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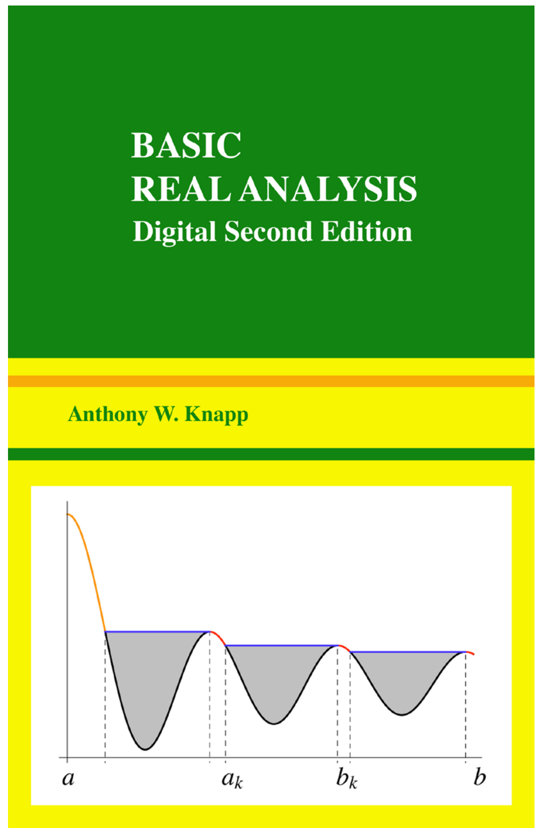

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CHAPTER X

Topological Spaces

Abstract. This chapter extends considerably the framework for discussing convergence, limits, and continuity that was developed in Chapter II: topological spaces replace metric spaces.

Section 1 makes various definitions, including definitions for the terms topology, open set, closed set, continuous function, base for a topology, separable, and subspace. It introduces two general kinds of constructions useful in analysis and other fields for forming new topological spaces out of old ones—weak topologies and quotient topologies. The section gives several examples of each.

Sections 2–3 develop standard facts, mostly elementary, about how certain combinations of properties of topological spaces imply others. Examples show some limitations to such implications. Properties that are studied include Hausdorff, regular, normal, dense, compact, locally compact, Lindelöf, and σ-compact.

Section 4 discusses product topologies on arbitrary product spaces, an example of a weak topology. The main theorem, the Tychonoff Product Theorem, says that the product of compact spaces is compact.

Section 5 introduces nets, a generalization of sequences. Sequences by themselves are inadequate for detecting convergence in general topological spaces, and nets are a substitute. The use of nets in many cases provides an easier way of establishing properties of subsets of a topological space than direct arguments with open and closed sets.

Section 6 elaborates on quotient topologies as introduced in Section 1. Conditions under which a quotient space is Hausdorff are of particular interest.

Sections 7–8 prove and apply Urysohn's Lemma, which says that any two disjoint closed sets in a normal topological space may be separated by a real-valued continuous function. This result is fundamental to serious uses of topological spaces in analysis. One application is to showing that every separable Hausdorff regular topology arises from a metric.

Section 9 extends Ascoli's Theorem and the Stone–Weierstrass Theorem from their settings in compact metric spaces in Chapter II to the wider setting of compact Hausdorff spaces.

1. Open Sets and Constructions of Topologies

In applications involving metric spaces, we have seen several times that the explicit form of a metric may not at all be one of objects of interest for the space. Instead, we may be interested in the open sets, or in convergence, or in continuity, or in some other aspect of the space. The same open sets, convergence, and continuity may come from two different metrics, and we have even encountered notions of convergence that are not associated with any metric at all. We saw in Section II.5, for example, that we could associate three different natural-looking metrics to the product $X \times Y$ of two metric spaces, and the three metrics led to the same open sets, the same convergence of sequences, and the same continuous functions. On the other hand, the notions in Chapter V of pointwise convergence, convergence almost everywhere, and weak-star convergence were defined without reference to a metric, and depending on the details of the situation, there need not be metrics yielding these notions of convergence. We have brushed against further, more subtle situations with one or the other of these phenomena—no special distinguished metric or no metric at all—but there is no need to produce a complete list. The present chapter introduces and studies an abstract generalization of the notion of a metric space, namely a "topology," that makes it unnecessary to have the kind of explicit formula demanded by the definition of metric space.

The framework for a "topological space" consists of a nonempty set and a collection of "open sets" satisfying the conditions of Proposition 2.5. Thus let X be a nonempty set. A set $\mathcal{T}$ of subsets of X is called a **topology** for X if

(i) X and $\varnothing$ are in $\mathcal{T}$,
(ii) any union of members of $\mathcal{T}$ is a member of $\mathcal{T}$,
(iii) any finite intersection of members of $\mathcal{T}$ is a member of $\mathcal{T}$.

The members of $\mathcal{T}$ are called **open sets**, and $(X, \mathcal{T})$ is called a **topological space**. When there is no chance for ambiguity, we may refer to X itself as a topological space.

Every metric space furnishes an example of a topological space by virtue of Proposition 2.5; we refer to the topology in question as the **metric topology** for the space. Two other examples of general constructions leading to topological spaces will be given later in this section, and some specific examples of other kinds will be given in Section 2.

Neighborhoods, open neighborhoods, interior, closed sets, limit points, and closure may be defined in the same way as in Section II.2. As remarked after Corollary 2.11, the proofs of certain results relating these notions depended only on the definitions and the three properties of open sets listed above. These results are Proposition 2.6 and Corollary 2.7 characterizing interior, Proposition 2.8 giving properties of the family of all closed sets, Proposition 2.9 relating closed sets to limit points, and Proposition 2.10 and Corollary 2.11 characterizing closure. Thus we may take all those results as known for general topological spaces, and it is not necessary to repeat their statements here.

The notion of continuity extends to topological spaces in straightforward fashion. Specifically the definition of continuity at a point is extracted from the statement of Proposition 2.13: if X and Y are topological spaces, a function

$X \to Y$ is **continuous at a point** $x \in X$ if for any open neighborhood V of $f(x)$ in Y, there is a neighborhood U of x such that $f(U) \subseteq V$. Then Corollary 2.14 is immediately available, saying that if $f : X \to Y$ is continuous at x and $g : Y \to Z$ is continuous at $f(x)$, then the composition $g \circ f$ is continuous at x.

Proposition 2.15 and its proof are available also, saying that the function $f : X \to Y$ is continuous at every point of X if and only if the inverse image under f of every open set in Y is open in X, if and only if the inverse image under f of every closed set in Y is closed in X. We say that $f : X \to Y$ is **continuous** if these equivalent conditions are satisfied. The function $f : X \to Y$ is said to be a **homeomorphism** if f is continuous, f is one-one and onto, and $f^{-1} : Y \to X$ is continuous. The relation "is homeomorphic to" is an equivalence relation.

Now let us come to the two general constructions of topological spaces, known as "weak topologies" and "quotient topologies." Both of these have many applications in real analysis.

The notion of "weak topology" starts from the fact that the intersection of a nonempty collection of topologies for a set is a topology; this fact is evident from the very definition. The prototype of a weak topology is the "product topology" for the product of a nonempty set of topological spaces. In the terminology of Section A1 of Appendix A, if S is a nonempty set and if X_s is a nonempty set for each s in S, then the Cartesian product $X = \bigtimes_{s \in S} X_s$ is the set of all functions f from S into $\bigcup_{s \in S} X_s$ such that $f(s)$ is in X_s for all $s \in S$. Now suppose that each X_s is a topological space, and let $p_s : X \to X_s$ be the s^{th} coordinate function, given by $p_s(f) = f(s)$. If X is given the **discrete topology** $\mathcal{D}$, in which every subset of X is open, then each p_s is continuous; in fact, the inverse image of an open set in X_s is some subset of X, and every subset of X is in $\mathcal{D}$. Form the collection of all topologies $\mathcal{T}_\alpha$ on X such that each $p_s : X \to X_s$ is continuous relative to $\mathcal{T}_\alpha$. The collection is nonempty since $\mathcal{D}$ is one. Let $\mathcal{T}$ be their intersection. The inverse image of any open set in X_s under p_s lies in $\mathcal{T}_\alpha$ for each α and hence lies in $\mathcal{T}$. Therefore each p_s is continuous relative to $\mathcal{T}$. We speak of $\mathcal{T}$ as the "weakest topology" on X such that all p_s are continuous, and this topology for X is called the **product topology** for X. We shall study product topologies in more detail in Section 4.

More generally let X be a nonempty set, let S be a nonempty set, let X_s be a topological space for each s in S, and suppose that we are given a function $f_s : X \to X_s$ for each s in S. If X is given the discrete topology, then every f_s is continuous. Arguing as in the previous paragraph, we see that there exists a smallest topology for X making all the functions f_s continuous. This is called the **weak topology for** X **determined by** $\{f_s\}_{s \in S}$.

EXAMPLES.

(1) Let (X, d) be a metric space. Then the weak topology for X determined by all functions $x \mapsto d(x, y)$ as y varies through X is the usual metric topology on X, as we readily check from the definitions.

(2) Let X be a normed linear space with field of scalars $\mathbb{F}$, such as an L^p space for $1 \le p \le \infty$, and let X^* be the vector space of continuous linear functionals on X, as introduced in Section V.9. (For $X = L^p$ with $1 \le p < \infty$ and with the assumption that the underlying measure is σ-finite, Theorem 9.19 identified X^* explicitly as $L^{p'}$, where p' is the dual index to p.) Each member x of X defines a function $f_x : X^* \to \mathbb{F}$ by the formula $f_x(x^*) = x^*(x)$. The weak topology on X^* determined by X is called the **weak-star topology** on X^* relative to X. The words "relative to X" are included in the terminology because two normed linear spaces X might have the same set X^* of continuous linear functionals. In Section V.9 we introduced a notion of weak-star convergence but no metric associated to it. In problems at the ends of Chapters VI, VIII, and IX, this kind of convergence became a powerful tool for working with harmonic functions, Poisson integrals, and positive definite functions. Later in the present chapter we shall relate topologies to convergence of sequences,[1] and it will be apparent that weak-star convergence as defined in Section V.9 is the appropriate notion of convergence for the newly defined weak-star topology.

(3) The construction in Example 2 can be transposed to other situations in which a topology is to be imposed on a vector space. For example, let X be a normed linear space with field of scalars $\mathbb{F}$ equal to $\mathbb{R}$ or $\mathbb{C}$, and let X^* be the vector space of continuous linear functionals on X. Then X^* indexes a set of functions $x^* : X \to \mathbb{F}$. The weak topology on X determined by X^* is known as the **weak topology** on X. This topology arises in some advanced situations, but we shall not have occasion to make use of it in the present volume.

(4) We have encountered three vector spaces of scalar-valued smooth functions on open sets of Euclidean space—in Section III.2 the space $C^\infty(U)$ of all smooth functions on U, in Section VIII.4 the space $C^\infty_{\text{com}}(U)$ of all smooth functions on U with compact support contained in U, and in Section VIII.4 the space $\mathcal{S}(\mathbb{R}^N)$ of Schwartz functions defined on $\mathbb{R}^N$. The subject of partial differential equations makes extensive use of functions of all three of these kinds, and it is necessary to be able to discuss convergence for them. The easiest convergence to describe is for $C^\infty(U)$, where convergence is to mean uniform convergence of the function and all of its partial derivatives on each compact subset of U. Uniform convergence by itself is captured by the supremum norm, and somehow we want to work here with the supremum norms of the function and each of its partial derivatives on each compact subset. The appropriate topology turns out to be the weak topology determined by all the functions $f \mapsto \|f - g\|$, where $\|\cdot\|$ is the supremum of some iterated partial derivative on some compact subset of U. This construction is carried out in detail in the companion volume, *Advanced Real Analysis*. A topology for the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ is obtained in a qualitatively similar way.

[1] And to "nets," which are a generalization of sequences.

A topology for $C^\infty_{\text{com}}(U)$ is more subtle, and it too is constructed in the companion volume.

The second general construction of topological spaces is the "quotient topology" for the set of equivalence classes on X when X is a topological space and some equivalence relation[2] has been specified on X. If the relation is written as $\sim$, the set of equivalence classes may be written as $X/\sim$, and the **quotient map**, i.e., passage from each member of X to its equivalence class, is a well-defined function $q : X \to X/\sim$. With a topology in place on X, define a subset U of $X/\sim$ to be open if $q^{-1}(U)$ is open. Since inverse images of functions preserve set-theoretic operations, it is immediate that the resulting collection of open subsets of $X/\sim$ is a topology for $X/\sim$ and that this topology makes q continuous. This topology is called the **quotient topology** for $X/\sim$. In any other topology $\mathcal{T}'$ on $X/\sim$, any subset V of $X/\sim$ that is open in $\mathcal{T}'$ but not open in the quotient topology must have the property that $q^{-1}(V)$ is not open; this condition implies that q is not continuous when $\mathcal{T}'$ is the topology on $X/\sim$. Therefore the quotient topology is the **finest topology** on $X/\sim$ that makes the quotient map continuous—in the sense that it contains all topologies making q continuous.

EXAMPLES.

(1) Let (X, d) be a pseudometric space such as the set of all integrable functions on some measure space $(S, \mathcal{A}, \mu)$ with $d(g, h) = \int_S |g - h|\, d\mu$. The pseudometric on X gives X a topology. For x and y in X, define $x \sim y$ if $d(x, y) = 0$. The result is an equivalence relation, and we know from Proposition 2.12 that the pseudometric d descends to be a metric on the set $X/\sim$ of equivalence classes. The quotient topology on $X/\sim$ coincides with the topology defined by this metric.

(2) Let X be the interval $[-\pi, \pi]$ with its usual topology from the metric on $\mathbb{R}$, let S^1 be the unit circle in $\mathbb{C}$ with its usual topology from the metric on $\mathbb{C}$, and let $q : X \to S^1$ be given by $q(x) = e^{ix}$. We can consider S^1 as the set of equivalence classes of X under the relation that lets $-\pi$ and π be the only nontrivial pair of elements of X that are equivalent. The function q is continuous, and it carries compact sets to compact sets. In Problem 11 at the end of the chapter, we shall see that q exhibits S^1 as having the quotient topology.

(3) Let X be the line $\mathbb{R}$ with its usual metric, let S^1 be the unit circle as in the previous example, and let $q : X \to S^1$ be given by $q(x) = e^{ix}$. The domain X is a group, and the function q identifies S^1 set-theoretically as the quotient group $\mathbb{R}/2\pi\mathbb{Z}$, where $\mathbb{Z}$ is the subgroup of integers. This example illustrates the natural

[2]Equivalence relations and their connection with equivalence classes are discussed in Section A6 of Appendix A.

topology to impose on any quotient of a group when the group has a topology for which all translations are homeomorphisms.[3]

In many situations the problem of describing what sets are to be open sets in a topological space is simplified by the notion of a base for a topology. By a **base** $\mathcal{B}$ for the topology $\mathcal{T}$ on X is meant a subfamily of members of $\mathcal{T}$ such that every member of $\mathcal{T}$ is a union of sets in $\mathcal{B}$. In Chapter II the topology for a metric space was really introduced by specifying that the family of all open balls is to be a base. Arguing as with Proposition 2.31, we obtain the following result.

Proposition 10.1. A family $\mathcal{B}$ of subsets of a nonempty set X is a base for some topology $\mathcal{T}$ on X if and only if

(a) $X = \bigcup_{B \in \mathcal{B}} B$ and
(b) whenever U and V are in $\mathcal{B}$ and x is in $U \cap V$, then there is a B in $\mathcal{B}$ such that x is in B and $B \subseteq U \cap V$.

In this case the topology $\mathcal{T}$ is necessarily the set of all unions of members of $\mathcal{B}$, and hence $\mathcal{T}$ is determined by $\mathcal{B}$. A family $\mathcal{B}$ of subsets of X is a base for a particular given topology $\mathcal{T}_0$ on X if and only if (a) holds and

(b′) for each $x \in X$ and member U of $\mathcal{T}_0$ containing x, there is some member B of $\mathcal{B}$ such that x is in B and B is contained in U.

REMARK. Condition (b) is satisfied if $\mathcal{B}$ is closed under finite intersections. Thus any family of subsets of X that is closed under finite intersections and has union X is a base for some topology on X.

A topological space $(X, \mathcal{T})$ is said to be **separable** if $\mathcal{T}$ has a base consisting of only countably many sets.[4] A separable metric space has a countable base consisting entirely of open balls.

As with metric spaces, there is a natural definition of subspaces for general topological spaces. If $(X, \mathcal{T})$ is a topological space and if A is a nonempty subset of X, then the **relative topology** for A is the family of all sets $U \cap A$ with U in $\mathcal{T}$. We can write $\mathcal{T} \cap A$ for this family. It is a simple matter to check that $\mathcal{T} \cap A$ is indeed a topology for A, and we say that $(A, \mathcal{T} \cap A)$ is a **topological subspace** of $(X, \mathcal{T})$. If there is no possibility of confusion and if the relative topology is understood, we may say that "A is a subspace of X."

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[3] The definition of "topological group," which is given in the companion volume, *Advanced Real Analysis*, imposes further conditions beyond the fact that every translation is a homeomorphism.

[4] Some authors use the word "separable" to mean that X has a countable dense set, but the meaning in the text here is becoming more and more common. The existence of a countable dense set is not a particularly useful property for a general topological space.

Proposition 10.2. If A and B are subspaces of a topological space X with $B \subseteq A \subseteq X$, then the relative topology of B considered as a subspace of X is identical to the relative topology of B considered as a subspace of A.

PROOF. The relative topology of B considered as a subspace of X consists of all sets $U \cap B$ with U open in X, and the relative topology of B considered as a subspace of A consists of all sets $(U \cap A) \cap B$ with U open in X. Thus the result follows from the identity $(U \cap A) \cap B = U \cap (A \cap B) = U \cap B$. □

The next two propositions are proved in the same way as Proposition 2.26 and Corollary 2.27.

Proposition 10.3. If A is a subspace of a topological space X, then the closed sets of A are all sets $F \cap A$, where F is closed in X. Consequently B is closed in A if and only if $B = B^{\text{cl}} \cap A$.

Proposition 10.4. If X and Y are topological spaces and $f : X \to Y$ is continuous at a point a of a subspace A of X, then the restriction $f\big|_A : A \to Y$ is continuous at a. Also, f is continuous at a if and only if the function $f_0 : X \to f(X)$ obtained by redefining the range to be the image is continuous at a.

2. Properties of Topological Spaces

Proposition 2.30 listed certain properties of metric spaces as "separation properties." These properties are not shared by all topological spaces, and instead we list them in this section as definitions. After giving the definitions, we shall examine implications among them and some roles that they play. The disproofs of certain implications provide an opportunity to introduce some further examples of topological spaces beyond those obtained from the constructions in Section 1.

Let $(X, \mathcal{T})$ be a topological space. We say that

- (i) X is a $\mathbf{T}_1$ **space** if every one-point set in X is closed,
- (ii) X is **Hausdorff** if for any two distinct points x and y of X, there are disjoint open sets U and V with $x \in U$ and $y \in V$,
- (iii) X is **regular** if for any point $x \in X$ and any closed set $F \subseteq X$ with $x \notin F$, there are disjoint open sets U and V with $x \in U$ and $F \subseteq V$,
- (iv) X is **normal** if for any two disjoint closed subsets E and F of X, there are disjoint open sets U and V such that $E \subseteq U$ and $F \subseteq V$.

Proposition 2.30 listed one further property of an arbitrary metric space X, namely that any two disjoint closed sets can be separated by a continuous function from X into [0, 1]. Urysohn's Lemma in Section 7 will establish this property for any normal topological space.

Proposition 10.5. If $(X, \mathcal{T})$ is a topological space, then

(a) X is $\mathbf{T}_1$ if and only if for any pair of distinct points x and y, there are open sets U and V such that $x \in U$, $y \notin U$, $x \notin V$, and $y \in V$,
(b) X is regular if and only if for any point x and any closed set F with $x \notin F$, there is an open set U such that $x \in U$ and $U^{\text{cl}} \cap F = \varnothing$,
(c) X is normal if and only if for any pair of disjoint closed sets E and F, there is an open set U such that $E \subseteq U$ and $U^{\text{cl}} \cap F = \varnothing$.

PROOF. If X is $\mathbf{T}_1$ and if x and y are given, we can choose $U = \{y\}^c$ and $V = \{x\}^c$. In the reverse direction, if x is given, choose, for each $y \neq x$, an open set V_y such that $x \notin V_y$ and $y \in V_y$; then $\{x\}^c = \bigcup_y V_y$ is open, and hence $\{x\}$ is closed.

If X is regular and if x and F are given, we can choose disjoint open sets U and V with $x \in U$ and $F \subseteq V$. Then the closed set V^c has $V^c \supseteq U$ and $V^c \cap F = \varnothing$; therefore also $V^c \supseteq U^{\text{cl}}$ and $U^{\text{cl}} \cap F = \varnothing$. In the reverse direction, suppose that x and F are given and that U is an open set with $x \in U$ and $U^{\text{cl}} \cap F = \varnothing$; choosing $V = (U^{\text{cl}})^c$, we see that $x \in U$, $F \subseteq V$, and $U \cap V = \varnothing$.

If X is normal and if E and F are given, we can choose disjoint open sets U and V with $E \subseteq U$ and $F \subseteq V$. Then the closed set V^c has $V^c \supseteq U$ and $V^c \cap F = \varnothing$; therefore also $V^c \supseteq U^{\text{cl}}$ and $U^{\text{cl}} \cap F = \varnothing$. In the reverse direction, suppose that E and F are given and that U is an open set with $E \subseteq U$ and $U^{\text{cl}} \cap F = \varnothing$; choosing $V = (U^{\text{cl}})^c$, we see that $E \subseteq U$, $F \subseteq V$, and $U \cap V = \varnothing$. □

Proposition 10.6. If $(X, \mathcal{T})$ is a topological space and

(a) if X is $\mathbf{T}_1$ and normal, then X is regular,
(b) if X is $\mathbf{T}_1$ and regular, then X is Hausdorff,
(c) if X is Hausdorff, then X is $\mathbf{T}_1$.

PROOF. In (a), if x and a disjoint closed set F are given, then $\{x\}$ is closed, and the fact that X is normal implies that we can separate the closed sets $\{x\}$ and F by disjoint open sets. In (b), if x and y are distinct points in X, then $\{y\}$ is closed and the fact that X is regular implies that we can separate the point x and the disjoint closed set $\{y\}$ by disjoint open sets. In (c), the fact that X is Hausdorff means that for any two distinct points x and y, there are disjoint open sets U and V with $x \in U$ and $y \in V$. Then X satisfies the condition in Proposition 10.5a that was shown to be equivalent to the $\mathbf{T}_1$ property. □

EXAMPLES.

(1) A space that is not $\mathbf{T}_1$, regular, or normal. Let $X = \{a, b, c\}$, and let $\mathcal{T} = \{\varnothing,\ \{a\},\ \{a, b\},\ \{a, c\},\ \{a, b, c\}\}$.

(2) A space that is $\mathbf{T}_1$ but not Hausdorff. Let X be an infinite set, and let $\mathcal{T}$ consist of the empty set and all complements of finite sets.

(3) A Hausdorff space that is not regular. Let X be the real line. A subset U of X is to be in $\mathcal{T}$ if for each point x of U, there is an open interval I_x containing x such that every rational number in I_x is in U. Then every open interval is in $\mathcal{T}$, and hence X is certainly Hausdorff. On the other hand, the set of rationals is open in this topology, and therefore the set of irrationals is closed. The set of irrationals cannot be separated from the point 0 by disjoint open sets.

(4) A Hausdorff regular space that is not normal. Let X be the closed upper half plane $\{\operatorname{Im} z \geq 0\}$ in $\mathbb{C}$. A base for $\mathcal{T}$ consists of all open disks in X that do not meet the x axis, together with all open disks in X that are tangent to the x axis; the latter sets are to include the point of tangency. It is easy to see that X is Hausdorff, but a little argument is needed to see that X is regular. To begin with, every open set in the usual metric topology for X is in $\mathcal{T}$, and hence every closed set in the usual metric topology for X is closed relative to $\mathcal{T}$. Let p be a point in X, and let F be a $\mathcal{T}$ closed subset of X not containing p. There is no difficulty in separating p and F by disjoint open sets if p has y coordinate positive, and we therefore assume that p lies on the x axis. Since F is closed, Proposition 10.1 produces a basic open set U tangent to the x axis at p such that $U \cap F = \varnothing$. If D denotes a strictly smaller basic open set tangent to the x axis at p, then the only point of the ordinary boundary of U that lies in D^{cl} is p itself. Thus $F \cap D^{\mathrm{cl}} = \varnothing$, and it follows that D and $(D^{\mathrm{cl}})^c$ are disjoint open sets separating p and F. Consequently X is regular. We postpone the argument that X is not normal until Section 7, when Urysohn's Lemma will be available.

(5) A normal space that is not regular. Let $X = \{a, b\}$, and let $\mathcal{T}$ consist of $\varnothing$, $\{a\}$, and $\{a, b\}$.

We shall see in Section 5 that the Hausdorff property is exactly the right condition to make limits be unique, hence to allow a reasonable notion of convergence. Also, in the construction of a quotient space, it is often a subtle matter to decide whether the quotient space is Hausdorff; we shall obtain sufficient conditions in Section 6.

The property of regularity makes possible a generalization of the passage from a pseudometric space of points to a metric space of equivalence classes. The point of departure is the following proposition; we shall examine the resulting quotient space further in Section 6.

Proposition 10.7. Let X be a regular topological space. For points x and y in X, define $x \sim y$ if x is in $\{y\}^c$. Then $\sim$ is an equivalence relation.

PROOF. Certainly x lies in $\{x\}^{\text{cl}}$, and if x lies in $\{y\}^{\text{cl}}$ and y lies in $\{z\}^{\text{cl}}$, then x lies in $\{z\}^{\text{cl}}$. For the symmetry property, we argue by contradiction and use the regularity of X. Suppose that x lies in $\{y\}^{\text{cl}}$ but y does not lie in $\{x\}^{\text{cl}}$. Regularity allows us to find disjoint open sets U and V such that $y \in U$ and $\{x\}^{\text{cl}} \subseteq V$. Then the closed set V^c contains y and hence also $\{y\}^{\text{cl}}$. Since x lies in $\{y\}^{\text{cl}}$, x lies in V^c. But this relationship contradicts the fact that x lies in V. We conclude that $\sim$ is symmetric and is therefore an equivalence relation. □

Subspaces of topological spaces inherit certain properties if the original space has them. Among these are $\mathbf{T}_1$, Hausdorff, and separable. A subspace of a normal space need not be normal, as is seen by taking $X = \{a, b, c, d\}$, and $\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c, d\}\}$, the subspace being $\{a, b, c\}$ and the relatively closed subsets of interest being $\{b\}$ and $\{c\}$. Let us state the result for regularity as a proposition.

Proposition 10.8. A subspace of a regular topological space is regular.

PROOF. Within a subspace A of X, let F be a relatively closed set, and let x be a point of A not in F. By Proposition 10.3 we have $F = F^{\text{cl}} \cap A$, the closure being taken in X. Since x is in A but not F, x is not in F^{cl}. Since X is regular, we can find disjoint open sets U and V in X with $x \in U$ and $F^{\text{cl}} \subseteq V$. Then $U \cap A$ and $V \cap A$ are disjoint relatively open sets containing x and F. □

As with metric spaces, a subset D of a topological space X is **dense in** A if $D^{\text{cl}} \supseteq A$; D is **dense** if D is dense in X. A set D is dense if and only if there is some point of D in each nonempty open set of X. If X is separable, then X has a countable dense set; we have only to select one point from each nonempty member of the base.

The properties of bases of a topological space X become more transparent with the aid of the notion of a local base. A set $\mathcal{U}_x$ of open neighborhoods of x is a **local base** at x if each open set containing x contains some member of $\mathcal{U}_x$. If $\mathcal{B}$ is a base, then the members of $\mathcal{B}$ containing x form a local base at x. Conversely if $\mathcal{U}_x$ is a local base for each x, then the union of all the $\mathcal{U}_x$'s is a base. We say that X has a **countable local base** at each point[5] if a countable such $\mathcal{U}_x$ can be chosen for each x in X. Metric spaces have this property; the open balls of rational radii centered at a point form a local base at the point.

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[5]Some authors say instead that "X satisfies the first axiom of countability" or "X is first countable" if this condition holds. In the same kind of terminology, one says that "X satisfies the second axiom of countability" or "X is second countable" if X is separable in the sense of Section 1.

EXAMPLE 4, CONTINUED. A space that has a countable dense set and has a countable local base at each point and yet is not separable. As in Example 4 earlier in this section, let X be the closed upper half plane $\{\operatorname{Im} z \geq 0\}$ in $\mathbb{C}$. A base for $\mathcal{T}$ consists of all open disks in X that do not meet the x axis, together with all open disks in X that are tangent to the x axis; the latter sets are to include the point of tangency. For a point p on the x axis, the open disks of rational radii with point of tangency p form a countable local base, and for a point p off the x axis, the open disks within the open upper half plane having center p and rational radius form a countable local base. A countable dense set consists of all points with rational coordinates and with y coordinate positive. We shall see in Corollary 10.10 in the next section that a separable regular space has to be normal, and this X is not normal, according to the statement in Example 4 and the proof to be given in Section 7. Thus X cannot be separable.

3. Compactness and Local Compactness

Let X be a topological space. In this section we carry over to a general topological space X some definitions made in Section II.7 for metric spaces. A collection $\mathcal{U}$ of open sets is an **open cover** of X if its union is X. An **open subcover** of $\mathcal{U}$ is a subset of $\mathcal{U}$ that is itself an open cover.

We begin with a new term, saying that the topological space X is a **Lindelöf space** if every open cover of X has a countable subcover. Proposition 2.32 showed that a metric space X is separable if and only if X is a Lindelöf space. For general topological spaces it is still true that any separable X is a Lindelöf space, by the same argument as for the implication that condition (a) implies condition (b) in Proposition 2.32. In fact, every subspace of a separable space is separable, and hence every subspace of a separable space is Lindelöf. However, a Lindelöf space need not be separable, as the following example shows rather emphatically.

EXAMPLE. We construct a topological space $(X, \mathcal{T})$ that is Hausdorff and normal, has a countable dense set, has a countable local base at each point, is Lindelöf, yet is not separable. Take X as a set to be the real line. The intersection of any two bounded intervals of the form $[a, b)$ is an interval of the same kind, and the union of all such intervals is the whole line. Hence the bounded intervals $[a, b)$ form a base for some topology on the line, and this topology we take to be $\mathcal{T}$. It is called the **half-open interval topology** for the real line. Since every ordinary open interval of the line is the union of intervals $[a, b)$, any open set in the usual metric topology is open in the half-open interval topology. Any two distinct points of X may be separated by ordinary disjoint open intervals, and therefore X is Hausdorff. To see that X is regular, let a point x and a closed set

F with x not in F be given. Since x is in the open set F^c, some $[x, x+\epsilon)$ is disjoint from F. Then $U = [x, x+\epsilon)$ and $V = (-\infty, x) \cup [x+\epsilon, +\infty)$ are disjoint open sets separating x and F, and we conclude that X is regular. Once we prove that X is Lindelöf, it will follow from Proposition 10.9 below that X is normal. The rationals form a countable dense subset of X, and the set of all intervals $\big[x, x+\frac{1}{n}\big)$ is a countable local base at x. The space X is not separable. In fact, if $\mathcal{B}$ is any base, we can find, for each x, some open neighborhood B_x of x that is in $\mathcal{B}$ and is contained in $[x, x+1)$. If $x < y$, then x cannot lie in B_y and hence $B_x \neq B_y$; therefore $\mathcal{B}$ has to be uncountable. Finally let us see that X is Lindelöf. Let an open cover $\mathcal{U}$ of X be given, and fix a negative real number x_0. Consider the set $S(x_0)$ of all real numbers x such that some countable collection of members of $\mathcal{U}$ covers $[x_0, x]$. Since x_0 is covered by some member of $\mathcal{U}$, the set $S(x_0)$ contains x_0. If the set contains an element x_1, then the member of the countable collection that covers x_1 must contain $[x_1, x_1+\epsilon)$ for some $\epsilon > 0$. Thus $x_1 + \frac{\epsilon}{2}$ is in $S(x_0)$, and $S(x_0)$ contains no largest element. We shall show that $S(x_0) = [x_0, +\infty)$. If the contrary is true, then $S(x_0)$ must be bounded. In this case, let c be the least upper bound. For large enough n, $c - \frac{1}{n}$ is in $S(x_0)$. Taking the union of the countable collections that cover $\big[x_0, c-\frac{1}{n}\big]$, together with one more set to cover c, we obtain a countable collection that covers $[x_0, c]$, and we see that c is in $S(x_0)$. Since c is in $S(x_0)$, we have a contradiction to the fact that $S(x_0)$ contains no largest element. We conclude that some countable subcollection of $\mathcal{U}$ covers $[x_0, +\infty)$, no matter what x_0 is. Taking the union of the countable subcollections corresponding to each negative integer, we obtain a countable subcollection of $\mathcal{U}$ covering $(-\infty, +\infty)$. Thus X is Lindelöf.

It is not always so obvious when a topological space is normal. The next result provides one sufficient condition.

Proposition 10.9 (Tychonoff's Lemma). Every regular Lindelöf space is normal.

PROOF. Let X be regular and Lindelöf, and let disjoint closed subsets E and F of X be given. By regularity and Proposition 10.5b each point of E has an open neighborhood whose closure is disjoint from F. Therefore the class $\mathcal{U}$ of open sets with closures disjoint from F covers E. Similarly the class $\mathcal{V}$ of open sets with closures disjoint from E covers F. Thus $\mathcal{U} \cup \mathcal{V} \cup \{X - (E \cup F)\}$ is an open cover of X. Since X is Lindelöf, there exist sequences of sets U_n in $\mathcal{U}$ and V_n in $\mathcal{V}$ such that $E \subseteq \bigcup_{n=1}^{\infty} U_n$ and $F \subseteq \bigcup_{n=1}^{\infty} V_n$. Put

$$U_n' = U_n - \bigcup_{k \leq n} V_k^{\mathrm{cl}} \qquad \text{and} \qquad V_n' = V_n - \bigcup_{k \leq n} U_k^{\mathrm{cl}}.$$

When $m \le n$, we have $V_m \subseteq \bigcup_{k\le n} V_k^{\mathrm{cl}}$. Then $U_n' \cap V_m = \varnothing$, and hence the smaller set $U_n' \cap V_m'$ is empty. Reversing the roles of the U's and the V's shows that $U_n' \cap V_m'$ is empty for $m \ge n$. Therefore $U_n' \cap V_m' = \varnothing$ for all n and m. Define

$$U = \bigcup_{n=1}^{\infty} U_n' \qquad \text{and} \qquad V = \bigcup_{m=1}^{\infty} V_m'.$$

Then $U \cap V = \bigcup_{n,m} (U_n' \cap V_m') = \varnothing$. Also,

$$E \cap U = E \cap \bigcup_{n=1}^{\infty} \Big(U_n - \bigcup_{k\le n} V_k^{\mathrm{cl}}\Big) \supseteq E \cap \bigcup_{n=1}^{\infty} \Big(U_n - \bigcup_{k=1}^{\infty} V_k^{\mathrm{cl}}\Big) = E \cap \Big(X - \bigcup_{k=1}^{\infty} V_k^{\mathrm{cl}}\Big),$$

the last equality holding since $\{U_n\}$ covers E. The right side here equals E since $V_k^{\mathrm{cl}} \subseteq X - E$ for all k, and therefore $E \subseteq U$. Similarly $F \subseteq V$. The proof is complete. □

Corollary 10.10. Every regular separable space is normal.

PROOF. A separable space is automatically Lindelöf, and thus the corollary follows from Proposition 10.9. □

Let us return to the concluding example in Section 2, in which X as a set is the closed upper half plane $\{\operatorname{Im} z \ge 0\}$ but in which the topology is nonstandard near the real axis. It was shown in Section 2 that this particular X is regular, and it was stated that Urysohn's Lemma would be used in Section 7 to show that X is not normal. By Corollary 10.10, X cannot be separable. This completes the argument that X has a countable dense set and has a countable local base at each point yet is not separable.

We can now proceed with carrying over some definitions from Section II.7, valid there for metric spaces, to a general topological space X. We call X **compact** if every open cover of X has a finite subcover. A subset E of X is **compact** if it is compact as a subspace of X, i.e., if every collection of open sets in X whose union contains E has a finite subcollection whose union contains E. It is immediate from the definition that the union of two compact subsets is compact.

This definition generalizes the property of closed bounded sets of $\mathbb{R}^n$ given by the Heine–Borel Theorem. We shall see that the Heine–Borel property, rather than the Bolzano–Weierstrass property for sequences, is the useful property to carry over to more general situations in real analysis. In fact, in several places in this book, we have combined an iterated application of the Bolzano–Weierstrass property with the Cantor diagonal process to obtain some conclusion. This construction is tantamount to proving that the product of countably many compact metric spaces, which is a metric space essentially by Proposition 10.28 below, is compact. There will be situations for which we want to consider an uncountable product of compact metric spaces, and then arguments with sequences are not decisive. Instead, it is the Heine–Borel property that is relevant. The Tychonoff Product Theorem of Section 4 will be the substitute for the Cantor diagonal process, and the use of nets, considered in Section 5, will be analogous to the use of sequences.

A number of the simpler results in Section II.7 generalize easily from compact metric spaces to all compact topological spaces or at least to all compact Hausdorff spaces. We list those now. A consequence of Proposition 10.12 below is that compactness is preserved under homeomorphisms.

A set of subsets of a nonempty set is said to have the **finite-intersection property** if each intersection of finitely many of the subsets is nonempty.

Proposition 10.11. A topological space X is compact if and only if each set of closed subsets of X with the finite-intersection property has nonempty intersection.

PROOF. Closed sets with the finite-intersection property have complements that are open sets, no finite subcollection of which is an open cover. □

Proposition 10.12. Let X and Y be topological spaces with X compact. If $f : X \to Y$ is continuous, then $f(X)$ is a compact subset of Y.

PROOF. If $\{U_\alpha\}$ is an open cover of $f(X)$, then $\{f^{-1}(U_\alpha)\}$ is an open cover of X. Let $\{f^{-1}(U_j)\}_{j=1}^n$ be a finite subcover. Then $\{U_j\}_{j=1}^n$ is a finite subcover of $f(X)$. □

Corollary 10.13. Let X be a compact topological space, and let $f : X \to \mathbb{R}$ be a continuous function. Then f attains its maximum and minimum values.

PROOF. By Proposition 10.12, $f(X)$ is a compact subset of $\mathbb{R}$. Arguing as in the proof of Corollary 2.39, we see that $f(X)$ has a finite supremum and a finite infimum and that both of these must lie in $f(X)$. □

Proposition 10.14. A closed subset of a compact topological space is compact.

PROOF. Let E be a closed subset of the compact space X, and let $\mathcal{U}$ be an open cover of E. Then $\mathcal{U} \cup \{E^c\}$ is an open cover of X. Passing to a finite subcover and discarding E^c, we obtain a finite subcover of E. Thus E is compact. □

Lemma 10.15. Let K and E be subsets of a topological space X, and let K be compact. Suppose that to each point x of K there are disjoint open sets U_x and V_x such that x is in U_x and $E \subseteq V_x$. Then there exist disjoint open sets U and V such that $K \subseteq U$ and $E \subseteq V$.

PROOF. As x varies through K, the open sets U_x form an open cover of K. By compactness, a finite subcollection of the U_x's is a cover, say $U_{x_1}, \dots, U_{x_n}$. Put $U = \bigcup_{k=1}^n U_{x_k}$ and $V = \bigcap_{k=1}^n V_{x_k}$. Then $K \subseteq U$ and $E \subseteq V$. Also, $U \cap V = \big(\bigcup_{k=1}^n U_{x_k}\big) \cap \big(\bigcap_{k=1}^n V_{x_k}\big) = \bigcup_{k=1}^n \big(U_{x_k} \cap \big(\bigcap_{l=1}^n V_{x_l}\big)\big) \subseteq \bigcup_{k=1}^n (U_{x_k} \cap V_{x_k}) = \varnothing$, and thus U and V have the required properties. □

Proposition 10.16. Every compact Hausdorff space is regular and normal.

PROOF. Let X be compact Hausdorff. If a point x and a closed set F with $x \notin F$ are given, we observe by Proposition 10.14 that F is compact. The Hausdorff property of X allows us to take $E = \{x\}$ and $K = F$ in Lemma 10.15, and we obtain disjoint open sets U and V such that x is in V and $F \subseteq U$. Thus X is regular.

If disjoint closed sets E and F are given, then F is compact by Proposition 10.14. The fact that X has been shown to be regular allows us to take $K = F$ in Lemma 10.15, and we obtain disjoint open sets U and V such that $E \subseteq V$ and $F \subseteq U$. Thus X is normal. □

Proposition 10.17. In a Hausdorff space every compact set is closed.

PROOF. Let X be a Hausdorff space, and let K be a compact subset of X. Fix x in K^c. The Hausdorff property of X allows us to take $E = \{x\}$ in Lemma 10.15, and we obtain disjoint open sets U_x and V_x such that x is in V_x and $K \subseteq U_x$. Letting x now vary, we see that $K^c = \bigcup_{x \in K^c} V_x$. Hence K^c is open and K is closed. □

Corollary 10.18. Let X and Y be topological spaces with X compact and with Y Hausdorff. If $f : X \to Y$ is continuous, one-one, and onto, then f is a homeomorphism.

PROOF. We are to show that $f^{-1} : Y \to X$ is continuous. Let E be a closed subset of X, and consider $(f^{-1})^{-1}(E) = f(E)$. The set E is compact in X by Proposition 10.14, $f(E)$ is compact by Proposition 10.12, and $f(E)$ is closed by Proposition 10.17. Since the inverse image under f^{-1} of any closed set is closed, f^{-1} is continuous. □

A topological space is **locally compact** if every point has a compact neighborhood. Compact spaces are locally compact, but the real line with its usual topology is locally compact and not compact. In a sense to be made precise in the next two propositions, locally compact Hausdorff spaces are just one point away from being compact Hausdorff.

Let $(X, \mathcal{T})$ be an arbitrary topological space. Define a new set X^* by $X^* = X \cup \{\infty\}$, where ∞ is not already a member of X, and define $\mathcal{T}^*$ to be the union of $\mathcal{T}$ and the set of all complements in X^* of closed compact subsets of X. We shall verify in Proposition 10.19 that $\mathcal{T}^*$ is a topology for X^*. The topological space $(X^*, \mathcal{T}^*)$ is called the **one-point compactification** of $(X, \mathcal{T})$. By way of examples, the one-point compactification of $\mathbb{R}$ may be visualized as a circle and the one-point compactification of $\mathbb{R}^2$ may be visualized as a sphere.

Proposition 10.19. If $(X, \mathcal{T})$ is a topological space, then $(X^*, \mathcal{T}^*)$ is a compact topological space, X is an open subset of X^*, and the relative topology for X in X^* is $\mathcal{T}$.

PROOF. To see that $\mathcal{T}^*$ is a topology, we observe first that $\varnothing$ and X^* are in $\mathcal{T}^*$. If U and V are in $\mathcal{T}^*$, there are three cases in checking that $U \cap V$ is in $\mathcal{T}^*$: If U and V are both in $\mathcal{T}$, then $U \cap V$ is in $\mathcal{T}$ since $\mathcal{T}$ is closed under finite intersections. If U is in $\mathcal{T}$ and V is not, then V^c is closed compact in X, and $X - V^c$ is thus open in X; since $\mathcal{T}$ is closed under finite intersections, $U \cap V = U \cap (X - V^c)$ is in $\mathcal{T}$. If U and V are not in $\mathcal{T}$, then the complements U^c and V^c in X^* are closed compact subsets of X; so is their union $(U \cap V)^c$, and hence $U \cap V$ is in $\mathcal{T}^*$.

We still have to check closure of $\mathcal{T}^*$ under arbitrary unions. Suppose that U_α is in $\mathcal{T}$ for α in an index set A and V_β has closed compact complement for β in an index set B. Then $\bigcup_{\alpha \in A} U_\alpha$ is in $\mathcal{T}$, and if B is nonempty, $\bigcap_{\beta \in B} V_\beta^c$ is a closed subset of one V_β^c and hence is compact; in this case, $\big(\bigcup_{\beta \in B} V_\beta\big)^c$ is closed compact in X, and hence $\bigcup_{\beta \in B} V_\beta$ is in $\mathcal{T}^*$. Thus we have only to check that $U \cup V$ is in $\mathcal{T}^*$ if U is in $\mathcal{T}$ and V^c is closed compact in X. As the intersection of two closed sets, one of which is compact, $(X - U) \cap V^c = (X - U) \cap (X - V)$ is closed and compact in X, and thus $U \cup V = ((X - U) \cap V^c)^c$ is in $\mathcal{T}^*$. Thus $\mathcal{T}^*$ is a topology.

To see that X^* is compact, let $\mathcal{U}$ be an open cover of X^*. Find some V in $\mathcal{U}$ containing the point ∞. The members of $\mathcal{U} \cap \mathcal{T}$ cover the compact subset V^c of X, and there is a finite subcollection $\mathcal{V}$ that covers V^c. Then $\mathcal{V} \cup \{V\}$ is a finite subcollection of $\mathcal{U}$ that covers X^*.

The set X is in $\mathcal{T}$ and is therefore in $\mathcal{T}^*$. Thus X is open in X^*. To complete the proof, we are to show that $\mathcal{T}^* \cap X = \mathcal{T}$. We know that $\mathcal{T}^* \cap X \supseteq \mathcal{T}$. If V is a member of $\mathcal{T}^*$ that does not lie in $\mathcal{T}$, then V^c is closed compact in X, and its complement $X - V^c = V \cap X$ in X is open in X. Hence $V \cap X$ is in $\mathcal{T}$.

Proposition 10.20. If X^* is the one-point compactification of a topological space X, then X^* is Hausdorff if and only if X is both locally compact and Hausdorff.

PROOF. Suppose that X is locally compact and Hausdorff. Since X is Hausdorff, any two points of X can be separated by disjoint open sets in X, and these sets will be open in X^*. To separate a point x in X from ∞, let C be a compact neighborhood of x in X. Since X is Hausdorff, C is closed in X. Thus C^c is in $\mathcal{T}^*$. Then C^o and C^c are disjoint open sets in X^* such that x is in C^o and ∞ is in C^c, and X^* is Hausdorff.

Conversely suppose that X^* is Hausdorff. Proposition 10.19 shows that X is a subspace of X^*. Since any subspace of a Hausdorff space is Hausdorff, X is Hausdorff. To see that X is locally compact, let x be in X, and find disjoint open sets U and V in X^* such that x is in U and ∞ is in V. Then U must be in $\mathcal{T}$, and V^c must be closed compact in X. Since $U \cap V = \varnothing$, $U \subseteq V^c$. This inclusion exhibits V^c as a compact neighborhood of x, and thus X is locally compact. □

Corollary 10.21. Every locally compact Hausdorff space is regular.

PROOF. If X is locally compact Hausdorff, Propositions 10.19 and 10.20 show that the one-point compactification X^* is compact Hausdorff and allow us to regard X as a subspace of X^*. Proposition 10.16 shows that X^* is regular, and Proposition 10.8 shows that X is therefore regular. □

A locally compact Hausdorff space need not be normal; an example is given in Problem 5 at the end of the chapter. The remainder of this section concerns senses in which a locally compact Hausdorff space is almost normal.

Corollary 10.22. If K and F are disjoint closed sets in a locally compact Hausdorff space and if K is compact, then there exist disjoint open sets U and V such that $K \subseteq U$ and $F \subseteq V$.

PROOF. This is immediate from Lemma 10.15 and Corollary 10.21. □

Corollary 10.23. If K is a compact set in a locally compact Hausdorff space, then there is a compact set L such that $K \subseteq L^o$.

PROOF. Let X be locally compact Hausdorff, and form the one-point compactification X^*. Since X^* is compact Hausdorff by Proposition 10.20, Proposition 10.17 shows that K is closed in X^* and Proposition 10.16 shows that X^* is regular. Thus Proposition 10.5b shows that we can find an open set U in X^* such that ∞ is in U and $U^{\text{cl}} \cap K = \varnothing$. Then $K \subseteq X^* - U^{\text{cl}} \subseteq X^* - U$. By definition of the topology of X^*, the set $L = X^* - U$ is compact in X. Its subset $X^* - U^{\text{cl}}$ is open and is therefore contained in L^o. Thus $K \subseteq L^o \subseteq L$ with L compact. □

A topological space is called σ-**compact** if there is a sequence of compact sets with union the whole space. The real line with its usual topology is σ-compact. For that matter, so is the subspace of rationals since each finite subset is compact.

Proposition 10.24. A locally compact topological space is σ-compact if and only if it is Lindelöf. Consequently every σ-compact locally compact Hausdorff space is normal.

PROOF. If X is σ-compact, write $X = \bigcup_{n=1}^{\infty} K_n$ with K_n compact. If $\mathcal{U}$ is an open cover of X, then $\mathcal{U}$ is an open cover of each K_n, and there is a finite subcover $\mathcal{U}_n$ of K_n. Then $\bigcup_{n=1}^{\infty} \mathcal{U}_n$ is a countable subcover of $\mathcal{U}$, and X is Lindelöf.

Conversely if X is locally compact and Lindelöf, choose, for each x in X, a compact neighborhood K_x of x, and let U_x be the interior of K_x. As x varies, the U_x form an open cover of X. Since X is Lindelöf, there is a countable subcover $\{U_{x_n}\}_{n=1}^{\infty}$. Since we have $U_{x_n} \subseteq K_{x_n}$ for all n, $\{K_{x_n}\}_{n=1}^{\infty}$ is a sequence of compact sets with union X. Hence X is σ-compact.

Finally if X is locally compact Hausdorff and σ-compact, hence also Lindelöf, then Corollary 10.21 shows that X is regular, and Tychonoff's Lemma (Proposition 10.9) shows that X is normal. □

Proposition 10.25. In a σ-compact locally compact Hausdorff space, there exists an increasing sequence $\{K_n\}$ of compact sets with union the whole space and with $K_n \subseteq K_{n+1}^o$ for all n.

PROOF. Let X be a locally compact Hausdorff space such that $X = \bigcup_{n=1}^{\infty} L_n$ with L_n compact. Replacing L_n by the union of the previous members of the sequence, we may assume that $L_n \subseteq L_{n+1}$ for all $n \geq 1$. Put $L_0 = K_0 = \varnothing$. Use Corollary 10.23 to choose K_1 compact with $L_1 \subseteq K_1^0$.

Inductively suppose that $n > 0$ and that for all k with $0 < k \leq n$, a compact set K_k has been defined such that $L_k \cup K_{k-1} \subseteq K_k^o$. Applying Corollary 10.23, we can find a compact set K_{n+1} such that the compact set $L_{n+1} \cup K_n$ is contained in K_{n+1}^o. Then $K_{k-1} \subseteq K_k^o$ for all $k \geq 1$ as required, and $X = \bigcup_{n=1}^{\infty} K_n$ since $K_n \subseteq L_n$ and $\bigcup_{n=1}^{\infty} L_n = X$. □

4. Product Spaces and the Tychonoff Product Theorem

The **product topology** for the product of topological spaces was discussed briefly in Section 1. If S is a nonempty set and if X_s is a topological space for each s in S, then the Cartesian product $X = \bigtimes_{s \in S} X_s$, as a set, is the set of all functions f from S into $\bigcup_{s \in S} X_s$ such that $f(s)$ is in X_s for all $s \in S$. The topology that is imposed on X is, by definition, the weakest topology that makes the s^{th} coordinate function $p_s : X \to X_s$ be continuous for every s.

Let us investigate what sets have to be open in this topology, and then we can look at examples and see better what the topology is. If U_s is any open subset of X_s, then $p_s^{-1}(U_s)$ has to be open in X since p_s is continuous. For example, if $S = \{1, 2\}$, we are considering $X = X_1 \times X_2$. A set $p_1^{-1}(U_1)$ is of the form $U_1 \times X_2$, and a set $p_2^{-1}(U_2)$ is of the form $X_1 \times U_2$. These have to be open if U_1 is open in X_1 and U_2 is open in X_2. The intersection of any two such sets, which is of the form $U_1 \times U_2$, has to be open in X, as well. We do not need to intersect these sets further, since $p_1^{-1}(U_1) \cap p_1^{-1}(V_1) = p_1^{-1}(U_1 \cap V_1)$. By the remark with Proposition 10.1, the sets $p_1^{-1}(U_1) \cap p_2^{-1}(U_2)$ with U_1 open in X_1 and U_2 open in X_2 form a base for some topology on $X = X_1 \times X_2$. These sets have to be open in the product topology, and p_1 and p_2 are indeed continuous in this topology. Therefore the product topology on $X = X_1 \times X_2$ has

$$\big\{p_1^{-1}(U_1) \cap p_2^{-1}(U_2) \,\big|\, U_1 \text{ open in } X_1,\, U_2 \text{ open in } X_2\big\}$$

as a base. More generally the product topology on $X = X_1 \times \cdots \times X_n$ has

$$\bigg\{\bigcap_{k=1}^{n} p_k^{-1}(U_k) \,\bigg|\, U_k \text{ open in } X_k \text{ for each } k\bigg\}$$

as a base.

When the index set S is the set of positive integers, the product $X = \bigtimes_{n \in S} X_n$, as a set, is the set of sequences $\{f(n)\}_{n \in S}$. Again any set $p_n^{-1}(U_n)$ with U_n open in X_n has to be open in X. Hence any finite intersection of such sets as n varies has to be open. But there is no need for infinite intersections of such sets to be open, and a base for the product topology in fact consists of all *finite* intersections of sets $p_n^{-1}(U_n)$ with U_n open in X_n.

The use of finite intersections, and not infinite intersections, persists for all S and gives us a description of a base for the product topology in general. When $S = [0, 1]$ and all X_s are $[0, 1]$, the description of the product topology has a helpful geometric interpretation. The set X consists of all functions from the closed unit interval to itself, and we can visualize these in terms of their graphs. A basic open set of such functions imposes restrictions at finitely many values of s, i.e., at finitely many points of the domain. At such values of s, the graph of a function in the basic open set is to pass through a certain window U_s depending on s. At all other values of s, the function is unrestricted.

Proposition 10.26. The topological product of Hausdorff topological spaces is Hausdorff.

PROOF. Let a product $X = \bigtimes_{n \in S} X_n$ be given, let $p_s : X \to X_s$ be the s^{th} coordinate function, and let two distinct members f and g of X be given. Members of X are functions of a certain kind, and these two functions, being distinct, have $f(s) \neq g(s)$ for some $s \in S$. Since X_s is Hausdorff, we can choose disjoint open sets U_s and V_s in X_s such that $f(s)$ is in U_s and $g(s)$ is in V_s. Then $p_s^{-1}(U_s)$ and $p_s^{-1}(V_s)$ are disjoint open sets in X such that f is in $p_s^{-1}(U_s)$ and g is in $p_s^{-1}(V_s)$. $\square$

Theorem 10.27 (Tychonoff Product Theorem). The topological product of compact topological spaces is compact.

REMARKS. This theorem is a fundamental tool in real analysis. We shall give the proof and then discuss how the theorem can be regarded as a generalization of the Cantor diagonal process used in the proofs earlier of the fact that any totally bounded complete metric space is compact (Theorem 2.46), the Helly Selection Principle (Problem 10 at the end of Chapter I), Ascoli's Theorem (Theorems 1.22 and 2.56), and, by implication, the Cauchy–Peano Existence Theorem for differential equations (Problems 24–29 at the end of Chapter IV). The proof will make use of Zorn's Lemma (Section A9 of Appendix A), which is one formulation of the Axiom of Choice. Actually, the Axiom of Choice arises in two more transparent ways in the proof as well. One is simply in the statement that the topological product is a topological space; for this to be the case, the product has to be nonempty, and that is the content of the Axiom of Choice. The other is the construction of a particular element x in the product that occurs near the beginning of the proof below.

PROOF. Let $X = \bigtimes_{s\in S} X_s$ be given with each X_s compact, and let $p_s : X \to X_s$ be the s^{th} coordinate function. We are to prove that any open cover of X has a finite subcover, and we begin by proving a special case. Let $\mathcal{S}$ be the family of all sets $p_s^{-1}(U_s)$ as U_s varies through all open sets of X_s and as s varies. We know that finite intersections of members of $\mathcal{S}$ form a base for the product topology on X. For the special case let $\mathcal{U}$ be an open cover of X by members of $\mathcal{S}$; we shall produce a finite subcover. For each s, let $\mathcal{B}_s$ be the family of all open sets U_s in X_s such that $p_s^{-1}(U_s)$ is in $\mathcal{U}$. We may assume for each s that no finite subfamily of $\mathcal{B}_s$ covers X_s, since otherwise the corresponding finitely many sets $p_s^{-1}(U_s)$ would cover X. By compactness of X_s, $\mathcal{B}_s$ does not cover X_s; say that x_s is not covered. The point x of X whose s^{th} coordinate is x_s then belongs to no member of $\mathcal{U}$, and $\mathcal{U}$ cannot be a cover. This contradiction shows that the special $\mathcal{U}$ has a finite subcover.

Now let $\mathcal{U}$ be any open cover of X, and suppose that no finite subfamily of $\mathcal{U}$ covers X. Let $\mathcal{C}$ be the system of all open covers $\mathcal{V}$ of X such that $\mathcal{U} \subseteq \mathcal{V}$ and such that no finite subfamily of $\mathcal{V}$ covers X. The set $\mathcal{C}$ is partially ordered by inclusion upward and is nonempty, having $\mathcal{U}$ as a member. If $\{\mathcal{V}_\alpha\}$ is a chain in $\mathcal{C}$, then we shall show that $\mathcal{V} = \bigcup_\alpha \mathcal{V}_\alpha$ is in $\mathcal{C}$ and hence is an upper bound in $\mathcal{C}$ for the chain $\{\mathcal{V}_\alpha\}$. In fact, $\mathcal{V}$ is certainly an open cover. If it has a finite subcover, then each member of the finite subcover lies in one of the covers, say $\mathcal{V}_{\alpha_j}$. Since $\{\mathcal{V}_\alpha\}$ is a chain, all members of the finite subcover lie in the largest of those $\mathcal{V}_{\alpha_j}$'s. Thus one of the $\mathcal{V}_{\alpha_j}$'s fails to be in $\mathcal{C}$, and we arrive at a contradiction. We conclude that every chain in $\mathcal{C}$ has an upper bound in $\mathcal{C}$. By Zorn's Lemma let $\mathcal{U}^*$ be a maximal cover from $\mathcal{C}$ of X.

The family $\mathcal{S} \cap \mathcal{U}^*$ of all members of $\mathcal{U}^*$ that are in the family $\mathcal{S}$ of the first paragraph of the proof has the property that no finite subfamily is a cover of X. By the result of the first paragraph, $\mathcal{S} \cap \mathcal{U}^*$ cannot be a cover of X. Hence we shall have arrived at a contradiction if we show that the union of the members of $\mathcal{U}^*$ is contained in the union of the members of $\mathcal{S} \cap \mathcal{U}^*$. Let U be a member of $\mathcal{U}^*$, and fix a point x in U. Since finite intersections of members of $\mathcal{S}$ form a base, Proposition 10.1 shows that there are members $S_1 \cap \cdots \cap S_n$ of $\mathcal{S}$ such that x is in $S_1 \cap \cdots \cap S_n$ and $S_1 \cap \cdots \cap S_n \subseteq U$. We shall show that one of the sets S_j is in $\mathcal{U}^*$, hence in $\mathcal{U}^* \cap \mathcal{S}$, and then the proof will be complete.

If S_1 is in $\mathcal{U}^*$, we are finished. Otherwise, by the maximality of $\mathcal{U}^*$, there are finitely many open sets $C_1, \ldots, C_k$ of $\mathcal{U}^*$ such that $X = S_1 \cup C_1 \cup \cdots \cup C_k$. Again by the maximality, no open set containing S_1 can belong to $\mathcal{U}^*$, since the union of that set with $C_1 \cup \cdots \cup C_k$ would be X. Proceeding inductively, suppose we have shown that no open set containing $S_1 \cap \cdots \cap S_i$ is in $\mathcal{U}^*$ and that there are open sets $D_1, \ldots, D_m$ in $\mathcal{U}^*$ with

$$X = (S_1 \cap \cdots \cap S_i) \cup (D_1 \cup \cdots \cup D_m).$$

If, as we may assume, S_{i+1} is not in $\mathcal{U}^*$, then by maximality of $\mathcal{U}^*$, there are open sets $E_1, \ldots, E_r$ in $\mathcal{U}^*$ such that $X = S_{i+1} \cup E_1 \cup \cdots \cup E_r$. Then

$$X - S_{i+1} \subseteq E_1 \cup \cdots \cup E_r,$$

and

$$\begin{aligned} S_{i+1} &= (S_1 \cap \cdots \cap S_{i+1}) \cup (S_{i+1} \cap (D_1 \cup \cdots \cup D_m)) \\ &\subseteq (S_1 \cap \cdots \cap S_{i+1}) \cup (D_1 \cup \cdots \cup D_m). \end{aligned}$$

Hence

$$X = S_{i+1} \cup (X - S_{i+1}) \subseteq \big((S_1 \cap \cdots \cap S_{i+1}) \cup (D_1 \cup \cdots \cup D_m)\big) \cup \big(E_1 \cup \cdots \cup E_r\big).$$

That is,

$$X = (S_1 \cap \cdots \cap S_{i+1}) \cup (D_1 \cup \cdots \cup D_m \cup E_1 \cup \cdots \cup E_r).$$

Therefore, once again by maximality of $\mathcal{U}^*$, no open set containing $S_1 \cap \cdots \cap S_{i+1}$ can be in $\mathcal{U}^*$, and the induction is complete. In particular, U, which is an open set containing $S_1 \cap \cdots \cap S_n$, is not in $\mathcal{U}^*$. This contradiction concludes the proof. □

As announced above, the Tychonoff Product Theorem is a generalization of the Cantor diagonal process. In fact, let us see how that diagonal process may be used to show directly that the product of a sequence of copies of $[0, 1]$ is compact. Denote the product as a set by $X = \bigtimes_{n=1}^{\infty} [0, 1]$. A member of X is a sequence $\{x_n\}$ with terms x_n. Let us impose on X the Hilbert-cube metric of Example 11 in Section II.1:

$$d(\{x_n\}, \{y_n\}) = \sum_n 2^{-n}|x_n - y_n|.$$

We show below in Corollary 10.29 that this metric on X yields the product topology. By Theorem 2.36 the space X will then be compact if every sequence in X has a convergent subsequence. A sequence in X means a system $\{x_n^{(m)}\}$ in which the n^{th} term of the m^{th} sequence is $x_n^{(m)}$. Convergence is term-by-term convergence. To produce a convergent subsequence of sequences, we iterate use of the Bolzano–Weierstrass property of $[0, 1]$. Remembering that m tells which sequence we are dealing with, we find first a subcollection m_k of the indices m such that we have convergence along the m_k's for $n = 1$, then a subcollection m_{k_l} of that such that we have convergence along the m_{k_l}'s for $n = 2$, and so on. Since the intersection of all these sequences may be empty, we instead obtain a convergent subsequence of our sequences by requiring that the k^{th} term of the desired subsequence be the k^{th} term of the k^{th} subsequence. This "diagonal process" thus shows that any sequence in X has a convergent subsequence. Hence X, being a metric space, is compact.

The general Tychonoff Product Theorem may thus be viewed as a topological generalization of the diagonal process to product spaces with an uncountable number of factors.

Here is one way in which the Tychonoff Product Theorem is used in real analysis. For the situation in which we have a set Y and a system of functions $f_s : Y \to \mathbb{C}$ for s in some set S, the first section of this chapter introduced the **weak topology for** Y **determined by** $\{f_s\}_{s\in S}$. This is the weakest topology making all the functions f_s continuous. Often in analysis a set Y and a system of functions f_s of this kind arise in a construction, and then this weak topology is imposed on Y. In favorable cases it turns out that each function f_s is bounded on Y. In this case if there are enough functions f_s to **separate points** of Y (i.e., enough so that for each x and y there is some s with $f_s(x) \neq f_s(y)$), then Y is a candidate for a compact Hausdorff space. To see what is needed for compactness, let X_s be a compact subset of $\mathbb{C}$ containing the image of f_s, and let $X = \times_{s\in S} X_s$. Define a function $F : Y \to X$ by "$F(y)$ is the function whose s^{th} coordinate is $f_s(y)$." It is readily verified that F is a homeomorphism of Y onto a subspace of the compact Hausdorff space X. Thus Y is compact if and only if $F(Y)$ is closed in X. Checking that a set is closed is much easier than checking compactness directly, and it is especially easy if one uses "nets," which are the objects introduced in the next section as a useful generalization of sequences.

To complete our discussion, we still need to prove that the Hilbert-cube metric on $X = \times_{n=1}^{\infty} [0, 1]$ yields the product topology. It will be helpful to prove the following more general result and to obtain the statement about the Hilbert cube as a special case.

Proposition 10.28. Suppose that X is a nonempty set and $\{d_n\}_{n\geq 1}$ is a sequence of pseudometrics on X such that $d_n(x, y) \leq 1$ for all n and for all x and y in X. Then $d(x, y) = \sum_{n=1}^{\infty} 2^{-n} d_n(x, y)$ is a pseudometric. If the open balls relative to d_n are denoted by $B_n(r; x)$ and the open balls relative to d are denoted by $B(r; x)$, then the B_n's and B's are related as follows:

(a) whenever some $B_n(r_n; x)$ is given with $r_n > 0$, there exists some $B(r; x)$ with $r > 0$ such that $B(r; x) \subseteq B_n(r_n; x)$,

(b) whenever $B(r; x)$ is given with $r > 0$, there exist finitely many $r_n > 0$, say for $n \leq K$, such that $\bigcap_{n=1}^{K} B_n(r_n; x) \subseteq B(r; x)$.

PROOF. For (a), choose $r = 2^{-n} r_n$. If $d(x, y) < r$, then $2^{-m} d_m(x, y) < r$ for all m and in particular $d_n(x, y) < 2^n r = r_n$.

For (b), choose K large enough so that $2^{-K} < r/2$, and put $r_n = r/2$ for $n \leq K$. If y is in $\bigcap_{n=1}^{K} B_n(r_n; x)$, then $d_n(x, y) < r_n = r/2$ for $n \leq K$. Hence $d(x, y) \leq \sum_{n=1}^{K} 2^{-n} d_n(x, y) + \sum_{n=K+1}^{\infty} 2^{-n} < \sum_{n=1}^{K} 2^{-n} r/2 + 2^{-K} < r/2 + r/2 = r$. Therefore y is in $B(r; x)$. □

Corollary 10.29. The Hilbert-cube metric on $X = \times_{n=1}^{\infty} [0, 1]$ yields the product topology.

PROOF. Proposition 10.28a implies that any basic open neighborhood of x in the product topology contains a basic open neighborhood in the Hilbert-cube metric topology. Proposition 10.28b shows that any basic open neighborhood of x in the Hilbert-cube metric topology contains a basic open neighborhood in the product topology. □

5. Sequences and Nets

Sequences are of limited interest in general topological spaces. Nets, which are generalized sequences of a certain kind, are a useful substitute, and we introduce them in this section. Using nets, we shall be able to see that product topologies are appropriate for detecting pointwise convergence in the same way that the metric topology obtained from the supremum norm is appropriate for detecting uniform convergence.

We begin with two examples that illustrate some of the difficulties with using sequences in general topological spaces. We use the natural definition suggested by Section II.4—that a sequence $\{x_n\}$ in X **converges** to x_0 if for each neighborhood of x_0, there is some N depending on the neighborhood such that x_n is in the neighborhood for $n \geq N$. We say that the sequence is **eventually** in the neighborhood. The point x_0 is a **limit** of the sequence.

EXAMPLES.

(1) Let X be the set of positive integers, and let a topology for X consist of the empty set and all sets whose complements are finite. If $x_n = 2n$, then the sequence $\{x_n\}$ converges to every point of X and hence does not have a unique limit. The space X is $\mathbf{T}_1$ and has a countable local base at each point, but X is not Hausdorff.

(2) Let X be the set of points (m, n) in the plane with m and n integers ≥ 0. Define a topology for X as follows. Any set not containing $(0, 0)$ is to be open. If a set U contains $(0, 0)$, then U is defined to be open if there are only finitely many columns $C_m = \{(m, n) \mid n = 0, 1, 2 \dots\}$ such that $C_m - (U \cap C_m)$ is infinite. Enumerate X, and define x_n to be the n^{th} point in the enumeration. It is easy to check that the image of the sequence $\{x_n\}$ has $(0, 0)$ as a limit point and that no subsequence of $\{x_n\}$ converges to $(0, 0)$. The space X is Hausdorff but does not have a countable local base at $(0, 0)$.

Thus the elementary results in Section II.4 do not generalize to all topological spaces. But Proposition 2.20 (the uniqueness of the limit of any sequence) is still valid if X is Hausdorff, and Proposition 2.22 and Corollary 2.23 (the characterization of limit points and of closed sets in terms of sequences) are still valid if X has a countable local base at each point. Nets will cure the problem about characterizing limit points and closed sets without countable local bases but not the problem about nonuniqueness of limits, and thus we shall be able to work well with nets in all Hausdorff spaces. In particular we shall be able to use nets in uncountable products of Hausdorff spaces, which arise frequently in real analysis and tend not to have a countable local base at each point.

Before defining nets, let us give one positive result whose statement mixes topological spaces and metric spaces. If S is any nonempty set, we have made $B(S)$, the vector space of all bounded scalar-valued functions on S, into a normed linear space—and hence a metric space—by means of the supremum norm. If S is a topological space, let $C(S)$ be the subset of continuous members of $B(S)$; this is a vector subspace and hence is itself a normed linear space.

Proposition 10.30. If S is a topological space and $\{f_n\}$ is a sequence of scalar-valued functions continuous at s_0 and converging uniformly to a function f, then f is continuous at x_0. Consequently the subspace $C(S)$ of $B(S)$ is a closed subspace, and $C(S)$ is complete as a metric space.

PROOF. Given $\epsilon > 0$, choose N such that $n \geq N$ implies $\|f_n - f\|_{\sup} < \epsilon$. For any s, we then have

$$\begin{aligned}|f(s) - f(s_0)| &\le |f(s) - f_N(s)| + |f_N(s) - f_N(s_0)| + |f_N(s_0) - f(s_0)| \\ &\le \|f_N - f\|_{\sup} + |f_N(s) - f_N(s_0)| + \|f_N - f\|_{\sup} \\ &< 2\epsilon + |f_N(s) - f_N(s_0)|.\end{aligned}$$

Since f_N is continuous at s_0, there exists a neighborhood of s_0 such that the right side is $< 3\epsilon$ for s in that neighborhood. Thus f is continuous at s_0.

If $\{f_n\}$ is a sequence in $C(S)$ converging uniformly to f in $B(S)$, then f is in $C(S)$, by the result of the previous paragraph. Since convergence of sequences in $B(S)$ is the same as uniform convergence, Corollary 2.23 shows that $C(S)$ is a closed subset of $B(S)$. Propositions 2.43 and 2.44 then show that $C(S)$ is complete as a metric space. □

Now we turn our attention to nets. In the indexing for a net, the set of positive integers is replaced by a "directed set," which we define first. Let D be a partially ordered set in the sense of Section A9 of Appendix A, the partial ordering being denoted by $\le$. We say that $(D, \le)$ is a **directed set** if for any α and β in D, there is some γ in D with $\alpha \le \gamma$ and $\beta \le \gamma$.

EXAMPLES.

(1) Take D to be the set of positive integers, and let $\le$ have the usual meaning.

(2) Let S be a nonempty set, take D to be the set of all finite subsets of S, and let $\alpha \le \beta$ mean that the inclusion $\alpha \subseteq \beta$ holds.

(3) Let X be a topological space, let x be a point in X, take D to be the set of all neighborhoods of x, and let $\alpha \le \beta$ mean that $\alpha \supseteq \beta$.

(4) Let $(D_1, \le_1)$ and $(D_2, \le_2)$ be two directed sets, take D to be $D_1 \times D_2$, and let $(\alpha_1, \alpha_2) \le (\beta_1, \beta_2)$ mean that $\alpha_1 \le_1 \beta_1$ and $\alpha_2 \le_2 \beta_2$.

If X is a nonempty set, a **net** in X is a function from a directed set D into X. If D needs to be specified to avoid confusion, we speak of a "net from D to X." The function will often be written $\alpha \mapsto x_\alpha$ or $\{x_\alpha\}$. If E is a subset of X, the net is **eventually in** E if there is some α_0 in D such that $\alpha_0 \le \alpha$ implies that x_α is in E. The net is **frequently in** E if for any α in D, there is a β in D with $\alpha \le \beta$ such that x_β is in E. It is important to observe that the negation of "the net is eventually in E" is that "the net is frequently in the complement of E."

The directedness of the set D plays an important role in the theory by allowing us to work simultaneously with finitely many conditions on a net. For example, if $\{x_\alpha\}$ is eventually in E_1 and eventually in E_2, then it is eventually in $E_1 \cap E_2$. In fact, the given conditions say that there are members α_1 and α_2 of D such that x_α is in E_1 for $\alpha_1 \le \alpha$ and x_α in E_2 for $\alpha_2 \le \alpha$. The directedness implies that

$\alpha_1 \le \alpha_0$ and $\alpha_2 \le \alpha_0$ for some α_0 in D. Then $\{x_\alpha\}$ is in $E_1 \cap E_2$ for $\alpha_0 \le \alpha$. This kind of argument will be used often without mention of the details.

If X is a topological space, a net $\{x_\alpha\}$ in X **converges** to x_0 in X if $\{x_\alpha\}$ is eventually in each neighborhood of x_0. In this case we write $x_\alpha \to x_0$, and we say that x_0 is a **limit** of $\{x_\alpha\}$. Because of the availability of Examples 3 and 4 above, it is an easy matter to characterize the terms "Hausdorff," "limit point," "closed set," and "continuous at a point" in terms of convergence of nets.

Proposition 10.31. A topological space X is Hausdorff if and only if every convergent net in X has only one limit.

PROOF. Suppose that X is Hausdorff and that $x_\alpha \to x_0$ and $x_\alpha \to y_0$ with $x_0 \ne y_0$. Choose disjoint open sets U and V with x_0 in U and y_0 in V. By the assumed convergence, $\{x_\alpha\}$ is in U eventually and is in V eventually. Then it is in $U \cap V = \varnothing$ eventually, and we have a contradiction.

Suppose that X is not Hausdorff. Find distinct points x_0 and y_0 such that every pair of neighborhoods U of x_0 and V of y_0 has nonempty intersection. For any such pair (U, V), define $x_{U,V}$ to be some point in the intersection. Combining Examples 3 and 4 above, we see that $(U, V) \mapsto x_{U,V}$ is a net in X converging to both x_0 and y_0. □

Proposition 10.32. If X is a topological space, then

(a) for any subset A of X and limit point x_0 of A, there exists a net in $A - \{x_0\}$ converging to x_0,

(b) any convergent net $\{x_\alpha\}$ in X with limit x_0 in X either has x_0 as a limit point of the image of the net or else is eventually constantly equal to x_0.

PROOF. For (a), the definition of limit point implies that for each neighborhood U of x_0, the set $U \cap (A - \{x_0\})$ is nonempty. If x_U denotes a point in the intersection, then $U \mapsto x_U$ is a net in $A - \{x_0\}$ converging to x_0.

For (b), suppose that x_0 is not a limit point of the image of the net. Then there exists a neighborhood U of x_0 such that $U - \{x_0\}$ is disjoint from the image of the net. Since the convergence implies that the net is eventually in U, it must be true that $x_\alpha = x_0$ eventually. □

Corollary 10.33. If X is a topological space, then a subset F of X is closed if and only if every convergent net in F has its limits in F.

PROOF. Suppose that F is closed and that $\{x_\alpha\}$ is a convergent net in F with limit x_0. By Proposition 10.32b, either x_0 is in the image of the net or x_0 is a limit point of the image of the net. In the latter case, x_0 is a limit point of the larger set F. In either case, x_0 is in F; thus the limit of any convergent net in F is in F.

Conversely suppose every convergent net in F has its limit in F. If x_0 is a limit point of F, then Proposition 10.32a produces a net in $F - \{x_0\}$ converging to x_0. By assumption, the limit x_0 is in F. Therefore F contains all its limit points and is closed. □

Proposition 10.34. Let $f : X \to Y$ be a function between topological spaces. Then f is continuous at a point x_0 in X if and only if whenever $\{x_\alpha\}$ is a convergent net in X with limit x_0, then $\{f(x_\alpha)\}$ is convergent in Y with limit $f(x_0)$.

REMARKS. This result needs to be used with caution if Y is not known to be Hausdorff. For example, let X and Y both be the set $\{a, b\}$. Let the topology for X be discrete and the topology for Y be indiscrete, consisting only of $\varnothing$ and the whole space. Every function $f : X \to Y$ is continuous. Suppose that $f(a) = f(b) = a$. Take $x_0 = b$ and $x_\alpha = b$ for all α. Then $\{f(x_\alpha)\}$ converges to both a and b. Hence we cannot evaluate $f(x_0)$ as just any limit of $\{f(x_\alpha)\}$; we have to pick the right limit.

PROOF. Suppose that f is continuous at x_0 and that $\{x_\alpha\}$ is a convergent net in X with limit x_0. Let V be any open neighborhood of $f(x_0)$. By continuity, there exists an open neighborhood U of x_0 such that $f(U) \subseteq V$. Since $x_\alpha \to x_0$, the members x_α of the net are eventually in U. Then $f(x_\alpha)$ is in $f(U) \subseteq V$ for the same α's, hence eventually. Therefore $\{f(x_\alpha)\}$ converges to $f(x_0)$.

Conversely suppose that $x_\alpha \to x_0$ always implies $f(x_\alpha) \to f(x_0)$. We are to show that f is continuous. If V is an arbitrary open neighborhood of $f(x_0)$, we seek some open neighborhood of x_0 that maps into V under f. Assuming that there is no such neighborhood for some V, we can find, for each neighborhood U of x_0, some x_U in U such that $f(x_U)$ is not in V. Then $x_U \to x_0$, but $f(x_U)$ does not have limit $f(x_0)$ because $f(x_U)$ is never in V. This is a contradiction, and we conclude that some U maps into V under f; thus f is continuous. □

Proposition 10.35. Let $X = \bigtimes_{s\in S} X_s$ be the product of topological spaces X_s, and let $p_s : X \to X_s$ be the s^{th} coordinate function. Then a net $\{x_\alpha\}$ in X converges to some x_0 in X if and only if the net $\{p_s(x_\alpha)\}$ in X_s converges to $p_s(x_0)$ for each s in S.

REMARK. This is the sense in which the product topology is the topology of pointwise convergence. In combination with Corollary 10.33, this proposition simplifies the problem of deciding when a subset of a product space is closed in the product topology.

PROOF. If $\{x_\alpha\}$ converges to x_0, then Proposition 10.34 and the continuity of p_s together imply that $\{p_s(x_\alpha)\}$ converges to $p_s(x_0)$.

Conversely suppose that $\{p_s(x_\alpha)\}$ converges to $p_s(x_0)$ for all s. Fix s. If U_s is an open neighborhood of $p_s(x_0)$ in X_s, then $\{p_s(x_\alpha)\}$ is eventually in U_s. Hence there is some α_0 such that $p_s(x_\alpha)$ is in U_s whenever $\alpha_0 \leq \alpha$. For the same values of α, $\{x_\alpha\}$ is in $p_s^{-1}(U_s)$. Thus $\{x_\alpha\}$ is eventually in $p_s^{-1}(U_s)$.

Any neighborhood N of x_0 in X contains some basic open neighborhood of the form $U = p_{s_1}^{-1}(U_{s_1}) \cap \cdots \cap p_{s_n}^{-1}(U_{s_n})$. It follows from the result of the previous paragraph that $\{x_\alpha\}$ is eventually in each $p_s^{-1}(U_s)$, hence is eventually in the intersection U, and hence is eventually in N. Therefore $\{x_\alpha\}$ converges to x_0. □

One can express also the notion of compactness in terms of nets, the idea being that compactness of X is equivalent to the fact that every net in X has a convergent subnet, for an appropriate definition of "subnet." The remainder of this section will deal with this question. Carrying out the details of this equivalence is harder than what we have done so far with nets. Actually, the main benefit of the equivalence is the resulting simplification to proofs of compactness, especially to the proof of the Tychonoff Product Theorem. Since we have already proved the Tychonoff Product Theorem without nets, the material in the remainder of this section will be used only in minor ways in the rest of the book.[6]

Let D and E be directed sets. A function from E to D, written $\mu \mapsto \alpha_\mu$, is **cofinal**[7] if for any β in D, there is a ν in E such that $\beta \leq \alpha_\mu$ whenever $\nu \leq \mu$. If $\mu \mapsto \alpha_\mu$ is cofinal and if $\alpha \mapsto x_\alpha$ is a net from D to X, then the composition $\mu \mapsto x_{\alpha_\mu}$ is a net from E to X and is called a **subnet** of the net $\alpha \mapsto x_\alpha$.

The prototype of a subnet is a subsequence. In this case, D and E are both the set of positive integers, and the function from E to D is $k \mapsto n_k$. If the sequence is $\{a_n\}$, then the subnet/subsequence is $\{a_{n_k}\}$. For a general subnet one might expect that it would suffice always to take E to be a subset of D and to let the function from E to D be inclusion. However, this definition of subnet is insufficient to prove the desired characterization of compactness in terms of nets and subnets.

A net from a directed set D to a nonempty set X is called **universal** if for any subset A of X, the net is eventually in A or eventually in A^c. It of course cannot be eventually in both, since otherwise it would eventually be in the intersection, namely the empty set.

Proposition 10.36. Each net in a nonempty set X has a universal subnet.

REMARK. The proof will use Zorn's Lemma. Apart from this one use, the only other uses of the Axiom of Choice in the remainder of this section are transparent ones.

[6]Nets play a more significant role in the companion volume, *Advanced Real Analysis*.

[7]This definition is not the standard one given in Kelley's *General Topology*, but it leads to the standard definition of "subnet."

PROOF. Let D be a directed set, and let $\alpha \mapsto x_\alpha$ be a net from D to X. Consider all families $\mathcal{C}_\beta$ of subsets of X that are closed under finite intersections and have the property, for each A in $\mathcal{C}_\beta$, that the net is frequently in A. There exists such a family, the singleton family $\{X\}$ being one. Partially order the set of such families by inclusion upward, saying that $\mathcal{C}_\beta \leq \mathcal{C}_{\beta'}$ when $\mathcal{C}_\beta \subseteq \mathcal{C}_{\beta'}$. In any chain of $\mathcal{C}_\beta$'s, let $\mathcal{C}_\gamma$ be the union of the sets in the various members of the chain. Since closure under intersection depends only on two sets at a time and since the other property of a $\mathcal{C}_\beta$ depends only on one set at a time, $\mathcal{C}_\gamma$ is again a family of this kind. By Zorn's Lemma let $\mathcal{C}$ be a maximal such family.

Let us prove for each subset A of X that either A or A^c is in $\mathcal{C}$. In fact, if for every B in $\mathcal{C}$, the net is frequently in $A \cap B$, then $\mathcal{C} \cup \{A\}$ is a family containing $\mathcal{C}$ and satisfying the two defining properties of one of our families. By maximality, $\mathcal{C} \cup \{A\} = \mathcal{C}$. Hence A is in $\mathcal{C}$. Assuming that A is not in $\mathcal{C}$, we obtain a set B in $\mathcal{C}$ such that the net fails to be frequently in $A \cap B$. Then B is a member of $\mathcal{C}$ such that the net is eventually in $(A \cap B)^c$.

Similarly if we assume that A^c is not in $\mathcal{C}$, we obtain a set B' in $\mathcal{C}$ such that the net is eventually in $(A^c \cap B')^c$. If neither A nor A^c is in $\mathcal{C}$, then the net is eventually in

$$\begin{aligned}(A \cap B)^c \cap (A^c \cap B')^c &= (A^c \cup B^c) \cap (A \cup B'^c)\\ &= (A^c \cap (A \cup B'^c)) \cup (B^c \cap (A \cup B'^c))\\ &= (A^c \cap B'^c) \cup (B^c \cap (A \cup B'^c))\\ &\subseteq B'^c \cup B^c = (B \cap B')^c,\end{aligned}$$

and it cannot be frequently in $B \cap B'$. This contradicts the fact that $B \cap B'$ is in $\mathcal{C}$ because $\mathcal{C}$ is closed under finite intersections. This completes the proof that either A or A^c has to be in $\mathcal{C}$.

The members of $\mathcal{C}$ form a directed set under inclusion downward, i.e., with partial ordering $A \leq B$ if $A \supseteq B$. Form $\mathcal{E} = \mathcal{C} \times D$ as a directed set under the definition in Example 4 earlier in this section. We construct a subnet as follows. For each ordered pair (A, β) in $\mathcal{C} \times D$, let $\alpha_{(A,\beta)}$ be an element of D with $\beta \leq \alpha_{(A,\beta)}$ and with $x_{\alpha_{(A,\beta)}}$ in A; this choice is possible since D is directed and the given net is frequently in A. The function $(A, \beta) \mapsto \alpha_{(A,\beta)}$ is cofinal because for any $\beta \in D$, the member (X, β) of $\mathcal{E} = \mathcal{C} \times D$ has $\beta \leq \alpha_{(B,\gamma)}$ whenever $(A, \beta) \leq (B, \gamma)$. Thus $(A, \beta) \mapsto x_{\alpha(A,\beta)}$ is a subnet.

To complete the proof, we show that this subnet is universal. For any subset A of X, we have seen that either A or A^c has to be in $\mathcal{C}$. Without loss of generality, assume that A is in $\mathcal{C}$. For any fixed β, the inequality $(A, \beta) \leq (B, \gamma)$ implies that $x_{\alpha_{(B,\gamma)}}$ is in the subset B of A, and hence the subnet is eventually in A. □

Proposition 10.37. The following three statements about a topological space X are equivalent:

(a) X is compact,
(b) every universal net in X is convergent,
(c) every net in X has a convergent subnet.

PROOF. To prove that (a) implies (b), let $\{x_\alpha\}$ be a universal net in X, and suppose that $\{x_\alpha\}$ is not convergent. For each x in X, there is then an open neighborhood U_x of x such that $\{x_\alpha\}$ is not eventually in U_x. Since the net is universal, it is eventually in $(U_x)^c$ for each x. The open sets U_x cover X. By compactness, let $\{U_{x_1}, \dots, U_{x_n}\}$ be a finite subcover. The net is eventually in each $(U_{x_j})^c$ and hence is eventually in their intersection. But their intersection is empty since $X = \bigcup_{j=1}^n U_{x_j}$. We have arrived at a contradiction, and thus $\{x_\alpha\}$ must be convergent.

Statement (b) implies statement (c) since every net has a universal subnet, by Proposition 10.36.

To prove that (c) implies (a), suppose that X is noncompact. We shall produce a net with no convergent subnet. If $\mathcal{U}$ is an open cover of X with no finite subcover, we shall use $\mathcal{U}$ to define a directed set. Let $\mathcal{F}$ be the set of all finite subcollections of members of $\mathcal{U}$. This is directed under inclusion upward: $\alpha \le \beta$ if $\alpha \subseteq \beta$. For each α in $\mathcal{F}$, the set $X - \bigcup_{U \in \alpha} U$ is not empty since $\mathcal{U}$ has no finite subcover, and we let x_α be an element of $X - \bigcup_{U\in\alpha} U$. Then $\alpha \mapsto x_\alpha$ is a net. Suppose that $\{x_\alpha\}$ has a convergent subnet, with some x_0 as limit. For any neighborhood N of x_0, $\{x_\alpha\}$ is frequently in N. Since $\mathcal{U}$ is a covering, there is some U in $\mathcal{U}$ with x_0 in U. By construction, $\{x_\alpha\}$ is not in U as soon as α has $\{U\} \le \alpha$. We conclude that no subnet of $\{x_\alpha\}$ converges. □

Proposition 10.37 gives the statement about general topological spaces that extends the equivalence of the Bolzano–Weierstrass property and the Heine–Borel property of closed bounded subsets of Euclidean space. To illustrate the power of nets, we can now use them to give a second proof of the Tychonoff Product Theorem (Theorem 10.27).

SECOND PROOF OF TYCHONOFF PRODUCT THEOREM. Let $X = \times_{s\in S} X_s$ be given with each X_s compact, let $p_s : X \to X_s$ be the s^{th} coordinate function, and let $\{x_\alpha\}$ be a universal net in X. Fix s, and let A_s be any subset of X_s. Since the net is universal, it is eventually in $p_s^{-1}(A_s)$ or in $(p_s^{-1}(A_s))^c$. Since $(p_s^{-1}(A_s))^c = p_s^{-1}((A_s)^c)$, the net $\{p_s(x_\alpha)\}$ is eventually in A_s or in $(A_s)^c$. Thus $\{p_s(x_\alpha)\}$ is a universal net in X_s. By Proposition 10.37 and the compactness of X_s, $\{p_s(x_\alpha)\}$ converges to some member x_s of X_s. Now let s vary. Forming the member x of X with $p_s(x) = x_s$ for all s and applying Proposition 10.35, we see that $x_\alpha \to x$. By Proposition 10.37, X is compact. □

6. Quotient Spaces

If X is a topological space and $\sim$ is an equivalence relation on X, then we saw in Section 1 that the set $X/\sim$ of equivalence classes inherits a natural topology known as the "quotient topology." If $q : X \to X/\sim$ is the **quotient map**, then a subset U of $X/\sim$ is defined to be open in the **quotient topology** if $q^{-1}(U)$ is open in X. The quotient topology is then the finest topology on $X/\sim$ that makes the quotient map continuous.

Without some assumption that relates the equivalence relation to the topology of X, we cannot expect much from general quotient spaces. In this section we shall investigate situations in which the quotient space does have reasonable properties. Ultimately our interest will be in four situations, some of which are hinted at in Section 1:

(i) the passage from a regular topological space to the quotient when the equivalence relation is that $x \sim y$ if x is in $\{y\}^{\text{cl}}$ (Proposition 10.7),
(ii) the passage from a compact Hausdorff space X to the quotient when the equivalence relation is closed as a subset of $X \times X$ (to be discussed in Problem 11 at the end of the chapter),
(iii) the passage from a "topological vector space" or "topological group" to a coset space (to be discussed in the companion volume, *Advanced Real Analysis*),
(iv) the piecing together of a "manifold," or a "vector bundle," or a "covering space" from local data (to be discussed in the companion volume, *Advanced Real Analysis*).

We begin with some general facts. The first is a kind of "universal mapping property" for all quotient spaces. Its corollary describes a situation in which we can recognize a given space as a quotient even if it was not constructed that way: we say that a function $F : X \to Y$ is **open** if F carries open sets to open sets.

Proposition 10.38.

(a) Let $F : X \to Y$ be a continuous function between topological spaces, let $\sim$ be an equivalence relation on X, and let $q : X \to X/\sim$ be the quotient map. Suppose that F has the property that $F(x_1) = F(x_2)$ whenever $x_1 \sim x_2$, so that there exists a well-defined function $f : X/\sim \to Y$ such that $F = f \circ q$. Then f is continuous.

(b) The quotient $X/\sim$ is characterized by the property in (a) in the following sense. Suppose that $q' : X \to Z$ is any continuous function of X onto a topological space Z such that

(i) $x_1 \sim x_2$ implies $q'(x_1) = q'(x_2)$,
(ii) whenever $F : X \to Y$ is a continuous function such that $x_1 \sim x_2$

implies $F(x_1) = F(x_2)$, there exists a continuous function $f' : Z \to Y$ with $F = f' \circ q'$.

Then Z is canonically homeomorphic to $X/\sim$.

PROOF. In (a), we want to know that $f^{-1}(U)$ is open in $X/\sim$ whenever U is open in Y. By definition of the quotient topology, $f^{-1}(U)$ is open in $X/\sim$ if and only if $q^{-1}(f^{-1}(U))$ is open in X. This set is $F^{-1}(U)$, which is open since F is assumed continuous.

In (b), suppose Z and q' are such that $q' : X \to Z$ has the stated properties. We apply the result of (a) with F taken to be $q : X \to X/\sim$. Property (ii) of Z gives us a continuous function $f' : Z \to X/\sim$ such that $q = f' \circ q'$. Then we apply the result of (a) with F taken to be $q' : X \to Z$, and (a) shows that the function $f : X/\sim \to Z$ with $q' = f \circ q$ is continuous. Combining these two equations gives us $q = f' \circ f \circ q$ and $q' = f \circ f' \circ q'$. Thus $f' \circ f$ is the identity on the image of q, and $f \circ f'$ is the identity on the image of q'. Since q is onto $X/\sim$ and q' is onto Z, $f : X/\sim \to Z$ is a homeomorphism. □

Corollary 10.39. Let $F : X \to Y$ be a continuous function from one topological space onto another, and define $x_1 \sim x_2$ if $F(x_1) = F(x_2)$. Let $q : X \to X/\sim$ be the quotient map, and let $f : X/\sim \to Y$ be the continuous map such that $F = f \circ q$. If F is open, then f is a homeomorphism and hence Y can be regarded as a quotient of X.

REMARK. The continuity of f is the conclusion of Proposition 10.38a.

PROOF. The function $f : X/\sim \to Y$ is continuous, one-one, and onto. To see that f is open and hence is a homeomorphism, let an open set U in $X/\sim$ be given. Then $F(q^{-1}(U))$ is open because q is continuous and F is open. Since $F(q^{-1}(U)) = f(q(q^{-1}(U))) = f(U)$, we see that $f(U)$ is open. Hence f is open. □

EXAMPLE. Let $X = \times_{s\in S} X_s$ be a product of topological spaces, fix s in S, and let $p_s : X \to X_s$ be the s^{th} coordinate function. We shall show that p_s is open, so that X_s can be regarded as the quotient of X by the relation that $x_1 \sim x_2$ if $p_{s'}(x_1) = p_{s'}(x_2)$ for all $s' \neq s$. If U is an open set in X and x is in U, then we can find a basic open set $V_x = p_{s_1}^{-1}(U_1) \cap \cdots \cap p_{s_n}^{-1}(U_n)$ about x that is contained in U. Then $p_s(V_x)$ equals U_j if $s = s_j$, and it equals X_s if s is not equal to any s_j. In either case, $p_s(V_x)$ is open. Thus $p_s(U)$ contains a neighborhood of each of its points and must be an open set. So p_s is open.

A key desirable property of a quotient space is that it is Hausdorff. The Hausdorff property is what makes limits unique, after all, and it therefore paves the way to doing some analysis with the space. The next proposition gives a useful necessary condition and a useful sufficient condition.

Proposition 10.40. Let X be a topological space, let $\sim$ be an equivalence relation on X, and let R be the subset $\{(x_1, x_2) \mid x_1 \sim x_2\}$ of $X \times X$. If the quotient topology on $X/\sim$ is Hausdorff, then R is a closed subset of $X \times X$. Conversely if R is a closed subset of $X \times X$ and if the quotient map $q : X \to X/\sim$ is open, then $X/\sim$ is Hausdorff.

PROOF. Suppose that $X/\sim$ is Hausdorff. If (x, y) is not in R, then $q(x)$ and $q(y)$ are distinct points in $X/\sim$. Find disjoint open sets U and V in $X/\sim$ such that $q(x)$ is in U and $q(y)$ is in V. Then $q^{-1}(U)$ and $q^{-1}(V)$ are open sets in X with the property that no member of $q^{-1}(U)$ is equivalent to any member of $q^{-1}(V)$. Thus $q^{-1}(U) \times q^{-1}(V)$ is an open neighborhood of (x, y) that does not meet R. Hence R is closed.

Conversely if R is closed and (x, y) is not in R, then there exists a basic open set $U \times V$ of $X \times X$ containing (x, y) that does not meet R. The sets $q(U)$ and $q(V)$ are open in $X/\sim$ since q is open, they are disjoint since no member of U is equivalent to a member of V, and they are neighborhoods of $q(x)$ and $q(y)$, respectively. Thus $X/\sim$ is Hausdorff. □

A special case is the situation with a pseudometric space in which the equivalence relation is that $x \sim y$ if x and y are at distance 0 from one another. A generalization of this relation was given in Proposition 10.7, which said that in a regular topological space the relation $x \sim y$ if x is in $\{y\}^{\text{cl}}$ is an equivalence relation. The corollary to follow gives properties of the quotient space when this equivalence relation is used.

Corollary 10.41. Let X be a regular topological space, let $\sim$ be the equivalence relation defined by saying that $x \sim y$ if x is in $\{y\}^{\text{cl}}$, and let $q : X \to X/\sim$ be the quotient map. Then

(a) q is open, and every open set in X is the union of equivalence classes,
(b) $X/\sim$ is regular and Hausdorff,
(c) X normal implies $X/\sim$ normal,
(d) X separable implies $X/\sim$ separable.

PROOF. First we show that every open set is a union of equivalence classes. Suppose that x is in an open set U in X. Let $x \sim y$. If y were not in U, then y would be in the closed set U^c and hence $\{y\}^{\text{cl}}$ would be contained in U^c. Since $x \sim y$, x is in $\{y\}^{\text{cl}}$, and we are led to the contradiction that x would be in U^c, hence in $U \cap U^c = \varnothing$. So U is a union of equivalence classes. Then it follows that $q^{-1}(q(U)) = U$, and the set $q(U)$ has the property that its inverse image is open in X. By definition of the quotient topology, $q(U)$ is open. Therefore q is an open map. This proves (a).

To prove the Hausdorff property in (b), we shall apply Proposition 10.40. Since (a) shows that q is open, it is enough to show that the subset $R = \{(x, y) \mid x \sim y\}$ of $X \times X$ is closed. If (x, y) is not in R, then x is not in $\{y\}^{\mathrm{cl}}$. By regularity of X, choose disjoint open sets U and V in X such that x is in U and $\{y\}^{\mathrm{cl}} \subseteq V$. Since U and V are unions of equivalence classes and are disjoint, no member of U is equivalent to any member of V. Therefore $(U \times V) \cap R = \varnothing$, and every point of R^c has an open neighborhood lying in R^c. Hence R is closed.

As a result of (a), the open sets in X are in one-one correspondence via q with the open sets in $X/\sim$, and the same thing is true for the closed sets. Under this correspondence disjoint sets correspond to disjoint sets. Then regularity in (b), as well as conclusions (c) and (d), follow immediately. $\square$

7. Urysohn's Lemma

According to Proposition 10.31, a Hausdorff topological space has unique limits for convergent sequences and nets. Corollary 10.41 shows that regularity of a space makes it possible to pass to a natural quotient space that is regular and Hausdorff. The following theorem exhibits a special role for the condition that a space be normal.

Theorem 10.42. (Urysohn's Lemma). If E and F are disjoint closed sets in a normal topological space X, then there exists a continuous function f from X into $[0, 1]$ that is 0 on E and is 1 on F.

PROOF. Proposition 10.5c shows in a normal space that between a closed set and a larger open set we can always interpolate an open set and its closure. Starting from $E \subseteq F^c$, we find an open set $U_{1/2}$ with

$$E \subseteq U_{1/2} \subseteq (U_{1/2})^{\mathrm{cl}} \subseteq F^c.$$

Then we can find open sets $U_{1/4}$ and $U_{3/4}$ with

$$E \subseteq U_{1/4} \subseteq (U_{1/4})^{\mathrm{cl}} \subseteq U_{1/2} \subseteq (U_{1/2})^{\mathrm{cl}} \subseteq U_{3/4} \subseteq (U_{3/4})^{\mathrm{cl}} \subseteq F^c.$$

Proceeding inductively on n, we obtain, for each diadic rational number $r = m/2^n$ with $0 < r < 1$, an open set U_r between E and F^c such that $r < s$ implies $(U_r)^{\mathrm{cl}} \subseteq U_s$. Put $U_1 = X$. For each x in X, define $f(x)$ to be the greatest lower bound of all r such that x is in U_r. Then f is 0 on E, is 1 on F, and has values in $[0, 1]$. To see that f is continuous, let x be given, let r and s be diadic rationals in $(0, 1)$ with $r < f(x) < s$, and choose diadic rationals r' and s' with $r < r' < f(x) < s' < s$. (If $f(x) = 0$, we omit r and r'; if $f(x) = 1$, we omit s and s'.) We are to produce an open neighborhood U of x with $f(U) \subseteq (r, s)$. If $U = U_{s'} - (U_{r'})^{\mathrm{cl}}$, then U is open with $r' \leq f(U) \leq s'$. Thus $r < f(U) < s$ as required. We conclude that f is continuous. $\square$

EXAMPLE. In Example 4 of Section 2, we produced a certain Hausdorff regular space X that is not normal, but we deferred the proof that X is not normal until we had Urysohn's Lemma in hand. We can now give that missing proof. As a set, X is the closed upper half plane $\{\operatorname{Im} z \geq 0\}$ in $\mathbb{C}$. A base for the topology in question consists of all open disks in X that do not meet the x axis, together with all open disks in X that are tangent to the x axis; the latter sets are to include the point of tangency. For a point p on the x axis, the open disks of rational radii with point of tangency p form a countable local base. Arguing by contradiction, suppose that X is normal. Any subset of the x axis in X is closed in X, and we take E to be the set of rationals on the axis and F to be the set of irrationals on the axis. Urysohn's Lemma (Theorem 10.42) supplies a continuous function $f : X \to [0,1]$ such that $f(E) = 0$ and $f(F) = 1$. Define a sequence of functions $f_n : \mathbb{R} \to [0,1]$ by $f_n(x) = f\big(x, \frac{1}{n}\big)$, the notation (x, y) indicating a point in the (x, y) plane. The functions f_n are continuous in the ordinary topology on $\mathbb{R}$ since the topology on X is the ordinary topology of the half plane as long as we stay away from the x axis. At any point $(x, 0)$ of the x axis, the sets

$$U_m = \{x, 0\} \cup B\big(\tfrac{1}{m}; \big(x, \tfrac{1}{m}\big)\big)$$

form a local base at $(x, 0)$, and $\big(x, \frac{1}{n}\big)$ is in U_m for $n \geq m$. The continuity of f therefore yields $\lim_n f\big(x, \frac{1}{n}\big) = f(x, 0)$. In other words, $\lim_n f_n$ exists pointwise on $\mathbb{R}$ and equals the indicator function of the set of irrationals. The sequence $\{f_n\}$ is therefore a sequence of continuous real-valued functions on $\mathbb{R}$ whose pointwise limit is everywhere discontinuous. However, Theorem 2.54 implies that the set of discontinuities of the limit function is of first category in $\mathbb{R}$, and the Baire Category Theorem (Theorem 2.53) implies that $\mathbb{R}$ is not of first category in itself. Thus we have a contradiction, and we conclude that X cannot be normal.

Corollary 10.43. If E and F are disjoint closed sets in a compact Hausdorff space X, then there exists a continuous function $f : X \to [0,1]$ that is 0 on E and is 1 on F.

PROOF. This follows by combining Proposition 10.16 and Theorem 10.42. □

Corollary 10.44. If K and F are disjoint closed sets in a locally compact Hausdorff space X and if K is compact, then there exists a continuous function $f : X \to [0,1]$ that is 1 on K, is 0 on F, and has compact support.

PROOF. Using Proposition 10.19, regard X as an open subset of the one-point compactification X^*. Proposition 10.20 shows that the compact space X^* is Hausdorff. Choose disjoint open sets U and V in X by Corollary 10.22 such that $K \subseteq U$ and $F \subseteq V$. Choose L compact in X by Corollary 10.23 such that $K \subseteq L^o$. Then $M = L \cap (X - V)$ is compact in X by Proposition 10.14, and $K \subseteq L^o \cap U \subseteq L^o \cap (X - V)^o \subseteq (L \cap (X - V))^o = M^o$. Hence K and $X^* - M^o$ are disjoint compact sets in X^*. Corollary 10.43 produces a continuous $g : X^* \to [0, 1]$ such that g is 1 on K and is 0 on $X^* - M^o$. Since $F \subseteq V \subseteq (X - L) \cup V = X - (L \cap (X - V)) = X - M \subseteq X - M^o \subseteq X^* - M^o$, the function $f = g\big|_X$ has the required properties. □

8. Metrization in the Separable Case

A problem about topological spaces, now completely solved, is to characterize those topologies that arise from metric spaces. Such a space is said to be **metrizable**. We consider only the separable case and prove the following theorem.

Theorem 10.45 (Urysohn Metrization Theorem). Any separable regular Hausdorff space X is homeomorphic to a subspace of the Hilbert cube $C = \times_{n=1}^{\infty} [0, 1]$ and is therefore metrizable.

PROOF. The Hilbert cube C is seen as a metric space in Example 11 in Section II.1, Corollary 10.29 identifies it as a product space, and the Tychonoff Product Theorem (Theorem 10.27) shows that it is compact. Let $p_n : X \to [0, 1]$ be the n^{th} coordinate function.

By Corollary 10.10, X is normal. Fix a countable base $\mathcal{B}$ for the open sets. Enumerate the countable set of pairs (U, V) of members of $\mathcal{B}$ such that $U^{\text{cl}} \subseteq V$. To the n^{th} pair, associate by Urysohn's Lemma (Theorem 10.42) a continuous function $f_n : X \to [0, 1]$ such that f_n is 1 on U^{cl} and is 0 on V^c. Let $F : X \to C$ be defined by "$F(x)$ is the sequence whose n^{th} term is $f_n(x)$." We are to show that F is continuous, is one-one, and is open as a function onto $F(X)$.

The continuity of $p_n \circ F = f_n$ for each n means that $F^{-1} p_n^{-1}$ of any open set in $[0, 1]$ is open in C. Since F^{-1} of a basic open set in C is the finite intersection of the various $F^{-1} p_n^{-1}$'s of open sets, F is continuous.

To see that F is one-one, let x and y be distinct points of X. By Proposition 10.6c, X Hausdorff implies that $\{y\}$ is closed and hence that $\{y\}^c$ is an open neighborhood of x. Choose a basic open set V containing x and contained in $\{y\}^c$. By Proposition 10.5b and the regularity of X, choose a basic open set U containing x such that $U^{\text{cl}} \subseteq V$. Then (U, V) is one of our pairs, and the corresponding function f_n has $f_n(x) = 1$ and $f_n(y) = 0$. Hence $F(x) \neq F(y)$, and F is one-one.

To see that F carries open sets of X to open sets in $F(X)$, let W be open in X, and fix x in W. Arguing as in the previous paragraph, we can find basic open sets U and V such that x is in U and $U^{\text{cl}} \subseteq V \subseteq W$. The corresponding f_n then has $f_n(x) = 1$ and $f_n(V^c) = 0$. Hence $f_n(W^c) = 0$. The set N_x of y's such that $f_n(y) > 0$ is open in X and contains x. The product $(0, 1]_n \times \big(\times_{k \neq n} [0, 1]_k\big)$ is open in C, and its intersection with $F(X)$ is the same as $F(N_x) \cap F(X)$. Thus $F(N_x) \cap F(X)$ is relatively open in $F(X)$. Then $F(x)$ lies in this relatively open set, which in turn lies in $F(W)$, and it follows that $F(W)$ is a relatively open neighborhood of each of its members. □

Corollary 10.46. Every separable compact Hausdorff space is metrizable.

PROOF. This is immediate from Proposition 10.16 and Theorem 10.45. □

9. Ascoli–Arzelà and Stone–Weierstrass Theorems

In Section II.10 we studied Ascoli's Theorem (Theorem 2.56) and the Stone–Weierstrass Theorem (Theorem 2.58) as tools for working with continuous functions on compact metric spaces. In turn, these theorems were illuminating generalizations of results about continuous functions on closed bounded intervals of the line, particularly the classical version of Ascoli's Theorem (Theorem 1.22) and the Weierstrass Approximation Theorem (Theorem 1.52). In this section we shall extend these results to the setting of continuous functions on compact Hausdorff spaces. The proof of the extended Ascoli theorem will be our first example of how the Cantor diagonal process gets replaced by an application of the Tychonoff Product Theorem (Theorem 10.27) when one is dealing with an uncountable number of limiting situations at once. The Stone–Weierstrass Theorem in the more general setting becomes in part a tool for dealing with large abstract compact Hausdorff spaces that arise in functional analysis. The starting point for this investigation is the general form of Alaoglu's Theorem,[8] which says that the closed unit ball in the dual X^* of a normed linear space X is compact in the weak-star topology; closed subsets of this space play a foundational role in the theory of Banach algebras.

We work in this section with a compact Hausdorff space X and with the algebra $C(X)$ of bounded continuous scalar-valued functions on X. The scalars may be real or complex. Corollary 10.13 shows that if f is a continuous scalar-valued function on X, then $|f|$ attains its maximum value on X. The set $C(X)$ is a subspace of the normed linear space $B(X)$ of bounded scalar-valued functions on X, the norm being $\|f\|_{\sup} = \sup_{x\in X} |f(x)|$. Convergence in $B(X)$ is uniform convergence. Proposition 10.30 shows that $C(X)$ is a closed subspace of $B(X)$ and is complete as a metric space.

We begin with the extended Ascoli theorem. Let $\mathcal{F} = \{f_\alpha\}$ be a set of scalar-valued functions on the compact Hausdorff space X. We say that $\mathcal{F}$ is **equicontinuous** at x in X if for each $\epsilon > 0$, there is an open neighborhood $U_{x,\epsilon}$

[8] A preliminary form of this theorem was given as Theorem 5.58. The general form appears in the companion volume, *Advanced Real Analysis*.

of x such that $|f_\alpha(y) - f_\alpha(x)| < \epsilon$ for all y in $U_{x,\epsilon}$ and all f_α in $\mathcal{F}$. We say that $\mathcal{F}$ is **equicontinuous** if it is equicontinuous at each point. Not having a metric to compare different points of X, we no longer define a notion of "uniform equicontinuity."

It is immediate from the definition that any subset of an equicontinuous family is equicontinuous. The definition of equicontinuity at x reduces to the definition of continuity if $\mathcal{F}$ has just one member, and therefore every member of an equicontinuous family is continuous.

As in Section II.10 the set $\mathcal{F}$ is **uniformly bounded** on X if it is pointwise bounded at each $x \in X$ and if the bound for the values $|f(x)|$ with $f \in \mathcal{F}$ can be taken independent of x.

Lemma 10.47. If $\mathcal{F} = \{f_\alpha\}$ is equicontinuous at x in X, then the closure $\mathcal{F}^{\text{cl}}$ of $\mathcal{F}$ in the product topology on $\mathbb{C}^X$ is equicontinuous at x.

REMARK. Consequently every member of $\mathcal{F}^{\text{cl}}$ is continuous at x.

PROOF. Let $U_{x,\epsilon}$ be as in the definition of equicontinuity of $\mathcal{F}$ at x. For each $\epsilon > 0$, the set of functions $f \in \mathbb{C}^X$ such that

$$|f(y) - f(x)| \le \epsilon$$

for a particular y in X is a closed subset of $\mathbb{C}^X$. Thus the set of functions $f \in \mathbb{C}^X$ such that this inequality holds for all y in $U_{x,\epsilon}$, being an intersection of closed sets, is closed, and it contains $\mathcal{F}$. In turn, the intersection G of these sets taken over all $\epsilon > 0$ is closed in $\mathbb{C}^X$ and contains $\mathcal{F}$. For each $\epsilon > 0$, each g in this closed set G satisfies the inequality $|g(y) - g(x)| < 2\epsilon$ whenever y is in $U_{x,\epsilon}$. Therefore G is equicontinuous at x, and so is its subset $\mathcal{F}^{\text{cl}}$. □

Theorem 10.48 (Ascoli–Arzelà Theorem). If $\{f_n\}$ is an equicontinuous family of scalar-valued functions defined on a compact Hausdorff space X and if $\{f_n\}$ has the property that $\{f_n(x)\}$ is bounded for each x, then $\{f_n\}$ has a uniformly convergent subsequence.

PROOF. We may assume that there are infinitely many distinct functions f_n, since otherwise the assertion is trivial. Let $|f_n(x)| \le c_x$ for all n, and form the product space $C = \times_{x \in X} \{z \in \mathbb{C} \mid |z| \le c_x\}$. The space C is compact by the Tychonoff Product Theorem (Theorem 10.27), and we are now assuming that there are infinitely many members of the sequence $\{f_n\}$ in the space. Let S be the image of the sequence as a subset of C. If S were to have no limit point in C, then each f_n would have an open neighborhood in C disjoint from the rest of S; these open sets and S^c would form an open cover of C with no finite subcover, in contradiction to compactness of C. Thus S has a limit point f in C. By Lemma 10.47 and the remarks before it, the family $S \cup \{f\}$ is equicontinuous.

Let $\epsilon > 0$. We shall complete the proof by producing an f_N in S such that $|f_N(x) - f(x)| < \epsilon$ for all x. By equicontinuity find an open neighborhood U_x for each x such that $y \in U_x$ implies

$$|f_n(y) - f_n(x)| < \epsilon/3 \qquad \text{for all } n$$

and

$$|f(y) - f(x)| < \epsilon/3.$$

The open sets U_x cover X, and finitely many of them suffice to cover, by the compactness of X. Thus there are finitely many points $x_1, \dots, x_k$ in X with the property that for each y in X, there is some x_j with $1 \leq j \leq k$ such that

$$|f_n(y) - f_n(x_j)| < \epsilon/3 \qquad \text{and} \qquad |f(y) - f(x_j)| < \epsilon/3$$

for all n. Since f is a limit point of S, choose N such that

$$|f_N(x_j) - f(x_j)| < \epsilon/3 \qquad \text{for } 1 \leq j \leq k.$$

Then for every y in X, there is an x_j such that

$$|f_N(y) - f(y)| \leq |f_N(y) - f_N(x_j)| + |f_N(x_j) - f(x_j)| + |f(x_j) - f(y)| < \epsilon.$$

Thus f_N is within distance ϵ of f, as asserted. □

Corollary 10.49. If X is a compact Hausdorff space, then a subset $\mathcal{F} = \{f_\alpha\}$ of $C(X)$ is compact if and only if

(a) $\mathcal{F}$ is closed in $C(X)$,
(b) the set $\{f_\alpha\}$ is pointwise bounded at each point in X, and
(c) $\mathcal{F}$ is equicontinuous.

In this case, $\mathcal{F}$ is uniformly bounded.

PROOF. Suppose that the three conditions hold. Being a subset of $C(X)$, $\mathcal{F}$ is a metric space under the restriction of the metric. By Theorem 2.36, $\mathcal{F}$ will be compact if we prove that every sequence has a convergent subsequence. Because of (b) and (c), Theorem 10.48 shows that every sequence in $\mathcal{F}$ has a uniformly Cauchy subsequence. By (a) and the completeness of $C(X)$ given in Proposition 10.30, $\mathcal{F}$ is complete as a metric space. Hence the Cauchy subsequence converges to an element of $\mathcal{F}$.

Conversely suppose that $\mathcal{F}$ is compact. Property (a) follows since compact sets are closed in any metric space. For (b) and the stronger conclusion that $\mathcal{F}$ is uniformly bounded, the function $f \mapsto \|f\|_{\sup}$ is a continuous function on the compact set $\mathcal{F}$, and Corollary 10.13 shows that it is bounded. For the equicontinuity in (c), let $\epsilon > 0$ and x be given. Theorem 2.46 shows that $\mathcal{F}$ is totally bounded as a metric space. Hence we can find a finite set $f_1, \dots, f_l$

in $\mathcal{F}$ such that each member f of $\mathcal{F}$ has $\sup_{y \in X} |f(y) - f_j(y)| < \epsilon$ for some j. By continuity of each f_i, choose an open neighborhood $U_{x,\epsilon}$ of x such that $|f_i(x) - f_i(y)| < \epsilon$ for $1 \le i \le l$ for all y in $U_{x,\epsilon}$. If f is some member of $\mathcal{F}$ and if f_j is the member of the finite set associated with f, then $y \in U_{x,\epsilon}$ implies

$$|f(y) - f(x)| \le |f(y) - f_j(y)| + |f_j(y) - f_j(x)| + |f_j(x) - f(x)| < 3\epsilon.$$

Hence $\mathcal{F}$ is equicontinuous at each x in X. □

Now we come to the extended Stone–Weierstrass Theorem. We are interested in showing that certain subalgebras of the algebra $C(X)$ of continuous scalar-valued functions on a compact Hausdorff space X are dense in $C(X)$. Except for the dropping of the assumption that X is metric, the assumptions and notation are the same as in Section II.10. In particular the scalars for the subalgebra and for $C(X)$ may be real or complex, and the statement of the theorem is slightly different in the two cases.

Theorem 10.50 (Stone–Weierstrass Theorem). Let X be a compact Hausdorff space.

(a) If $\mathcal{A}$ is a real subalgebra of real-valued members of $C(X)$ that separates points and contains the constant functions, then $\mathcal{A}$ is dense in the algebra of real-valued members of $C(X)$ in the uniform metric.

(b) If $\mathcal{A}$ is a complex subalgebra of $C(X)$ that separates points, contains the constant functions, and is closed under complex conjugation, then $\mathcal{A}$ is dense in $C(X)$ in the uniform metric.

REMARKS. Curiously, Urysohn's Lemma (Corollary 10.43) does not play a role in the proof. Instead, the role of Urysohn's Lemma is to ensure that $C(X)$ is large in applications, and then the present theorem has serious content. The actual proof of Theorem 10.50 is word-for-word the same as for Theorem 2.58, and there is no need to repeat it.

10. Problems

1. Let f and g be continuous functions from a topological space into a Hausdorff space Y.
   (a) Prove that the set of all points x in X for which $f(x) = g(x)$ is closed.
   (b) Prove that if $f(x) = g(x)$ for all x in a dense subset of X, then $f = g$.
2. **(Dini's Theorem)** Let X be a compact Hausdorff space. Suppose that the function $f_n : X \to \mathbb{R}$ is continuous, that $f_1 \le f_2 \le f_3 \le \cdots$, and that $f(x) = \lim f_n(x)$ is continuous and is nowhere $+\infty$. Use the defining property of compactness to prove that $\{f_n\}$ converges to f uniformly on X.

3. **(Baire Category Theorem)** Prove that a locally compact Hausdorff space cannot be the countable union of closed nowhere dense sets.
4. Prove that a locally compact dense subset of a Hausdorff space is open.
5. This problem produces a locally compact Hausdorff space that is not normal. Verify the details of the construction. Let X be a countably infinite discrete space, and let Y be an uncountable discrete space. Let X^* and Y^* be their one-point compactifications, with the added points denoted by x_∞ and y_∞. The locally compact Hausdorff space is $Z = X^* \times Y^* - \{(x_\infty, y_\infty)\}$ with the relative topology. Two closed subsets that cannot be separated by disjoint open sets are $A = (\{x_\infty\} \times Y^*) - \{(x_\infty, y_\infty)\}$ and $B = (X^* \times \{y_\infty\}) - \{(x_\infty, y_\infty)\}$.
6. If X is compact, prove that each infinite subset of X has a limit point.
7. Let $\mathcal{U}$ be the family of subsets of $\mathbb{R}$ consisting of all sets $\{x \in \mathbb{R} \mid x < a\}$, together with $\varnothing$ and $\mathbb{R}$.
   (a) Prove that $\mathcal{U}$ is a topology for $\mathbb{R}$ and that it is not Hausdorff. (It is called the **upper topology** of $\mathbb{R}$.)
   (b) If $\{t_n\}_{n\in D}$ is a net in $\mathbb{R}$, define $\limsup_n t_n$ to be the infimum over n of $\sup_{m\in D,\ m\geq n} t_n$. Prove that a net $\{t_n\}_{n\in D}$ in $\mathbb{R}$ converges to t relative to $\mathcal{U}$ if and only if $\limsup_n t_n \leq t$.
8. Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{U}$ be the upper topology of $\mathbb{R}$ as in the previous problem. A function $f : X \to \mathbb{R}$ is said to be **upper semicontinuous** if it is continuous with respect to $\mathcal{T}$ and $\mathcal{U}$.
   (a) Prove that upper semicontinuity of $f : X \to \mathbb{R}$ is equivalent to the condition that $\limsup f(x_n) \leq f(x)$ whenever $x_n \to x$ in X.
   (b) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ that is 1 at $x = 0$ and is 0 elsewhere is upper semicontinuous.
   (c) Prove that if f and g are upper semicontinuous functions on X and if c is nonnegative real, then $f + g$ and cf are upper semicontinuous.
   (d) Prove that if $\{f_s\}_{s\in S}$ is a nonempty set of upper semicontinuous functions on X such that $\inf_{s\in S} f(x) > -\infty$ for all $x \in X$, then $\inf_{s\in S} f_s$ is upper semicontinuous.
   (e) Prove that if f is a bounded real-valued function on X, then there exists a unique smallest upper semicontinuous function f^- with $f^-(x) \geq f(x)$ for all x.
9. Let $(X, \mathcal{T})$ be a topological space. A function $f : X \to \mathbb{R}$ is **lower semicontinuous** if $-f$ is upper semicontinuous. In this case if f is bounded, let $f_- = -(-f)^-$, with the right side defined as in the previous problem. Let the **oscillation** Q_f of f be defined by $Q_f(x) = f^-(x) - f_-(x)$ for x in X.
   (a) Why is Q_f upper semicontinuous?
   (b) Prove that this definition agrees with the one in Section II.9.
   (c) Prove that f is continuous if and only if Q_f is identically 0.

10. Let X be a Hausdorff topological space in which there are two disjoint nonempty closed sets A and B. Let $\sim$ be the equivalence relation that identifies all elements of A with each other, identifies all elements of B with each other, and otherwise identifies no distinct points of X.
    (a) Prove that the subset of pairs (x, y) in $X \times X$ with $x \sim y$ is closed.
    (b) Give an example of this kind in which $X/\sim$ is not Hausdorff.
11. Let X be a compact Hausdorff space, and let $\sim$ be an equivalence relation on X such that the subset $R \subseteq X \times X$ of pairs (x, y) with $x \sim y$ is closed. Let $q : X \to X/\sim$ be the quotient map.
    (a) Prove for each $x \in X$ that $q^{-1}q(x)$ is a closed subset of X.
    (b) If $U \subseteq X$ is open, prove that $V = \{x \in X \mid q^{-1}q(x) \subseteq U\}$ is open by first proving that $V^c = p_2((U^c \times X) \cap R)$, where $p_2 : X \times X \to X$ is the projection to the second coordinate.
    (c) Prove that the compact quotient $X/\sim$ is Hausdorff.
    (d) Prove that the quotient map is **closed**, i.e., that closed sets map to closed sets.
    (e) Is the quotient map necessarily open?
    (f) As in one of the examples in Section 1, let X be the interval $[-\pi, \pi]$, and let S^1 be the unit circle in $\mathbb{C}$. Let $\sim$ be the equivalence relation that lets $-\pi$ and π be the only nontrivial pair of elements of X that are equivalent, and form $X/\sim$. Prove that $X/\sim$ is homeomorphic to S^1 and that under this identification the quotient map may be taken to be the function $p : X \to S^1$ given by $p(x) = e^{ix}$.

Problems 12–15 concern connectedness and connected components. Most of the definitions and proofs in the first three are rather similar to those in Chapter II (§II.8 and Problems 11–13) for the special case of metric spaces. A topological space X is **connected** if X cannot be written as $X = U \cup V$ with U and V open, disjoint, and nonempty. A subset E of X is **connected** if E is connected as a subspace of X, i.e., if E cannot be written as a disjoint union $(E \cap U) \cup (E \cap V)$ with U and V open in X and with $E \cap U$ and $E \cap V$ both nonempty.

12. (a) Prove that a continuous function between topological spaces carries connected sets to connected sets.
    (b) A **path** in a topological space X is a continuous function from a closed bounded interval $[a, b]$ into X. Why is the image of a path necessarily connected?
13. (a) If X is a topological space and $\{E_\alpha\}$ is a system of connected subsets of X with a point x_0 in common, prove that $\bigcup_\alpha E_\alpha$ is connected.
    (b) If X is a topological space and E is a connected subset of X, prove that the closure E^{cl} is connected.
14. (a) A topological space X is **pathwise connected** if for any two points x_1 and x_2 in X, there is some continuous $p : [a, b] \to X$ with $p(a) = x_1$ and $p(b) = x_2$. Why is a pathwise-connected space X necessarily connected?

(b) A topological space X is called **locally pathwise connected** if each point has arbitrarily small open neighborhoods that are pathwise connected. Prove that if X is connected and locally pathwise connected, then it is pathwise connected.

15. In a topological space X, define two points to be equivalent if they lie in a connected subset of X.
    (a) Show that this notion of equivalence is indeed an equivalence relation. The equivalence classes are called the **connected components** of X.
    (b) Prove that the connected components of X are closed sets.
    (c) Prove that the connected components of X are open sets if X is **locally connected**, i.e., if each point has arbitrarily small connected neighborhoods.

Problems 16–17 concern partitions of unity, which were introduced in Section III.5. An open cover $\mathcal{U}$ of a topological space is said to be **locally finite** if each point of x has a neighborhood that lies in only finitely many members of $\mathcal{U}$.

16. Suppose that $\mathcal{U}$ is a locally finite open cover of a normal space X. By applying Zorn's Lemma to the class of all functions F defined on subfamilies of $\mathcal{U}$ such that $F(U)$, for each U in the domain of F, is an open set with $F(U)^{\text{cl}} \subseteq U$ and
$$\Big(\bigcup_{U\in\text{domain}(F)} F(U)\Big) \cup \Big(\bigcup_{\substack{V\in\mathcal{U},\\ V\notin\text{domain}(F)}} V\Big) = X,$$
prove that it is possible to select, for each U in $\mathcal{U}$, an open set V_U such that $V_U^{\text{cl}} \subseteq U$ and such that $\{V_U \mid U \in \mathcal{U}\}$ is an open cover of X.

17. Prove that if $\mathcal{U}$ is a locally finite open cover of a normal space X, then it is possible to select, for each U in $\mathcal{U}$, a continuous function $f_U : X \to [0, 1]$ such that f_U is 0 outside U and such that $\sum_{U\in\mathcal{U}} f_U(x) = 1$ for all $x \in X$.

Problems 18–20 establish the Tietze Extension Theorem. Let X be a normal topological space, and let C be a closed subset of X. Suppose that f is a bounded real-valued continuous function defined on C. The theorem is that there exists a continuous function $F : X \to \mathbb{R}$ such that $F\big|_C = f$ and $\sup_{x\in X} |F(x)| = \sup_{x\in C} |f(x)|$.

18. Let $g_0 = f$, $c_0 = \sup_{x\in C} |g_0(x)|$, $P_0 = \{x \in C \mid g_0(x) \geq c_0/3\}$, and $N_0 = \{x \in C \mid g_0(x) \leq -c_0/3\}$. Show that there is a continuous function F_0 from X into $[-c_0/3, c_0/3]$ that is $c_0/3$ on P_0 and $-c_0/3$ on N_0.

19. In the previous problem, put $g_1 = g_0 - F_0$ on C, and let $c_1 = \sup_{x\in C} |g_1(x)|$. Show that $c_1 \leq \frac{2}{3}c_0$. When the result of the previous problem is applied to g_1 in order to produce a function F_1, what properties does F_1 have?

20. Show that iteration of the above results produces a sequence of continuous functions $F_n : X \to \mathbb{R}$ such that the series $\sum_{n=0}^{\infty} F_n(x)$ is uniformly convergent on X and such that the sum $F(x) = \sum_{n=0}^{\infty} F_n(x)$ is continuous. Show also that F has $F\big|_C = f$ and satisfies $\sup_{x\in X} |F(x)| = \sup_{x\in C} |f(x)|$.

Problems 21–28 concern order topologies. Suppose that X is a set with at least two elements and having a **total ordering**, i.e., a partial ordering $\leq$ such that

(i) $x \leq y$ and $y \leq x$ together imply $x = y$,
(ii) any x and y in the set have either $x \leq y$ or $y \leq x$.

Define $x < y$ to mean that $x \leq y$ and $x \neq y$. The **order topology** on X is the topology for which a base consists of all sets $\{x \mid x < b\}$, $\{x \mid a < x\}$, and $\{x \mid a < x < b\}$. For a nonempty subset Y of X, the terms "lower bound," "upper bound," "greatest lower bound," and "least upper bound" are defined in the expected way. Examples are given by the real line $\mathbb{R}$ with its usual topology, the set Ω of countable ordinals (as defined in Problems 25–33 at the end of Chapter V) with its order topology, and other examples given below.

21. Prove that every open interval $\{x \mid a < x < b\}$ in X is open and every closed interval $\{x \mid a \leq x \leq b\}$ is closed.
22. Prove that X is Hausdorff and regular in its order topology.
23. Prove that every nonempty subset with an upper bound has a least upper bound if and only if every every nonempty subset with a lower bound has a greatest lower bound. In this case, X is said to be **order complete**.
24. Suppose that X is order complete.
    (a) Prove that a nonempty subset Y of X is compact if and only if Y is closed and has a lower bound and an upper bound.
    (b) Prove that X is locally compact.
25. (a) Prove that if there exist a and b in X with $a < b$ and with no c such that $a < c < b$, then X is not connected, in the sense of Problems 12–15. Let us say that X has a **gap** when such a and b exist.
    (b) Prove that if X is order complete and has no gaps, then X is connected.
26. The set $X = [0, 1) \cup [2, 3)$ is totally ordered. Prove that this X is connected in its order topology, and conclude that the order topology is different from the relative topology for X as a subspace of $\mathbb{R}$.
27. The set $X = [0, 1) \cup (1, 2]$ is totally ordered. Prove that this X is not connected in its order topology but has no gaps.
28. Let X and Y be two totally ordered sets with at least two elements apiece. Define the **lexicographic ordering** on $X \times Y$ to be the total ordering given by $(x_1, y_1) \leq (x_2, y_2)$ if $x_1 < x_2$ or else $x_1 = x_2$ and $y_1 \leq y_2$.
    (a) Prove that the lexicographic ordering on $[0, 1] \times [0, 1]$ makes the space compact connected but not separable.
    (b) The **long line** is defined to be the product $\Omega \times [0, 1)$ with the lexicographic ordering, where Ω is the set of countable ordinals as defined in Problems 25–33 at the end of Chapter V. Prove that the long line is locally compact and connected but not separable.