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QUATERNIONS AND ROTATION SEQUENCES

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Abstract. In this paper we introduce and define the quaternion; we give a brief introduction to its properties and algebra, and we show, what appears to be, its primary application — the quaternion rotation operator. The quaternion rotation operator competes with the conventional matrix rotation operator in a variety of rotation sequences.

1. Introduction

The 1950's post World War II period was a time in world history when large nations were again driven by Minds of Fear — fear of each other. The development of many new technologies continued to flourish, perhaps because of this fear. In these post-war years I was involved in the aerospace industry. On various occasions, I would meet with several people, each of whom represented one of several companies. These companies together formed a Consortium with a common goal — that of designing an anti-Inter Continental Ballistic Missile (anti-ICBM).

My interest at that time (for the Consortium) was Inertial Guidance. The proposed anti-ICBM Guidance strategies suggested by members of the Consortium often encountered orientations which approached gimbal-lock and therefore it would introduce its associated errors. At one point someone asked whether *quaternions* might offer an alternative computational approach. I didn't know — in fact, that was my first encounter with the term *quaternion*. That was quite long time ago, but it was the start of my personal foray into these matters.

In this paper we introduce and define the quaternion, give a brief introduction to its properties and its algebra. We then illustrate what is perhaps its primary application in a quaternion rotation operator. And, finally, we use these quaternion operators in a variety of rotation sequence applications.

2. Hamiltons Quaternions

While numbers of the form $a + ib$, that is, complex numbers of rank 2, were gaining general acceptance, some mathematicians of that day sought other mathematical systems over the hyper-complex numbers of, say, rank $3, 4, \ldots, n$. In 1843 after years of struggling, in an effort to create such a system, a sudden stroke of mathematical insight came upon William Rowan Hamilton. History says he happened to be out walking with his wife and, reputedly, carved these now famous equations in the stone wall of the bridge, in Dublin, over which they happened to be walking:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1\,.$$

Implicit in these equations, of course, is that

$$\begin{aligned}\mathbf{ij} = \mathbf{i} \times \mathbf{j} = \mathbf{k} = -\mathbf{j} \times \mathbf{i} = -\mathbf{ji}\\ \mathbf{jk} = \mathbf{j} \times \mathbf{k} = \mathbf{i} = -\mathbf{k} \times \mathbf{j} = -\mathbf{kj}\\ \mathbf{ki} = \mathbf{k} \times \mathbf{i} = \mathbf{j} = -\mathbf{i} \times \mathbf{k} = -\mathbf{ik}\,.\end{aligned}$$

All of quaternion algebra proceeds from these equations, e.g̃. the product of two quaternions $\mathbf{p}$ and $\mathbf{q}$ where

$$\begin{aligned}p = p_0 + \mathbf{p} = p_0 + \mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3\\ q = q_0 + \mathbf{q} = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3\end{aligned}$$

can be reduced to

$$pq = p_0 q_0 - \mathbf{p} \cdot \mathbf{q} + p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q}$$

all in accordance with Hamiltons foregoing original equations.

3. Quaternion Notation

Bold-faced letters denote ordinary vectors in three-dimensional space, $\mathbb{R}^3$; in particular we use $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ to denote the standard orthonormal basis for $\mathbb{R}^3$. Vectors in three dimensional space are written as triplets of real numbers (scalars), so we write the orthonormal basis as

$$\mathbf{i} = (1, 0, 0) \qquad \mathbf{j} = (0, 1, 0) \qquad \mathbf{k} = (0, 0, 1)\,.$$

A quaternion, as the name suggests, is a 4-tuple; it defines an element in $\mathbb{R}^4$. So for a quaternion we write

$$q = (q_0, q_1, q_2, q_3)$$

where q_0, q_1, q_2, and q_3 are simply real numbers or scalars.

As an alternative representation for a quaternion, we define a scalar part — some real number, say q_0, and associate with it a vector part, say $\mathbf{q}$, an ordinary vector in $\mathbb{R}^3$, namely

$$\mathbf{q} = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \, .$$

Here $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the standard orthonormal basis in $\mathbb{R}^3$. We then define the *quaternion* as the unlikely sum

$$q = q_0 + \mathbf{q} = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \, .$$

In summary, q_0 is called the *scalar part* of the quaternion while $\mathbf{q}$ is called the *vector part* of the quaternion. The scalars q_0, q_1, q_2, q_3 are called the *components* of the quaternion.

4. Some Quaternion Properties

4.1. Complex Conjugate

The *complex conjugate* of the quaternion

$$q = q_0 + \mathbf{q} = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$$

is denoted

$$q^* = q_0 - \mathbf{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3 \, .$$

It follows that

$$(pq)^* = q^*p^* \quad \text{and} \quad (p^*q)^* = q^*p \, ,$$

etc.

4.2. Quaternion Norm

The *norm* of a quaternion q is denoted by the scalar $N(q)$ where

$$N(q) = \sqrt{q^*q} \qquad \text{or} \qquad N^2(q) = q^*q \, .$$

4.3. Unit Quaternion

A *unit* quaternion, q, has a *norm* equal to one, that is

$$|q| = |q^*| = 1 \qquad \text{and} \qquad N^2(q) = q^*q = 1 \, .$$

The product of unit quaternions is a unit quaternion.

4.4. Quaternion Inverse

By definition of an inverse we have $q^{-1}q = qq^{-1} = 1$. Then pre-multplying both sides of, say, the 2nd equation from previous section by q^* we may write

$$q^* q q^{-1} = N^2(q) q^{-1} = q^*$$

from which it follows that

$$q^{-1} = \frac{q^*}{N^2} = \frac{q^*}{|q|^2}$$

and if q is a unit quaternion, then

$$q^{-1} = q^* .$$

5. Quaternion Algebra

Quaternion algebra proceeds from these fundamental equations:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

and implicit in these equations is that

$$\mathbf{ij} = \mathbf{i} \times \mathbf{j} = \mathbf{k} = -\mathbf{j} \times \mathbf{i} = -\mathbf{ji} \text{ , etc.}$$

The set of all quaternions with operations addition and multiplcation defines a *ring* — or more explicitly a *non-commutative division ring.* This longer title merely emphasizes that in the set of all quaternions every non-zero quaternion has an inverse and that quatenion products, in general, are non-commutative.

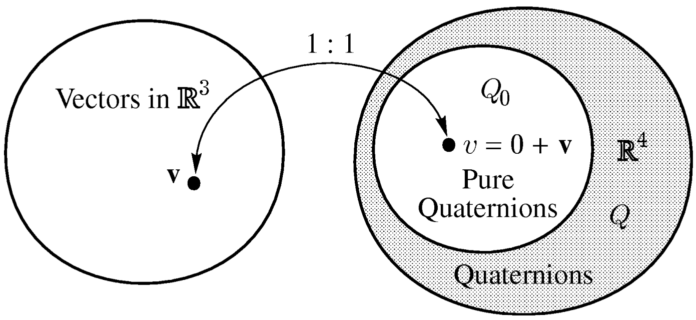

Figure 1. Correspondence: vectors ↔ quaternions

A *pure quaternion* is defined as a quaternion whose scalar part is zero. From the one-to-one relationship between all vectors in $\mathbb{R}^3$ and their corresponding pure quaternion, the meaning of the product of a vector and a quaternion merely becomes the *quaternion* product of two quaternions — one of which is a pure quaternion.

6. Special Quaternion Triple-Product — A Rotation Operator

First we note that any unit quaternion q may be written as

$$q = q_0 + \mathbf{q} = \cos\theta + \mathbf{u}\sin\theta$$

where

$$\mathbf{u} = \frac{\mathbf{q}}{|\mathbf{q}|} \quad \text{and} \quad \tan\theta = \frac{|\mathbf{q}|}{q_0}\,.$$

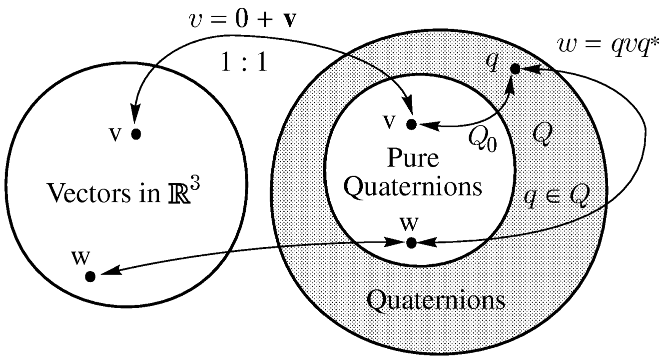

Figure 2. Quaternion operations on vectors

Theorem 1. *For any unit quaternion*

$$q = q_0 + \mathbf{q} = \cos\theta + \mathbf{u}\sin\theta$$

and for any vector $\mathbf{v} \in \mathbb{R}^3$ *the action of the operator*

$$L_q(\mathbf{v}) = q\mathbf{v}q^*$$

on $\mathbf{v}$ *may be interpreted geometrically as a rotation of the vector* $\mathbf{v}$ *through an angle* 2θ *about* $\mathbf{q}$ *as the axis of rotation.*

7. Rotation Operator Geometry

The quaternion rotation operator takes $v \to w$. That is,

$$w = qvq^* = (q_0^2 - |\mathbf{q}|^2)\mathbf{v} + 2(\mathbf{q}\cdot\mathbf{v})\mathbf{q} + 2q_0(\mathbf{q}\times\mathbf{v})\,.$$

If we write the vector $\mathbf{v}$ in the form

$$\mathbf{v} = \mathbf{a} + \mathbf{n}$$

where, $\mathbf{a}$ is the component of $\mathbf{v}$ *along* the vector part of q, and $\mathbf{n}$ is the component of $\mathbf{v}$ *normal* to the vector part of q. Then, it follows that

$$\mathbf{a} + \mathbf{m} = q(\mathbf{a}+\mathbf{n})q^* = (q_0^2 - |\mathbf{q}|^2)(\mathbf{a}+\mathbf{n}) + 2(\mathbf{q}\cdot\mathbf{a})\mathbf{q} + 2q_0(\mathbf{q}\times\mathbf{n}) = \mathbf{w}\,.$$

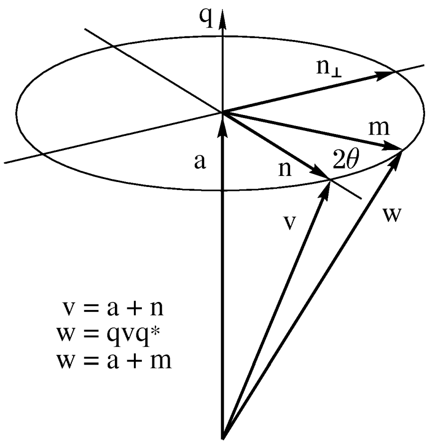

Figure 3. Rotation operator geometry

8. Rotation Operator Interpretation as a Frame or a Point-Set Rotation

Theorem 2. *For any unit quaternion*

$$q = q_0 + \mathbf{q} = \cos\theta + \mathbf{u}\sin\theta$$

and for any vector $\mathbf{v} \in \mathbb{R}^3$ *the action of the operator*

$$L_{q^*}(\mathbf{v}) = q^*\mathbf{v}q$$

may be interpreted geometrically as

- *a rotation of the coordinate frame with respect to the vector* $\mathbf{v}$ *through an angle* 2θ *about* $\mathbf{q}$ *as the axis, or,*
- *an opposite rotation of the vector* $\mathbf{v}$ *with respect to the coordinate frame through an angle* 2θ *about* $\mathbf{q}$ *as the axis.*

9. Open Rotation Sequences

We adopt the following useful symbol to represent the rotation R^z_α. Thus, this symbol in this case, represents a rotation about the z-axis through a positive angle α. The new frame c_{new} is related to the old frame c_{old} by the equations

$$\begin{aligned} x_{\text{new}} &= x_{\text{old}}\cos\alpha + y_{\text{old}}\sin\alpha \\ y_{\text{new}} &= y_{\text{old}}\cos\alpha - x_{\text{old}}\sin\alpha \\ z_{\text{new}} &= z_{\text{old}}\,. \end{aligned}$$

Clearly, the coordinate frame c_{new} consists of coordinate axes x_{new}, y_{new}, and z_{new} — where a positive rotation will always be regarded as a *right-handed* rotation about the indicated axis.

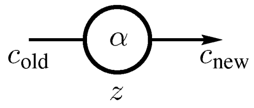

Figure 4. A rotation symbol

The notation for a sequence of rotation operators is then a string of such symbols, the order of the sequence is read from *left to right*. Thus a sequence of two coordinate frame rotations symbolically represents an *open rotation sequence*, as shown below. Proceeding from left to right, the first rotation is through an angle α about the z-axis, followed by the second rotation through an angle β about the new y-axis.

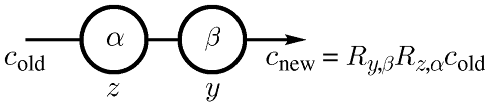

Figure 5. A tracking rotation sequence

10. Closed Rotation Sequences

It is clear that the inverse of a rotation through an angle α about some axis is simply a rotation through the angle $-\alpha$ about that same axis. Further, it is clear that the inverse of a sequence of rotation operators is simply the product of the inverses of the individual rotations in that sequence, written in reverse order.

Thus the inverse of the two angle sequence is a rotation through the angle $-\beta$ about the y-axis, followed by a rotation through the angle $-\alpha$ about the z-axis. Our new notation nicely represents this fact.

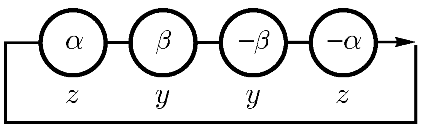

Figure 6. Tracking sequence and its inverse

In this figure, the *closed-loop* merely emphasizes that the final frame is the same as the initial frame. That is, being closed, this entire sequence represents an identity. Moreover, for analysis purposes the sequence may be opened at any point. This attribute of some application sequences is found to be very useful in the analysis of multi-coordinate relationships — whether related by rotation matrices or quaternion rotation operators.

11. Two-Rotation Tracking Sequence Using Matrices

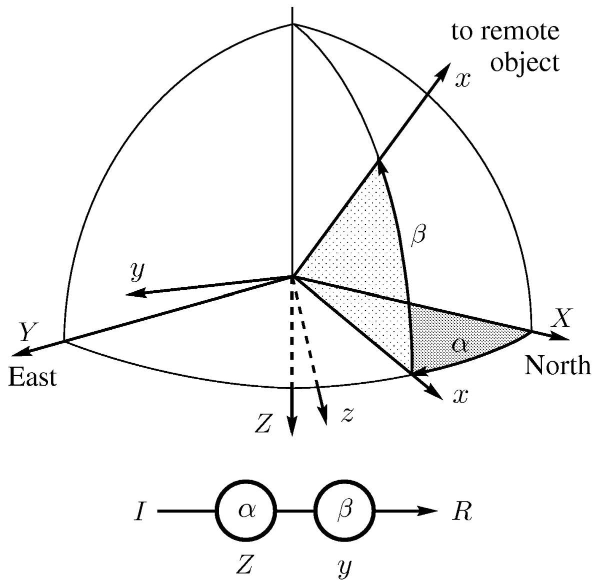

Figure 7. A rotation about Z-axis, followed by a rotation about new y-axis

$$\mathbf{R} = R_\beta^y R_\alpha^Z \mathbf{I} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{I}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta & \sin\alpha\cos\beta & -\sin\beta \\ -\sin\alpha & \cos\alpha & 0 \\ \cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta \end{bmatrix} \mathbf{I}$$

$\mathbf{I}$ = identity matrix in $\mathbb{R}^3$.

12. Two-Rotation Tracking Sequence Using Quaternions

$$\mathbf{R} = q_{y,\beta}^* q_{z,\alpha}^* \mathbf{E} q_{z,\alpha} q_{y,\beta} = p^* \mathbf{E} p$$

where

$$q_{z,\alpha} = \cos\frac{\alpha}{2} + \mathbf{k}\sin\frac{\alpha}{2}$$
$$q_{y,\beta} = \cos\frac{\beta}{2} + \mathbf{j}\sin\frac{\beta}{2}$$

and

$$\mathbf{E} = \begin{bmatrix} 0 & 0 & 0 \\ | & | & | \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{bmatrix} = \text{$\mathbb{R}^3$ basis vectors expressed as pure quaternions}$$

We now solve for what the quaternion operator does to the first column vector of the Identity matrix, here identified as the pure quaternion $i = 0 + \mathbf{i}$. We compute $p = q_{z,\alpha} q_{y,\beta}$ to give

$$\begin{aligned} p &= \left(\cos\frac{\alpha}{2} + \mathbf{k}\sin\frac{\alpha}{2}\right)\left(\cos\frac{\beta}{2} + \mathbf{j}\sin\frac{\beta}{2}\right) \\ &= \cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \mathbf{i}\sin\frac{\alpha}{2}\sin\frac{\beta}{2} + \mathbf{j}\cos\frac{\alpha}{2}\sin\frac{\beta}{2} + \mathbf{k}\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\,. \end{aligned}$$

Therefore

$$p^* = \cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \mathbf{i}\sin\frac{\alpha}{2}\sin\frac{\beta}{2} - \mathbf{j}\cos\frac{\alpha}{2}\sin\frac{\beta}{2} - \mathbf{k}\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\,.$$

Then

$$p^*\mathbf{i}p = \begin{bmatrix} \cos\alpha\cos\beta \\ -\sin\alpha \\ \cos\alpha\sin\beta \end{bmatrix}$$

however, gives only the 1st column of $\mathbf{R} = p^*\mathbf{E}p$ to confirm the matrix result.

Here, again $\mathbf{E} = \begin{bmatrix} 0 & 0 & 0 \\ | & | & | \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{bmatrix} =$ $\mathbb{R}^3$ basis vectors expressed as pure quaternions.

13. Three-Rotation Aerospace Sequence

This rotation can be represented as a

$$\begin{aligned} \text{matrix product:} &\quad M(\psi,\theta,\phi) \Rightarrow M_\phi M_\theta M_\psi \mathbf{I} \\ \text{or as a} & \\ \text{quaternion product:} &\quad q(\psi,\theta,\phi) \Rightarrow q_\phi^* q_\theta^* q_\psi^* \mathbf{I} q_\psi q_\theta q_\phi\,. \end{aligned}$$

14. Euler Angle-Axes Sequences

12 possible sequences

xyz	yzx	zxy	
xzy	yxz	zyx	← aerospace
xyx	yzy	zxz	← orbit
xzx	yxy	zyz	

In this listing, the *aerospace sequence* is the 2nd sequence in the 3rd column. It is a distinct axes or *non-repeating* axis, zyx, sequence. The *orbit sequence*, on the other hand, is a *repeated* axis sequence, zxz, which was perhaps first used by Euler himself. It is the 3rd sequence in the 3rd column. In this connection Euler is credited with the following two theorems for relating any two orthonormal coordinate frames:

Eulers 1st theorem:

Any two independent orthonormal coordinate frames may be related by a minimum sequence of rotations (less than four) about coordinate axes, where no two successive rotations may be about the same axis.

Eulers 2nd theorem:

Any two independent orthonormal coordinate frames may be related by a single rotation about some axis.

This single rotation ϕ, in Theorem$\tilde{2}$ would be, for example, about the eigen-direction of the composite matrix M (see following Fig.$\tilde{8}$), where

$$\cos\phi = \frac{\operatorname{trace} M - 1}{2}\,.$$

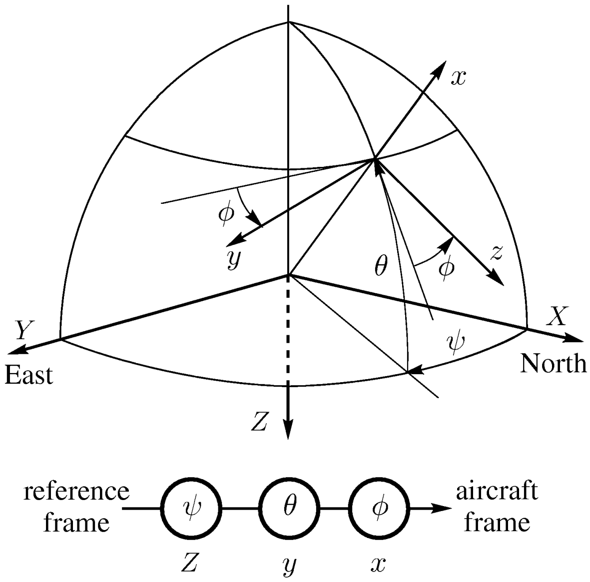

Figure 8. The aerospace Euler angle-axes sequence

15. Rotation of Vector Sets

In most applications it will be necessary to rotate either an entire coordinate frame which is defined by its set of three basis vectors, or an entire rigid body which is defined by a set of vectors or points. We adopt the following convenient matrix notation to represent a collection of n-column vectors.

$$V = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}$$

A quaternion rotation operator which operates on this set of vectors produces a new set, which we collect in the matrix, say W. We will *define* this operation to mean

$$W = L_q(V) = qVq^* = q \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix} q^*$$

$$= \begin{bmatrix} | & | & & | \\ q\mathbf{v}_1 q^* & q\mathbf{v}_2 q^* & \cdots & q\mathbf{v}_n q^* \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_n \\ | & | & & | \end{bmatrix}.$$

16. Near-Earth Orbit Sequence

We will now consider a near-earth orbit described in terms of its orbital parameters, namely, the location of its ascending node Ω, the orbital inclination angle ι, and the argument of the latitude ν, and we will relate them to the Orbit Ephemeris parameters, namely, its longitude λ, its latitude L, and the direction of its path α.

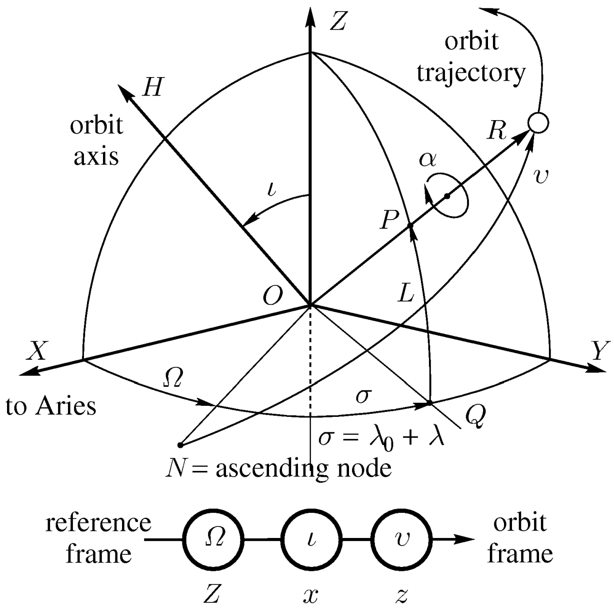

Figure 9. Near-Earth orbit sequence

17. Orbit Ephemeris Sequence

The ephemeris is defined by the location, longitude λ, and latitude L, of the *impact point* which is the intersection point defined on the surface of the earth by a geocentric line from the orbiting satellite. The angle α, is related to the path direction of that point on the surface of the earth.

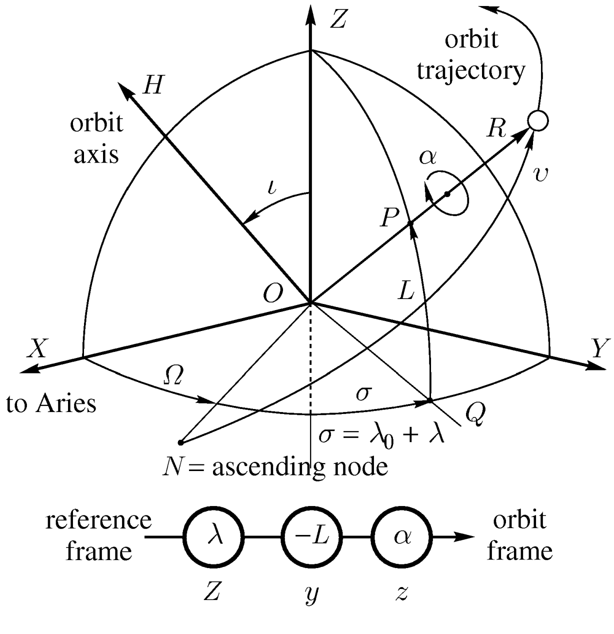

Figure 10. Near-Earth Ephemeris sequence

18. Orbit Closed Sequence

The two previous open sequences are now used to form a closed sequence. It is important to note that each of the composite 3-rotation sequences which, in this application, are represented by the orbit sequence and the ephemeris sequence are equal; so the product of one with the inverse of the other is an identity. Therefore, knowing one set of parameters allows one to predict the others. Rotation matrices and/or *quaternion* rotation operators may be used.

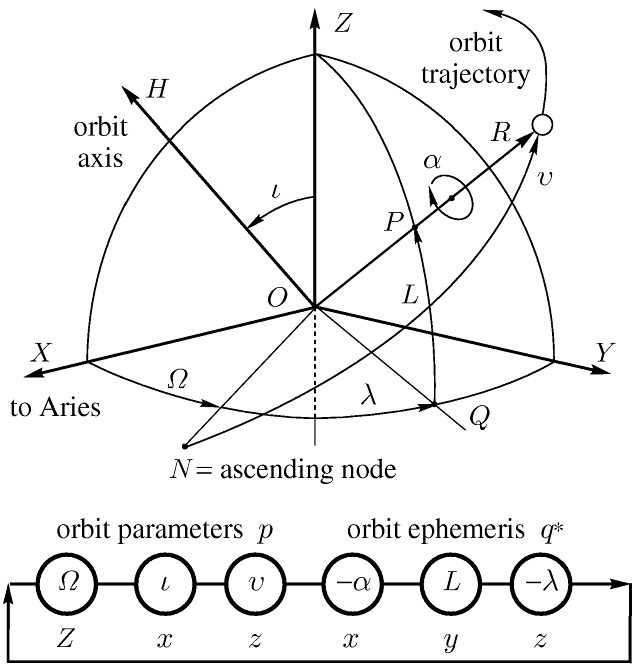

Figure 11. Orbit/Ephemeris closed sequence

19. Great Circle Paths

The *radian distance* θ, and *directions*, in this case ψ_1 and ψ_2, between two points on the sphere are computed. Points are defined, in this example, in terms of the familiar angles, longitude and latitude (λ, L); any other sequence would do.

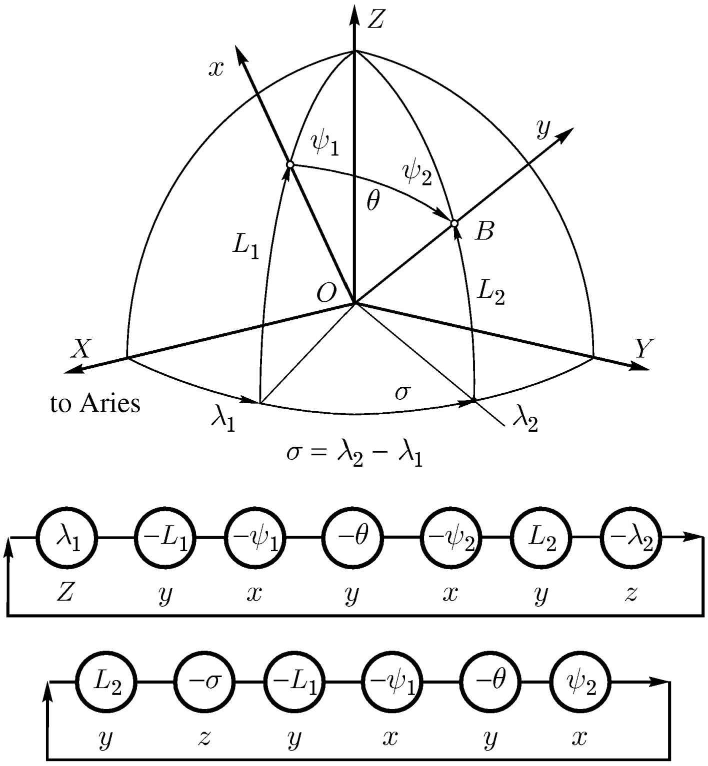

Figure 12. Great circle paths

The closed rotation sequence is partitioned and labeled. From this it is clear, since the sequence is closed, we can write in terms of rotation operators **M** and **N** using

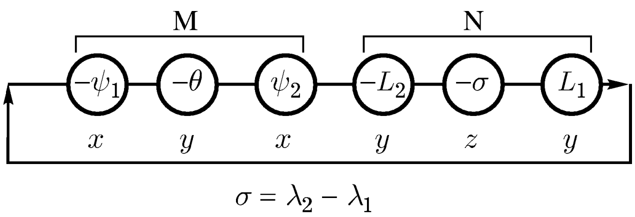

Figure 13. Partitioned great circle sequence

Rotation Matrices

$$\mathbf{MN} = \mathbf{I} = \text{identity}\,.$$

Then

$$\mathbf{M} = \mathbf{N}^{-1} = \mathbf{N}^{t}$$

or using

Quaternion Operators

$$q_{z,\lambda_1} q^*_{y,L_1} q^*_{x,\psi_1} q^*_{y,\theta} q_{x,\psi_2} q_{y,L_2} q^*_{z,\lambda_2} = \text{identity}$$
$$m = q^*_{x,\psi_1} q^*_{y,\theta} q_{x,\psi_2} = q_{y,L_1} q_{z,\sigma} q^*_{y,L_2} = n^*$$

In either case, we obtain the following results:

$$\cos\theta = \cos L_2 \cos\sigma \cos L_1 + \sin L_1 \sin L_2$$
$$\tan\psi_1 = \frac{\cos L_2 \sin\sigma}{\cos L_1 \sin L_2 - \cos L_2 \cos\sigma \sin L_1}$$
$$\tan\psi_2 = \frac{\cos L_1 \sin\sigma}{\sin L_2 \cos\sigma \cos L_1 - \sin L_1 \cos L_2}$$

20. Spherical Triangles

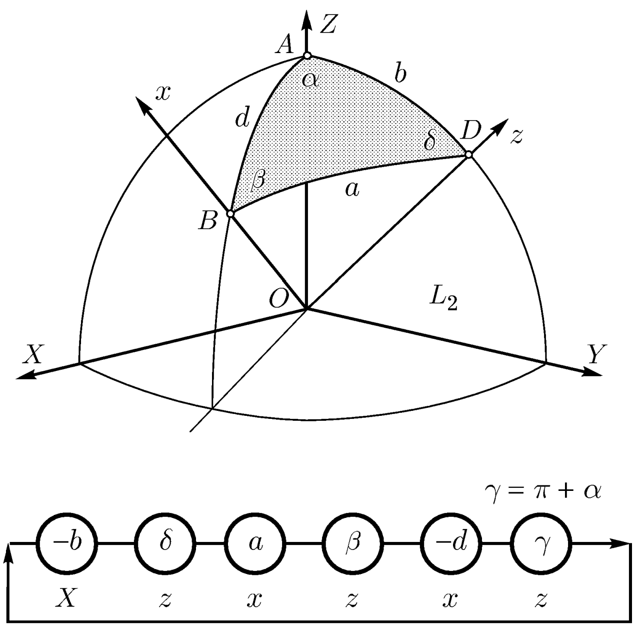

Figure 14. Triangles on the sphere

A rotation sequence will generate all of the equations and relationships one might encounter in spherical trigonometry. This sequence easily yields the three most familiar equations:

$$\frac{\sin a}{\sin\alpha} = \frac{\sin b}{\sin\beta} = \frac{\sin d}{\sin\delta}$$
$$\cos d = \sin a\cos\delta\sin b + \cos a\cos b$$
$$\cos\delta = \sin\alpha\cos d\sin\beta - \cos\alpha\cos\beta$$

Various half-angle equations can be found using quaternions.

21. Spherical Equilateral n-gons

Quaternion expressions are now used to determine *side-angle* parameter relationships for equilateral spherical triangles and regular 4-gons on the sphere. Any regular n-gon is produced from some q^n.

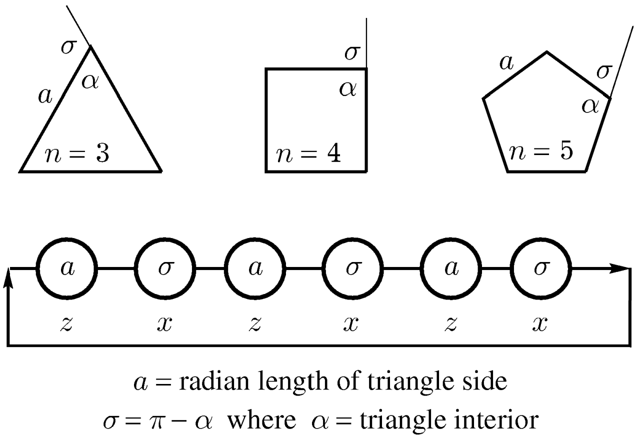

Figure 15. Equilateral n-gons on the sphere

We write

$$\begin{aligned} q &= r_0 + \mathbf{u}r_1 \\ &= \left(\cos\frac{a}{2} + \mathbf{k}\sin\frac{a}{2}\right)\left(\cos\frac{\sigma}{2} + \mathbf{i}\sin\frac{\sigma}{2}\right) \\ &= \cos\frac{a}{2}\cos\frac{\sigma}{2} + \mathbf{i}\cos\frac{a}{2}\sin\frac{\sigma}{2} + \mathbf{j}\sin\frac{a}{2}\sin\frac{\sigma}{2} + \mathbf{k}\sin\frac{a}{2}\cos\frac{\sigma}{2} \end{aligned}$$

Then

$$\begin{aligned} q^2 &= (r_0^2 - r_1^2) + \mathbf{u}(2r_0r_1) \\ q^3 &= r_0(r_0^2 - r_1^2) - (2r_0r_1^2) + \mathbf{u}[r_1(r_0^2 - r_1^2) + 2r_0^2r_1] \\ q^4 &= [(r_0^2 - r_1^2)^2 - 4r_0^2r_1^2] + \mathbf{u}[4r_0r_1(r_0^2 - r_1^2)]\,, \text{ etc.} \end{aligned}$$

The rotation sequence shown above must hold for a regular triangle or 3-gon on the sphere. Again, using the quaternion rotation operator, we may write

$$(q)^3\mathbf{E}(q^*)^3 = \mathbf{E}\,.$$

If

$$\begin{aligned} q &= r_0 + \mathbf{u}r_1 = \text{a unit quaternion} \\ &= \left(\cos\frac{a}{2} + \mathbf{k}\sin\frac{a}{2}\right)\left(\cos\frac{\sigma}{2} + \mathbf{i}\sin\frac{\sigma}{2}\right) \end{aligned}$$

then

$$q^3 = r_0(r_0^2 - r_1^2) - (2r_0r_1^2) + \mathbf{u}[r_1(r_0^2 - r_1^2) + 2r_0^2r_1] = 1 + \mathbf{u}0\,.$$

Therefore, $r_1(r_0^2-r_1^2)+2r_0^2r_1 = (r_0^2-r_1^2)+2r_0^2 = 0$ and since $r_0 = \cos\frac{a}{2}\cos\frac{\sigma}{2}$ upon substitution and simplification, the side and angle are related in the equilateral spherical triangle, by

$$\cos\frac{a}{2}\cos\frac{\sigma}{2} = \frac{1}{2}\,.$$

Similarly, if we set $q^4 = 1$ we will find

$$\cos\frac{a}{2}\cos\frac{\sigma}{2} = \frac{\sqrt{2}}{2}$$

which relates side and angle for *regular* spherical 4-gons, etc.

22. Un-manned Weather Satellite

This is a special application which describes a research effort supported by NASA several years ago. A strategy is defined for an unmanned Weather Predicting Satellite.

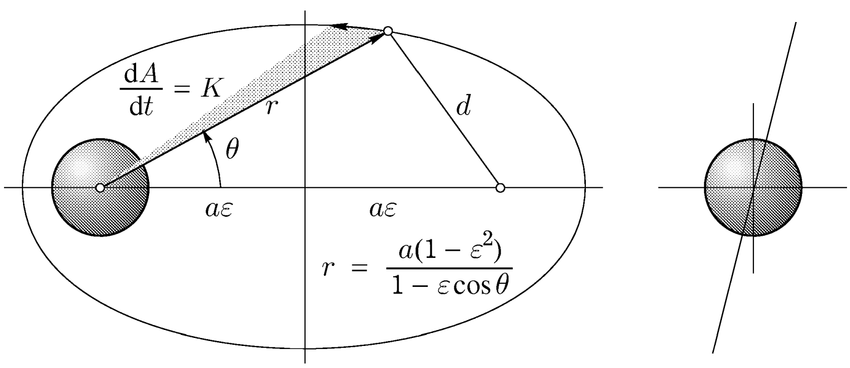

Figure 16. An Earth near-polar satellite orbit

Here, the objective is to choose the *near-polar* angle of inclination such that, due to the oblate spheroid geometry of the earth, the orbital plane would precess so that the orbital plane would always tend to contain the Earth–Sun line.

The strategy

The telescope in the satellite looks aft into dark and locks on a star near the orbital plane and also above but near the earth horizon. As satellite proceeds the line-of-sight approaches the edge of the earths surface, the line-of-sight refracts, continuing until occultation occurs. The telescope then picks up another star near the horizon and the process is repeated. For each star, a refraction profile is recorded until the star line-of-sight is occulted by the earth.

The refraction data collected for each star is reduced, using Snells Law, to provide meaningful weather information (at the impact point of the line-of-sight). As the satellite proceeds along its orbit the telescope would, of course, eventually be looking into the sun. So, only half of each orbit can be used to track stars and collect data.

Since the earth rotates, say 15 degrees or so, during every period of the near-earth orbit, a new portion of the earth surface (atmosphere) is ready for tracking, perhaps using the same or nearly the same stars.

23. Summary

The *quaternion* is a 4-tuple whose primary, if not its only, application is in a quaternion rotation operator. As such it competes with the more familiar 3×3 rotation matrices which are, of course, each comprised of 9 elements.

In certain analytical procedures and in some applications it is found that the quaternion can offer fundamental computational, operational and/or implementation and data handling advantages over the conventional rotation matrix.

References

[1] Kuipers J. B., *Quaternions & Rotation Sequences: A Primer with Applications to Orbits*, Aerospace and Virtual Reality, Princeton University Press, Princeton, New Jersey 1999.