\,\mathrm{e}^{\mathrm{i}\theta}\Lambda(g)\,, \qquad \theta = \arg(\varepsilon)\,, \tag{4.1}$$

in which the last two factors represent Bonnet and generalized Goursat transformations. One should note as well the commutativity of these transformations.

As our main concern here are the generalized Goursat transformations of the minimal surfaces they will be illustrated by their action on "Flachenstück" surface element of D infinite periodic minimal surface (IPMS). D IPMS was described by Schwarz in the last century, but recent years reveal that this surface is closely related with the structure of solid and liquid crystals and interfaces (cf. Andersson *et al.* 1988). Remarkable particular cases are the structures of diamond, cubic ice and monoolein–water system. In Fig. 1 an original surface element is presented.

It was drawn using Weierstrass–Enneper representation (3.5) with the function

$$R(w) = \frac{1}{\sqrt{w^8 - 14w^4 + 1}}\,. \tag{4.2}$$

Defining area for the complex variable $w = u + \mathrm{i}v$ has as boundary the arcs of the circles of radius $\sqrt{2}$, centered at the four points $w_{\pm,\pm} = \pm 1/\sqrt{2} \pm \mathrm{i}/\sqrt{2}$ in the complex plane. This is our initial surface that shall be subjected to various transformations.

Figure 2 represent the surface which is obtained by varying ϱ in (3.10) and fixing χ to be zero. These are just Goursat transformations of the D surface element. It is possible to take instead a real a complex number ϱ. The integral

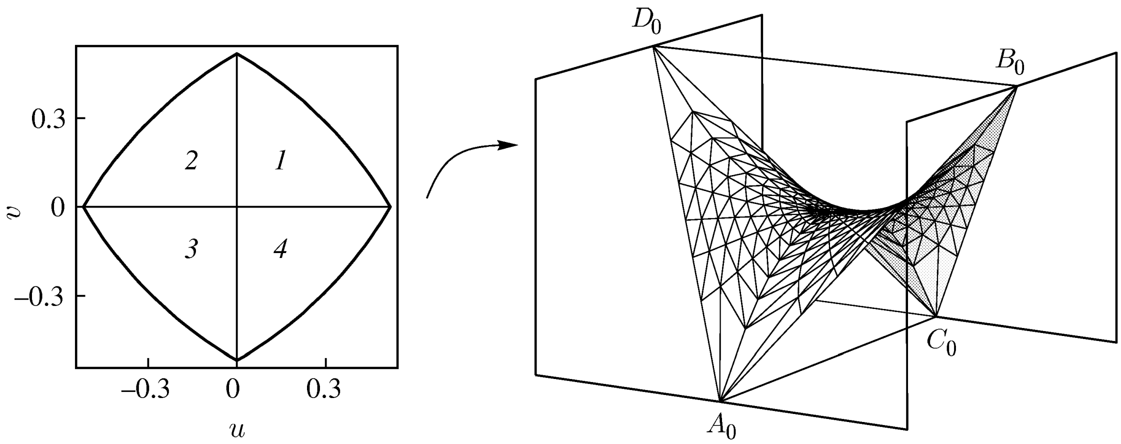

Figure 1. The surface element (Flachenstück) of the D IPMS

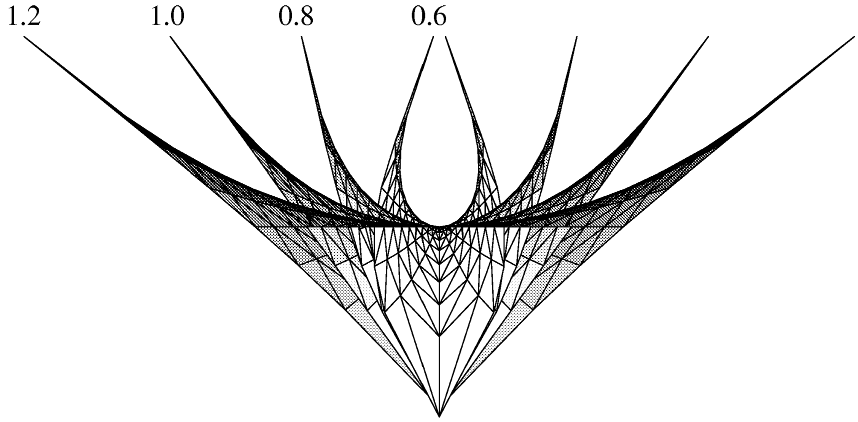

Figure 2. Goursat transformation of the D surface element at $\varrho = (1.2, 1, 0.8, 0.6)$, and $\chi = 0$. The images of the 1st and 2nd quadrant are shown

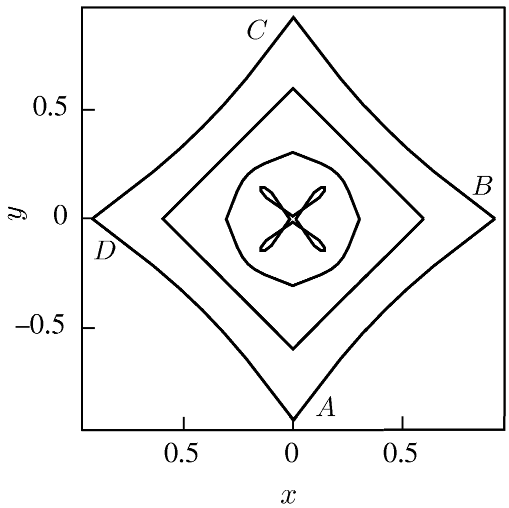

Figure 3. Same as Fig. 2.
View from the top of the z axis

effect of such change would be a rotation of the figure around the z axis. Another useful observations is that ϱ enters into transformation matrix in second power, which reduces the range of this variable to the interval $(0, \infty)$. In this interval $\varrho = 1$ is a point corresponding to identity transformation. For $\varrho > 1$, Goursat transformations correspond to stretching and increasing the distances in x–y plane, while for $\varrho < 1$, the resulting transformations are bending. In the bending interval two points are quite interesting from geometrical viewpoint. The first one is when the vertices A and B of the Schwarz skew quadrilateral $ABCD$ coincide respectively with C and D. To some extent this situation is illustrated in Fig. 2. The next one appears when A and B replace their positions respectively with C and D. The precise values of the deforming parameter ϱ can be written in terms of the complete elliptic integrals of first and second kind but the corresponding expressions are too complicated to be reproduced here. The significant digits in these two cases are 0.59 and 0.35. When ϱ approaches the zero the distances between vertices become infinite and the D surface element looks like the plane.

In Figure 3 boundaries of D surface element are presented for the same values of ϱ as in Fig. 2. The viewpoint is chosen to be from the top of z axis. While the boundaries of the original surface are straight lines, any non-trivial Goursat transformation make them curved.

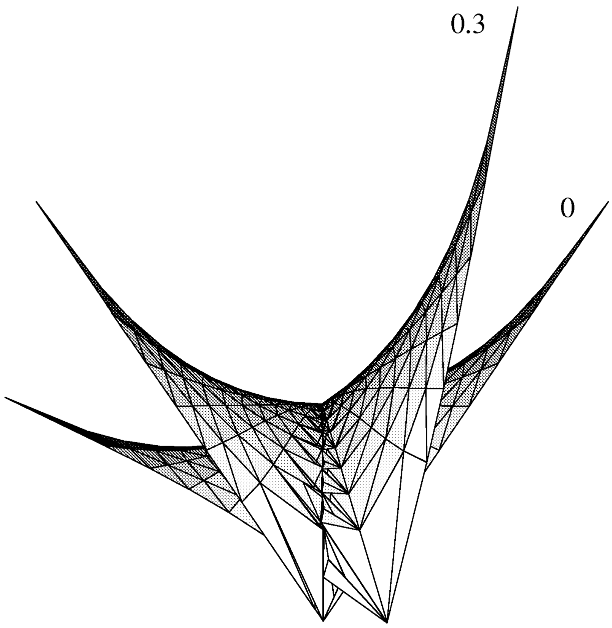

Figure 4. Generalized Goursat transformation of the D surface element at $\varrho = 1$, and $\chi = (0, 0.3)$. Shown are the images of the 1st and 2nd quadrant

Figure 4 represent the deformed surface obtained for $\varrho = 1$, $\chi = 0.3$ along an initial one for comparison. By its definition χ is a complex number. In order to simplify the systematization, χ is taken in a polar form. When varying its modulus an asymmetric deformations appear. In the direction of deformation defined by its argument the surface is stretched, but in the opposite direction it is compressed. This directional asymmetry is the main difference with the Goursat transformation. This allows us to consider χ as a plane vector of deformation. The modulus will corresponds to the amplitude of the vector, and the argument to its direction in the real space. This transformation do not preserve heights and it involves changes not only of x and y, but also of z coordinates.

In Figure 5 a view of boundaries as they are seen from the top of the z axis is presented. Straight line boundaries of the original surface become curved because stretching make them concave, while compressing result in convexity of the adjacent to the direction of the deformation boundaries.

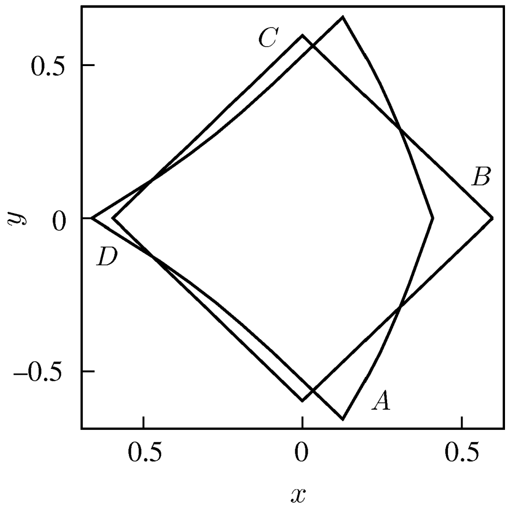

Figure 5. Same as Fig. 4. View from the top of the z axis

Surely, the above transformations can be combined in arbitrary order. This process increase tremendously the family of transformed surfaces that can be generated from the original one. In addition to Goursat and generalized Goursat transformations, a Bonnet transformations can be also executed following prescription in (4.1).

Some differential-geometric remarks at the end of this paper are in order as well. First of all, while bending the surface, the generalized Goursat transformations stretches it directionally. So, they change the metric and curvature but since any linear fractional transformation and the stereographic projection are conformal mappings their composition preserves angles. This means that such things like lines of curvature and asymptotic lines are also preserved.

Finally, it is obvious that this larger family of transformations of the minimal surfaces is a challenging one and deserves further systematic studies.

Acknowledgement

This work was partially supported by the National Science Fund of the Bulgarian Ministry of Education and Science, Grant No F-644/1996.

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