CHAPTER VI

Compact and Locally Compact Groups

Abstract. This chapter investigates several ways that groups play a role in real analysis. For the most part the groups in question have a locally compact Hausdorff topology.

Section 1 introduces topological groups, their quotient spaces, and continuous group actions. Topological groups are groups that are topological spaces in such a way that multiplication and inversion are continuous. Their quotient spaces by subgroups are of interest when they are Hausdorff, and this is the case when the subgroups are closed. Many examples are given, and elementary properties are established for topological groups and their quotients by closed subgroups.

Sections 2–4 investigate translation-invariant regular Borel measures on locally compact groups and invariant measures on their quotient spaces. Section 2 deals with existence and uniqueness in the group case. A left Haar measure on a locally compact group G is a nonzero regular Borel measure invariant under left translations, and right Haar measures are defined similarly. The theorem is that left and right Haar measures exist on G, and each kind is unique up to a scalar factor. Section 3 addresses the relationship between left Haar measures and right Haar measures, which do not necessarily coincide. The relationship is captured by the modular function, which is a certain continuous homomorphism of the group into the multiplicative group of positive reals. The modular function plays a role in constructing Haar measures for complicated groups out of Haar measures for subgroups. Of special interest are "unimodular" locally compact groups G, i.e., those for which the left Haar measures coincide with the right Haar measures. Every compact group, and of course every locally compact abelian group, is unimodular. Section 4 concerns translation-invariant measures on quotient spaces G/H. For the setting in which G is a locally compact group and H is a closed subgroup, the theorem is that G/H has a nonzero regular Borel measure invariant under the action of G if and only if the restriction to H of the modular function of G coincides with the modular function of H. In this case the G invariant measure is unique up to a scalar factor. Section 5 introduces convolution on unimodular locally compact groups G. Familiar results valid for the additive group of Euclidean space, such as those concerning convolution of functions in various L^p classes, extend to be valid for such groups G.

Sections 6–8 concern the representation theory of compact groups. Section 6 develops the elementary theory of finite-dimensional representations and includes some examples, Schur orthogonality, and properties of characters. Section 7 contains the Peter–Weyl Theorem, giving an orthonormal basis of L^2 in terms of irreducible representations and concluding with an Approximation Theorem showing how to approximate continuous functions on a compact group by trigonometric polynomials. Section 8 shows that infinite-dimensional unitary representations of compact groups decompose canonically according to the irreducible finite-dimensional representations of the group. An example is given to show how this theorem may be used to take advantage of the symmetry in analyzing a bounded operator that commutes with a compact group of unitary operators. The same principle applies in analyzing partial differential operators.

1. Topological Groups

The theme of this chapter is the interaction of real analysis with groups. We shall work with topological groups, their quotients, and continuous group actions, all of which are introduced in this section. A **topological group** is a group G with a Hausdorff topology such that multiplication, as a mapping $G \times G \to G$, and inversion, as a mapping $G \to G$, are continuous. A **homomorphism** of topological groups is a continuous group homomorphism. An **isomorphism** of topological groups is a group isomorphism that is a homeomorphism of topological spaces.

EXAMPLES.

(1) Any **discrete group**, i.e., any group with the discrete topology.

(2) The additive group $\mathbb{R}$ or $\mathbb{C}$ with the usual metric topology. The group operation is addition, and the inversion operation is negation.

(3) The multiplicative groups $\mathbb{R}^\times = \mathbb{R} - \{0\}$ and $\mathbb{C}^\times = \mathbb{C} - \{0\}$, with the relative topology from $\mathbb{R}$ or $\mathbb{C}$.

(4) Any subgroup of a topological group, with the relative topology. Thus, for example, the circle $\big\{z \in \mathbb{C} \bigm| |z| = 1\big\}$ is a subgroup of $\mathbb{C}^\times$.

(5) Any product of topological groups, with the product topology. Thus, for example, the additive groups $\mathbb{R}^N$ and $\mathbb{C}^N$ are topological groups. So is the countable product of two-element groups, each with the discrete topology; in this case the topological space in question is homeomorphic to the standard Cantor set in $[0, 1]$.

(6) The **general linear group** $GL(N, \mathbb{C})$ of all nonsingular N-by-N complex matrices, with matrix multiplication as group operation. The topology is the relative topology from $\mathbb{C}^{N^2}$. Each entry in a matrix product is a polynomial in the $2N^2$ entries of the two matrices being multiplied and is therefore continuous; thus matrix multiplication is continuous. Inversion is defined on the set where the determinant polynomial is not 0 and is given, according to Cramer's rule, in each entry by the quotient of a polynomial function and the determinant function; thus inversion is continuous. By (4), the general linear group $GL(N, \mathbb{R})$ is a topological group.

(7) The additive group of any topological vector space in the sense of Section IV.1. The additive groups of normed linear spaces are special cases.

In working with topological groups, we shall use expressions like

$$aU = \{au \mid u \in U\} \qquad \text{and} \qquad Ub = \{ub \mid u \in U\},$$
$$U^{-1} = \{u^{-1} \mid u \in U\} \qquad \text{and} \qquad UV = \{uv \mid u \in U,\ v \in V\}.$$

In any topological group every left translation $y \mapsto xy$ and every right translation $y \mapsto yx$ is a homeomorphism. The continuity of each translation follows by restriction from the continuity of multiplication, and the continuity of the inverse of a translation follows because the inverse of a translation is translation by the inverse element. For an abstract topological group, we write 1 for the identity element.

Continuity of the multiplication mapping $G \times G \to G$ at $(1, 1)$ implies, for any open neighborhood V of the identity in G, that there is an open neighborhood U of the identity for which $UU \subseteq V$. Inversion, being a continuous operation of order two, carries open sets to open sets; therefore if U is an open neighborhood of the identity, so is $U \cap U^{-1}$. Combining these facts, we see that if V is an open neighborhood of the identity, then there is an open neighborhood U of the identity such that $UU^{-1} \subseteq V$.

Conversely if whenever V is an open neighborhood of the identity, there is an open neighborhood U of the identity such that $UU^{-1} \subseteq V$, then it follows that the mapping $(x, y) \mapsto xy^{-1}$ is continuous at $(x, y) = (1, 1)$. If also all translations are homeomorphisms, then $(x, y) \mapsto xy^{-1}$ is continuous, and it follows easily that $x \mapsto x^{-1}$ and $(x, y) \mapsto xy$ are continuous.

Proposition 6.1. If G is a topological group, then G is regular as a topological space.

PROOF. We are to separate by disjoint open sets a point x and a closed set F with $x \notin F$. Since translations are homeomorphisms, we may assume x to be 1. Then $V = F^c$ is an open neighborhood of 1, and we can choose an open neighborhood U of 1 such that $UU \subseteq V$. Let us see that $U^{\text{cl}} \subseteq V$. From $UU \subseteq V$ and $1 \in U$, we have $U \subseteq V$. Thus let y be in $U^{\text{cl}} - U$. Since y is then a limit point of U and since $U^{-1}y$ is an open neighborhood of y, $U^{-1}y$ meets U. If z is in $U^{-1}y \cap U$, then $z = u^{-1}y$ for some u in U, and so $y = uz$ is in $UU \subseteq V$. Thus $U^{\text{cl}} \subseteq V$ and $U^{\text{cl}} \cap F = \varnothing$. Consequently G is regular. $\square$

If H is a subgroup of G, then the **quotient space** G/H of **left cosets** aH results from the equivalence relation that $a \sim b$ if there is some h in H with $a = bh$. The quotient space is given the quotient topology. Quotient spaces of topological groups are sometimes called **homogeneous spaces**.

Proposition 6.2. Let G be a topological group, let H be a *closed* subgroup, and let $q : G \to G/H$ be the quotient map. Then q is an open map, and G/H is a Hausdorff regular space such that the action of G on G/H given by $(g, aH) \mapsto (ga)H$ is continuous. Moreover,

(a) G separable implies G/H separable,
(b) G locally compact implies G/H locally compact,

(c) G is compact if and only if H and G/H are compact,
(d) H normal in the group-theoretic sense implies that G/H is a topological group.

PROOF. Let U be open. To show that $q(U)$ is open, we are to show that $q^{-1}(q(U))$ is open. But $q^{-1}(q(U)) = \bigcup_{h \in H} Uh$, which is open, being the union of open sets. Hence q is open.

To consider the action of G on H, we start from the continuous open mapping $1 \times q : G \times G \to G \times (G/H)$ given by $(g, a) \mapsto (g, aH)$. This descends to a well-defined one-one mapping $\widetilde{q} : (G \times G)/(1 \times H) \to G \times (G/H)$ given by $(g, a)(1 \times H) \mapsto (g, aH)$, and the quotient topology is defined in such a way that this is continuous. The image under $\widetilde{q}$ of an open set is the same as the image under $1 \times q$ of the same open set, and this is open. Therefore $\widetilde{q}$ is a homeomorphism.

The mapping $(g, a) \mapsto (ga)H$ is the composition of multiplication $(g, a) \mapsto ga$ followed by q and is therefore continuous. Hence it descends to a continuous map $(g, a)(1 \times H) \mapsto (ga)H$. If $\widetilde{q}^{-1}$ is followed by this continuous map, the resulting map is $(g, aH) \mapsto (ga)H$, which is the action of G on G/H. Hence the action is continuous.

To see that G/H is regular, we are to separate by disjoint open sets a point x in G/H and a closed set F with $x \notin F$. The continuity of the action shows that we may assume x to be $1H$. Then $M = F^c$ is an open neighborhood of $1H$ in G/H, and the continuity of the action at $(1, 1H)$ shows that we can choose an open neighborhood U of 1 in G and an open neighborhood N of $1H$ in G/H such that $UN \subseteq M$. Let us see that $N^{\text{cl}} \subseteq M$. Using the identity element of U, we see that $N \subseteq M$. Thus let y be in $N^{\text{cl}} - N$. Since y is then a limit point of N and since $U^{-1}y$ is an open neighborhood of y (q being open), $U^{-1}y$ meets N. If z is in $U^{-1}y \cap N$, then $z = u^{-1}y$ for some u in U, and so $y = uz$ is in $UN \subseteq M$. Thus $N^{\text{cl}} \subseteq M$ and $N^{\text{cl}} \cap F = \varnothing$. Consequently G/H is regular.

To see that G/H is Hausdorff, consider the inverse image under q of a coset xH. This inverse image is xH as a subset of G, and this subset is closed in G since H is closed and translations are homeomorphisms. Thus G/H is $\mathbf{T}_1$, as well as regular, and consequently it is Hausdorff.

Conclusion (a) follows from the fact that q is open, since the image under q of a countable base of open sets is therefore a countable base for G/H. Conclusion (b) is similarly immediate; the image of a compact neighborhood of a point is a compact neighborhood of the image point.

In (c), let G be compact. Then H is compact as a closed subset of a compact set, and G/H is compact as the continuous image of a compact set. In the converse direction let $\mathcal{U}$ be an open cover of G. For each x in G, $\mathcal{U}$ is an open cover of the subset xH of G, which is compact since it is homeomorphic to H. Let $\mathcal{V}_x$ be a finite subcover of xH, and let

$$V_x = \{y \in G \mid yH \text{ is covered by } \mathcal{V}_x\}.$$

We show that V_x is open in G. Let W_x be the open union of the members of $\mathcal{V}_x$. If y is in V_x, then yh is in W_x for all h in H. For each such h, we use the continuity of multiplication to find open neighborhoods U_h of 1 and N_h of h in G such that $U_h y N_h \subseteq W_x$. As h varies, the sets N_h cover H. If $\{N_{h_1}, \dots, N_{h_m}\}$ is a finite subcover, then each set $(U_{h_1} \cap \cdots \cap U_{h_m})yN_{h_j}$ lies in W_x and hence so does $(U_{h_1} \cap \cdots \cap U_{h_m})yH$. Thus $(U_{h_1} \cap \cdots \cap U_{h_m})y$ lies in V_x, and V_x is open.

The definition of V_x makes $V_x H = V_x$, and thus $q^{-1}qV_x = X_x$. The open sets V_x together cover G, and hence the open sets qV_x cover G/H. Since G/H is compact, some finite subcollection $\{qV_{x_1}, \dots, qV_{x_n}\}$ covers G/H. The equality $q^{-1}qV_{x_j} = V_{x_j}$ for all j implies that $\{V_{x_1}, \dots, V_{x_n}\}$ is an open cover of G. Then $\bigcup_{j=1}^n \mathcal{V}_{x_j}$ is a finite subcollection of $\mathcal{U}$ that covers G. This proves (c).

In (d), suppose that H is group-theoretically normal, and let V be an open neighborhood of 1 in G/H. Choose, by the continuity of the action on G/H, an open neighborhood U of 1 in G and an open neighborhood N of $1H$ in G/H such that $UN \subseteq V$. Then qU and N are open neighborhoods of the identity in G/H such that $(qU)N \subseteq V$. Hence multiplication in G/H is continuous at $(1, 1)$. Since the map $G \to G/H$ given for fixed aH by $g \mapsto (ga)H$ is continuous, the descended map $gH \mapsto (gH)(aH)$ is continuous. Thus left translations are continuous on G/H, and multiplication on G/H is continuous everywhere. To see continuity of inversion on G/H, let V be an open neighborhood of 1 in G/H, and let U be an open neighborhood of 1 in G with $U^{-1} \subseteq q^{-1}(V)$. Then $q(U^{-1}) \subseteq V$, and inversion is continuous at the identity. Since left and right translations are continuous on G/H, inversion is continuous everywhere. This completes the proof. □

Proposition 6.3. If G is a topological group, then

(a) any open subgroup H of G is closed and the quotient G/H has the discrete topology,

(b) any discrete subgroup H of G (i.e., any subgroup whose relative topology is the discrete topology) is closed.

REMARK. Despite (a), a closed subgroup need not be open. For example, the closed subgroup $\mathbb{Z}$ of integers is not open in the additive group $\mathbb{R}$.

PROOF. For (a), if H is an open subgroup, then every subset xH of G is open in G. Then the formula $H = G - \bigcup_{x \notin H} xH$ shows that H is closed. Also, since $G \to G/H$ is an open map, the openness of the subset xH of G implies that every one-element set $\{xH\}$ in G/H is open. Thus G/H has the discrete topology.

For (b), choose by discreteness an open neighborhood V of 1 in G such that $H \cap V = \{1\}$. By continuity of multiplication, choose an open neighborhood U of 1 with $UU \subseteq V$. If H is not closed, let x be a limit point of H that is not in H. Then the neighborhood $U^{-1}x$ of x must contain a member h of H, and h cannot equal x since x is not in H. Write $u^{-1}x = h$ with $u \in U$. Then $u = xh^{-1}$ is a limit point of H that is not in H, and we can find $h' \neq 1$ in H such that h' is in Uu. But $Uu \subseteq UU \subseteq V$, and so h' is in $H \cap V = \{1\}$, contradiction. We conclude that H contains all its limit points and is therefore closed. $\square$

A **compact group** is a topological group whose topology is compact Hausdorff. Similarly a **locally compact group** is a topological group whose topology is locally compact Hausdorff. Among the examples at the beginning of this section, the following are locally compact: any group with the discrete topology, the additive groups $\mathbb{R}$ and $\mathbb{C}$, the multiplicative groups $\mathbb{R}^\times$ and $\mathbb{C}^\times$, the circle as a subgroup of $\mathbb{C}^\times$, the additive groups $\mathbb{R}^N$ and $\mathbb{C}^N$, the general linear groups $GL(N, \mathbb{R})$ and $GL(N, \mathbb{C})$, and the additive groups of finite-dimensional topological vector spaces. An arbitrary direct product of compact groups, with the product topology, is a compact group. Similarly any *finite* direct product of locally compact groups is a locally compact group.

A number of interesting subgroups of $GL(N, \mathbb{R})$ and $GL(N, \mathbb{C})$ are defined as the sets of matrices where certain polynomials vanish. Since polynomials are continuous, these subgroups are closed in $GL(N, \mathbb{R})$ or $GL(N, \mathbb{C})$. The next proposition says that they provide further examples of locally compact groups.

Proposition 6.4. Any closed subgroup of a locally compact group is a locally compact in the relative topology.

PROOF. Let G be the given locally compact group, and let H be the closed subgroup. As a subgroup of a topological group, H is a topological group. For local compactness, choose a compact neighborhood U_h in G of any element h of H. Then $U_h \cap H$ is a compact set in H since H is closed, and it is a neighborhood of h in the relative topology. Thus h has a compact neighborhood, and H is a locally compact group. $\square$

EXAMPLES OF CLOSED SUBGROUPS OF $GL(N, \mathbb{R})$ AND $GL(N, \mathbb{C})$.

(1) Affine group of the line. This consists of all matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with a and b real and with $a > 0$.

(2) Upper triangular group over $\mathbb{R}$ or $\mathbb{C}$. This consist of all matrices whose entries on the diagonal are all nonzero, whose entries above the diagonal are arbitrary, and whose entries below the diagonal are 0.

(3) Commutator subgroup of previous example. This consists of all matrices whose entries on the diagonal are all 1, whose entries above the diagonal are arbitrary in $\mathbb{R}$ or $\mathbb{C}$, and whose entries below the diagonal are 0.

(4) Special linear group $SL(N, \mathbb{F})$ with $\mathbb{F}$ equal to $\mathbb{R}$ or $\mathbb{C}$. This consists of all N-by-N matrices with determinant 1.

(5) Symplectic group $Sp(N, \mathbb{F})$ with $\mathbb{F}$ equal to $\mathbb{R}$ or $\mathbb{C}$. This consists of all $2N$-by-$2N$ matrices g with determinant 1 such that $g^{\mathrm{tr}} \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix}$.

(6) Unitary group $U(N)$. This consists of all N-by-N complex matrices g that are **unitary** in the sense that $\bar{g}^{\mathrm{tr}} g = 1$. The group is compact; the compactness of the topology follows since the members of $U(N)$ form a closed bounded subset of a Euclidean space. The group $SU(N)$ is the subgroup of all g in $U(N)$ with determinant 1; it is a closed subgroup of $U(N)$ and hence is compact.

(7) Orthogonal group $O(N)$ and rotation group $SO(N)$. The group $O(N)$ consists of all N-by-N real matrices that are **orthogonal** in the sense that $g^{\mathrm{tr}} g = 1$; it is the intersection[1] of the unitary group $U(N)$ with $GL(n, \mathbb{R})$. Members of $O(N)$ have determinant ± 1. The subgroup $SO(N)$ consists of those members of $O(N)$ with determinant 1, i.e., the **rotations**. The groups $O(N)$ and $SO(N)$ are compact.

Proposition 6.5. If G is a locally compact group, then

(a) any compact neighborhood V of 1 with $V = V^{-1}$ has the property that $H = \bigcup_{n=1}^{\infty} V^n$ is a σ-compact open subgroup,

(b) G is normal as a topological space.

PROOF. The set V^n is the result of applying the multiplication mapping to $V \times \cdots \times V$ with n factors. This mapping is continuous, and hence V^n is compact. Thus H is σ-compact. Since $V^n V^m = V^{m+n}$, H is closed under multiplication. Since $V = V^{-1}$, we have $V^n = (V^{-1})^n = (V^n)^{-1}$, and H is closed under inversion. Thus H is a subgroup. Since V is a neighborhood of 1, Vx is a neighborhood of x. Therefore V^{n+1} is a neighborhood of each member of V^n, and H is open. This proves (a).

Let H be as in (a). The subspace H of G is σ-compact and hence Lindelöf, and Tychonoff's Lemma[2] shows that it is normal as a topological subspace. Let $\{x_\alpha\}$ be a complete system of coset representatives for H in G, so that $G = \bigcup_\alpha x_\alpha H$ is exhibited as the disjoint union of open closed sets, each of which is topologically normal. If E and F are disjoint closed sets in G, then $E \cap x_\alpha H$ and $F \cap x_\alpha H$ are disjoint closed sets in $x_\alpha H$. Hence there exist disjoint open sets U_α and V_α in $x_\alpha H$ such that $E \cap x_\alpha H \subseteq U_\alpha$ and $F \cap x_\alpha H \subseteq V_\alpha$. Then $U = \bigcup_\alpha U_\alpha$ and

[1] This fact provides justification for using the term "unitary" in Proposition 2.6 even when $\mathbb{F} = \mathbb{R}$.

[2] Proposition 10.9 of *Basic*.

$V = \bigcup_\alpha V_\alpha$ are disjoint open sets in G such that $E \subseteq U$ and $F \subseteq V$. This proves (b). $\square$

The final proposition of the section shows that members of $C_{\text{com}}(G)$ are uniformly continuous in a certain sense that can be defined without the aid of a metric.

Proposition 6.6. If G is a locally compact group and f is in $C_{\text{com}}(G)$, then for any $\epsilon > 0$, there is an open neighborhood W of the identity with $W = W^{-1}$ such that $xy^{-1} \in W$ implies $|f(x) - f(y)| < \epsilon$.

PROOF. Let S be the support of f, and let $\epsilon > 0$ be given. For each y in S, let U_y be an open neighborhood of y such that $x \in U_y$ implies $|f(x) - f(y)| < \epsilon/2$. Since $U_y y^{-1}$ is a neighborhood of 1, we can find an open neighborhood V_y of 1 with $V_y = V_y{}^{-1}$ and $V_y V_y \subseteq U_y y^{-1}$. As y varies through S, the sets $V_y y$ form an open cover of S. Let $\{V_{y_1} y_1, \dots, V_{y_n} y_n\}$ be a finite subcover, and put $W = V_{y_1} \cap \cdots \cap V_{y_n}$. This will be the required neighborhood of 1.

To see that W has the property asserted, let xy^{-1} be in W. If $f(x) = f(y) = 0$, then $|f(x) - f(y)| < \epsilon$. If $f(y) \neq 0$, then for some k, y is in $V_{y_k} y_k \subseteq U_{y_k} y_k^{-1} y_k = U_{y_k}$ and thus $|f(y_k) - f(y)| < \epsilon/2$. Also, $x = (xy^{-1})y$ is in $W V_{y_k} y_k \subseteq V_{y_k} V_{y_k} y_k \subseteq U_{y_k} y_k^{-1} y_k \subseteq U_{y_k}$ and thus $|f(x) - f(y_k)| < \epsilon/2$. Hence $|f(x) - f(y)| < \epsilon$. Finally if $f(x) \neq 0$, then $W = W^{-1}$ implies that yx^{-1} is in W, the roles of x and y are interchanged, and the proof that $|f(x) - f(y)| < \epsilon$ goes through as above. $\square$

Corollary 6.7. If G is a locally compact group and f is in $C_{\text{com}}(G)$, then the map of $G \times G$ into $C(G)$ given by $(g, g') \mapsto f(g(\,\cdot\,)g')$ is continuous.

PROOF. We first prove two special cases. If $g_0 \in G$ and $\epsilon > 0$ are given, then Proposition 6.6 produces an open neighborhood W of the identity such that $\sup_{x \in G} |f(gx - f(g_0 x)| < \epsilon$ for gg_0^{-1} in W, and hence $g \mapsto f(g(\,\cdot\,))$ is continuous. Applying this result to the function $\widetilde{f}$ given by $\widetilde{f}(x) = f(x^{-1})$ and using continuity of the inversion map $x \mapsto x^{-1}$ within G, we see that $g' \mapsto f((\,\cdot\,)g')$ is continuous.

Now we reduce the general case to these two special cases. If (g_0, g_0') is given in $G \times G$, then

$$\begin{aligned} |f(gxg') - f(g_0 x g_0')| &\leq |f(gxg') - f(g_0 x g')| + |f(g_0 x g') - f(g_0 x g_0')| \\ &\leq \sup_{x \in G} |f(gx) - f(g_0 x)| + \sup_{x \in G} |f(xg') - f(xg_0')|. \end{aligned}$$

The two special cases show that the right side tends to 0 as (g, g') tends to (g_0, g_0'), and the corollary follows. $\square$

If G is a group and X is a set, a **group action** of G on X is a function $G \times X \to X$, often written $(g, x) \mapsto gx$, such that

(i) $1x = x$ for all x in X,

(ii) $g_1(g_2x) = (g_1g_2)x$ for all x in X and all g_1 and g_2 in G.

If G is a topological group and X has a Hausdorff topology, a **continuous group action** is a group action such that the map $(g, x) \mapsto gx$ is continuous. In this case we say that G acts continuously on X. The fundamental example is the action of G on the quotient space G/H by a closed subgroup: $(g, g'H) \mapsto (gg')H$.

An **orbit** for a group action of G on X is any subset Gx of X. The action is **transitive** if there is just one orbit, i.e., if $Gx = X$ for some, or equivalently every, x in X. This is the situation with the fundamental example above. The action of the general linear group $GL(N, \mathbb{R})$ on $\mathbb{R}^N$ by matrix multiplication is a continuous group action that is not transitive; it has two orbits, one open and the other closed.

Let G act continuously on X, fix x_0 in X, and let H be the subgroup of elements h in G with $hx_0 = x_0$. This is the **isotropy subgroup** at x_0. It is a closed subgroup, being the inverse image in G of the closed set $\{x_0\}$ under the continuous function $g \mapsto gx_0$. Proposition 6.2 shows that the quotient topology on the set G/H of left cosets is Hausdorff. Since G/H has the quotient topology, the continuous map $G \to Gx_0$ given by $g \mapsto gx_0$ descends to a one-one continuous map $G/H \to Gx_0$. In favorable cases the map $G/H \to Gx_0$ is a homeomorphism with its image, and Problems 2–4 at the end of the chapter give sufficient conditions for it to be a homeomorphism. Sometimes the ability to do serious analysis on X depends on having the map be a homeomorphism. A case in which it is not a homeomorphism is the action of the discrete additive line G on the ordinary line $X = \mathbb{R}$ by translation.

2. Existence and Uniqueness of Haar Measure

The point of view in *Basic* in approaching the Riesz Representation Theorem for a locally compact Hausdorff space X was that the steps in the construction of Lebesgue measure work equally well with X. The only thing that is missing is some device to encode geometric data—to provide a generalization of length. That missing ingredient is captured by any positive linear functional on $C_{\text{com}}(X)$, but there is no universal source of interesting such functionals.

For the next few sections we shall impose additional structure on X, assuming now that X is a locally compact group in the sense of Section 1. We shall see in this case that a nonzero positive linear functional always exists with the property that it takes equal values on a function and any left translate of the function. In other words the positive linear functional has the same kind of invariance property under translation as the Riemann integral. The corresponding regular Borel measure, which is Lebesgue measure in the case of the line, is called a (left) "Haar measure" and is the main object of study in Sections 2–5 of this chapter.

Several examples of locally compact groups were given in Section 1. Among them are the circle group, the additive group $\mathbb{R}^N$, and the general linear groups $GL(N, \mathbb{C})$ and $GL(N, \mathbb{R})$, which consist of all N-by-N nonsingular matrices and have matrix multiplication as the group operation. Proposition 6.4 showed that any closed subgroup of a locally compact group is itself a locally compact group. Special linear groups, unitary groups, orthogonal groups, and rotation groups are among the examples that were mentioned.

Thus let G be a locally compact group. We shall write the group multiplicatively except when we are dealing with special examples where a different notation is more suitable. Ordinarily no special symbol will be used for a translation map in G. Thus left translations are simply the homeomorphisms $x \mapsto gx$ for g in G, and right translations are the maps $x \mapsto xg$.

Let us consider these as special cases of what any continuous mapping does. The notation will be clearer if we distinguish the domain from the image. Thus let Φ be a continuous mapping of a locally compact Hausdorff space X into a locally compact Hausdorff space Y. The mapping Φ carries subsets of X to subsets of Y by the rule $\Phi(E) = \{\Phi(x) \mid x \in E\}$.

If Φ is a homeomorphism, it preserves the topological character of sets. Thus compact sets go to compact sets, G_δ's go to G_δ's, and so on. Consequently Borel sets map to Borel sets, and Baire sets map to Baire sets.

By contrast a scalar-valued function f on Y **pulls back** to the scalar-valued function f^Φ on X given by $f^\Phi(x) = f(\Phi(x))$, with continuity being preserved. A Borel measure μ on X **pushes forward** to a measure μ_Φ on Y given by $\mu_\Phi(E) = \mu(\Phi^{-1}(E))$; the measure μ_Φ is defined on Borel sets but need not be finite on compact sets. If Φ is a homeomorphism, however, then μ_Φ is a Borel measure, and regularity of μ implies regularity of μ_Φ.

Of great importance for current purposes is the effect of Φ on integration, where the effect is that of a change of variables. The formula is

$$\boxed{\int_X f^\Phi \, d\mu = \int_Y f \, d\mu_\Phi}$$

if f is a Borel function ≥ 0, for example. To prove this formula, we first take f to be the indicator function I_E of a subset E of Y. On the left side we have $I_E^\Phi(x) = I_E(\Phi(x)) = I_{\Phi^{-1}(E)}(x)$. Hence the left side equals $\int_X I_E^\Phi \, d\mu = \mu(\Phi^{-1}(E)) = \mu^\Phi(E)$, which in turn equals the right side $\int_Y I_E \, d\mu_\Phi$. Linearity allows us to extend this conclusion to nonnegative simple functions, and monotone convergence allows us to pass to Borel functions ≥ 0.

An important consequence of the boxed formula is the formula

$$(F\,d\mu)_\Phi = F^{\Phi^{-1}}\,d\mu_\Phi.$$

In fact, if we set $f = F^{\Phi^{-1}} I_E$ in the boxed formula, then we obtain $\int_X F I_E^\Phi\,d\mu = \int_Y F^{\Phi^{-1}} I_E\,d\mu_\Phi$. Thus $\int_{\Phi^{-1}(E)} F\,d\mu = \int_E F^{\Phi^{-1}}\,d\mu_\Phi$ and $(F\,d\mu)_\Phi(E) = (F\,d\mu)(\Phi^{-1}(E)) = \int_{\Phi^{-1}(E)} F\,d\mu = \int_E F^{\Phi^{-1}}\,d\mu_\Phi = (F^{\Phi^{-1}}\,d\mu_\Phi)(E)$.

The Euclidean change-of-variables formula[3] is a special case of the boxed formula, and the content of the theorem amounts to an explicit identification of μ_Φ. Let $\varphi : U \to \varphi(U)$ be a diffeomorphism with $\det \varphi'(x)$ nowhere 0. If $y = \varphi(x)$, then the formula gives $dy = |\det \varphi'(x)|\,dx$. Since $dy = d(\varphi(x)) = (dx)_{\varphi^{-1}}$, the formula is saying that $(dx)_{\varphi^{-1}} = |\det \varphi'(x)|\,dx$. We recover the usual Euclidean integration formula by applying the boxed formula with $\Phi = \varphi^{-1}$, $X = \varphi(U)$, $Y = U$, $d\mu = dy$, and $d\mu_{\varphi^{-1}} = |\det \varphi'(x)|\,dx$, and then by letting $F = f^{\varphi^{-1}}$. The result is $\int_{\varphi(U)} F(y)\,dy = \int_U F(\varphi(x))|\det \varphi'(x)|\,dx$, as it should be.

The rule for composition for points and sets is that $(\Psi \circ \Phi)(x) = \Psi(\Phi(x))$ and $(\Psi \circ \Phi)(E) = \Psi(\Phi(E))$. But for functions and measures the rules are $f^{\Psi\circ\Phi} = (f^\Psi)^\Phi$ and $\mu_{\Psi\circ\Phi} = (\mu_\Phi)_\Psi$. In other words, when Φ is followed by Ψ in operating on points and sets, Φ is again followed by Ψ in pushing forward measures, but Ψ is followed by Φ in pulling back functions. In the special case that $X = Y = G$, this feature will mean that certain expressions that we might want to write as triple products do not automatically satisfy an expected associativity property without some adjustment to the notation.

First consider left translation. On points, left translation L_h by h sends x to hx, and left translation by g sends this to $g(hx) = (gh)x$. The behavior on sets is similar. On functions and measures we therefore have $f^{L_{gh}} = f^{L_g L_h} = (f^{L_h})^{L_g}$ and $\mu_{L_{gh}} = \mu_{L_g L_h} = (\mu_{L_h})_{L_g}$. To obtain group actions on functions and measures, we therefore define

$$(gf)(x) = f^{L_g^{-1}}(x) = f(g^{-1}x) \qquad \text{and} \qquad (g\mu)(E) = \mu_{L_g}(E) = \mu(g^{-1}E)$$

for g in G. With these definitions we have $g(hf) = (gh)f$ and $g(h\mu) = (gh)\mu$, consistently with the formulas for a group action.

With right translation the effect on points is that right translation by h sends x to xh, and right translation by g sends this to $(xh)g = x(hg)$. The behavior on sets is similar. We want the same kind of formula with functions and measures, and to get it we define

$$(fg)(x) = f(xg^{-1}) \qquad \text{and} \qquad (\mu g)(E) = \mu(Eg^{-1})$$

[3] Theorem 6.32 of *Basic*.

for g in G. With these definitions we have $(fh)g = f(hg)$ and $(\mu h)g = \mu(hg)$. These are the formulas of what we might view as a "right group action."

A nonzero regular Borel measure on G invariant under all left translations is called a **left Haar measure** on G. A **right Haar measure** on G is a nonzero regular Borel measure invariant under all right translations. The main theorem, whose proof will occupy much of the remainder of this section, is as follows.

Theorem 6.8. If G is a locally compact group, then G has a left Haar measure, and it is unique up to a multiplicative constant. Similarly G has a right Haar measure, and it is unique up to a multiplicative constant.

Before coming to the proof, we give some examples. Checking the invariance in each case involves using the boxed formula above for some homeomorphism Φ. In Euclidean situations we can often evaluate μ_Φ directly by the change-of-variables formula for multiple integrals. In an abelian group the left and right Haar measures are the same, and we speak simply of Haar measure; but this need not be true in nonabelian groups, as one of the examples will illustrate.

EXAMPLES.

(1) $G = \mathbb{R}^N$ under addition. Lebesgue measure is a Haar measure.

(2) $G = GL(N, \mathbb{R})$. Problem 4 in Chapter VI of *Basic* showed that if M_N is the N^2-dimensional Euclidean space of all real N-by-N matrices and if dx refers to its Lebesgue measure, then

$$\int_{M_N} f(gx)\,\frac{dx}{|\det x|^N} = \int_{M_N} f(x)\,\frac{dx}{|\det x|^N}$$

for each nonsingular matrix g and Borel function $f \geq 0$. In the formula, gx is the matrix product of g and x. Problem 10 in the same chapter showed that the zero locus of any polynomial that is not identically zero has Lebesgue measure 0. Thus the set where $\det x = 0$ has measure 0, and we can rewrite the above formula as

$$\int_{GL(N,\mathbb{R})} f(gx)\,\frac{dx}{|\det x|^N} = \int_{GL(N,\mathbb{R})} f(x)\,\frac{dx}{|\det x|^N},$$

where dx is still Lebesgue measure on the underlying Euclidean space of all N-by-N matrices. This formula says that $\frac{dx}{|\det x|^N}$ is a left Haar measure on $GL(N, \mathbb{R})$. This measure happens to be also a right Haar measure.

(3) $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right\}$ with real entries and $a > 0$. Then $a^{-2}\,da\,db$ is a left Haar measure and $a^{-1}\,da\,db$ is a right Haar measure. To check the first of these assertions, let φ be left translation by $\begin{pmatrix} a_0 & b_0 \\ 0 & 1 \end{pmatrix}$. Since $\begin{pmatrix} a_0 & b_0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_0 a & a_0 b + b_0 \\ 0 & 1 \end{pmatrix}$,

we can regard φ as the vector function $\varphi\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a_0 a \\ a_0 b + b_0 \end{pmatrix}$ with $\varphi'\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix}$ and $\left|\det \varphi'\begin{pmatrix} a \\ b \end{pmatrix}\right| = a_0^2$. Then $(da\,db)_{\varphi^{-1}} = a_0^2\,da\,db$ and $(a^{-2}\,da\,db)_{\varphi^{-1}} = (a^{-2})^{\varphi}(da\,db)_{\varphi^{-1}} = (a_0 a)^{-2} a_0^2\,da\,db = a^{-2}\,da\,db$. So $a^{-2}\,da\,db$ is indeed a left Haar measure. By a similar argument, $a^{-1}\,da\,db$ is a right Haar measure.

We shall begin the proof of Theorem 6.8 with uniqueness. The argument will use Fubini's Theorem for certain Borel measures on G, and we need to make two adjustments to make Fubini's Theorem apply. One is to work with Baire sets, rather than Borel sets, so that the product σ-algebra from the Baire sets of G times the Baire sets of G is the σ-algebra of Baire sets for $G \times G$.[4] The other is to arrange that the spaces we work with are σ-compact. The device for achieving the σ-compactness is Proposition 6.5, which shows that G always has an open σ-compact subgroup H. Imagine that we understand the restriction of a left Haar measure μ to H. We form the left cosets gH, all of which are open in G. Any compact set is covered by all these cosets, and there is a finite subcover. That means that any compact set K is contained in the union of finitely many cosets gH, say in $g_1 H \cup \cdots \cup g_n H$. We can compute μ on any gH by translating the set by g^{-1}. This fact and the formula $\mu(K) = \sum_{j=1}^n \mu(K \cap g_j H)$ together show that we can compute $\mu(K)$ from a knowledge of μ on H. Thus there is no loss of generality in the uniqueness question in assuming that G is σ-compact.

PROOF OF UNIQUENESS IN THEOREM 6.8. As remarked above, G has a σ-compact open subgroup H, and it is enough to prove the uniqueness for H. Changing notation, we may assume that our given group is σ-compact. We work with Baire sets in this argument.

Let μ_1 and μ_2 be left Haar measures. Then the sum $\mu = \mu_1 + \mu_2$ is a left Haar measure, and $\mu(E) = 0$ implies $\mu_1(E) = 0$. By the Radon–Nikodym Theorem,[5] there exists a Baire function $h_1 \geq 0$ such that $\mu_1 = h_1\,d\mu$. Fix g in G. By the left invariance of μ_1 and μ, we have

$$\begin{aligned} \int_G f(x) h_1(g^{-1}x)\,d\mu(x) &= \int_G f(gx) h_1(x)\,d\mu(x) = \int_G f(gx)\,d\mu_1(x) \\ &= \int_G f(x)\,d\mu_1(x) = \int_G f(x) h_1(x)\,d\mu(x) \end{aligned}$$

for every Baire function $f \geq 0$. Therefore the measures $h_1(g^{-1}x)\,d\mu(x)$ and $h_1(x)\,d\mu(x)$ are equal, and $h_1(g^{-1}x) = h_1(x)$ for almost every $x \in G$ (with respect to $d\mu$). We can regard $h_1(g^{-1}x)$ and $h_1(x)$ as functions of $(g, x) \in G \times G$,

[4] Proposition 11.17 of *Basic*.
[5] Theorem 9.16 of *Basic*.

and these are Baire functions since the group operations are continuous. For each g, they are equal for almost every x. By Fubini's Theorem they are equal for almost every pair (g, x) (with respect to the product measure), and then for almost every x they are equal for almost every g. Pick one such x, say x_0. Then it follows that $h_1(x) = h_1(x_0)$ for almost every x. Thus $d\mu_1 = h_1(x_0)\, d\mu$. So $d\mu_1$ is a multiple of $d\mu$, and so is $d\mu_2$. □

Now we turn our attention to existence. The shortest and best-motivated known proof dates from 1940 and modifies Haar's original argument in two ways that we shall mention. First let us consider that original argument, in which the setting is a locally compact separable metric topological group. In trying to construct an invariant measure, there is not much to work with, the situation being so general. We can get an idea how to proceed by examining $\mathbb{R}^N$, where we are trying to construct Lebesgue measure out of almost nothing. We do have some rough comparisons of size because of the compactness. If we take a compact geometric rectangle and an open geometric rectangle, the latter centered at the origin, the compactness ensures that finitely many translates of the open rectangle together cover the compact rectangle. The smallest such number of translates is a rough estimate of the ratio of their Lebesgue measures. This integer estimate in some sense becomes more refined as the open rectangle gets smaller, but the integer in question grows in size also. To take this scaling factor into account, we compare this integer ratio with the integer ratio for some standard compact rectangle as the open rectangle gets small. This ratio of two integer ratios appears to be a good approximation to the ratio of the measure of the general compact rectangle to the measure of the standard compact rectangle. In fact, one easily shows that this ratio of ratios is bounded above and below as the open rectangle shrinks in size through a sequence of rectangles to a point. The Bolzano–Weierstrass Theorem gives a convergent subsequence for the ratio of ratios. It turns out that this convergence has to be addressed only for countably many of the compact rectangles, and this we can do by the Cantor diagonal process. Then we obtain a value for the measure of each compact rectangle in the countable set and, as a result, for all compact rectangles. It then has to be shown that we can build a measure out of this definition of the measure on compact rectangles.

Two things are done to modify the above argument to obtain a general proof for locally compact groups. One is to replace the Cantor diagonal process by an application of the Tychonoff Product Theorem. The other is to bypass the long process of constructing a measure on Borel sets from its values on compact sets by instead using positive linear functionals and applying the Riesz Representation Theorem. Once an initial comparison can be made with continuous functions of compact support, rather than compact sets and open sets, the path to the theorem is fairly clear. It is Lemma 6.9 below that says that the initial comparison can be carried out with such functions. For a locally compact group G, let $C^+_{\text{com}}(G)$ be the set of nonnegative elements in $C_{\text{com}}(G)$.

Lemma 6.9. If f and φ are nonzero members of $C^+_{\text{com}}(G)$, then there exist a positive integer n, finitely many members $g_1, \dots, g_n$ of G, and real numbers $c_1, \dots, c_n$ all > 0 such that

$$f(x) \le \sum_{j=1}^{n} c_j \varphi(g_j x) \qquad \text{for all } x.$$

REMARK. We let $H(f, \varphi)$ be the infimum of all finite sums $\sum_j c_j$ as in the statement of the lemma. The expression $H(f, \varphi)$ is called the value of the **Haar covering function** at f and φ.

PROOF. Fix $c > \|f\|_{\text{sup}}/\|\varphi\|_{\text{sup}}$. The set $U = \{x \mid c\varphi(x) > \|f\|_{\text{sup}}\}$ is open and nonempty, and the sets hU, for $h \in G$, form an open cover of the support of f. Choose a finite subcover, writing

$$\text{support}(f) \subseteq h_1 U \cup \cdots \cup h_n U.$$

For $1 \le j \le n$, we then have

$$\begin{aligned} h_j U = \{x \mid h_j^{-1} x \in U\} &= \{x \mid c\varphi(h_j^{-1}x) > \|f\|_{\text{sup}}\} \\ &\subseteq \{x \mid f(x) \le \textstyle\sum_{j=1}^n c\varphi(h_j^{-1}x)\}. \end{aligned}$$

Hence

$$\text{support}(f) \subseteq \{x \mid f(x) \le \textstyle\sum_{j=1}^n c\varphi(h_j^{-1}x)\}.$$

The lemma follows with $g_j = h_j^{-1}$ and with all c_j equal to c. $\square$

Lemma 6.10. The Haar covering function has the properties that

(a) $H(gf, \varphi) = H(f, \varphi)$ for g in G,
(b) $H(f_1 + f_2, \varphi) \le H(f_1, \varphi) + H(f_2, \varphi)$,
(c) $H(cf, \varphi) = cH(f, \varphi)$ for $c > 0$,
(d) $f_1 \le f_2$ implies $H(f_1, \varphi) \le H(f_2, \varphi)$,
(e) $H(f, \psi) \le H(f, \varphi)H(\varphi, \psi)$,
(f) $H(f, \varphi) \ge \|f\|_{\text{sup}} \big/ \|\varphi\|_{\text{sup}}$.

PROOF. Properties (a) through (d) are completely elementary. For (e), the inequalities $f(x) \le \sum_i c_i \varphi(g_i x)$ and $\varphi(x) \le \sum_j d_j \psi(h_j x)$ together imply that $f(x) \le \sum_{i,j} c_i d_j \psi(h_j g_i x)$. Therefore

$$H(f, \psi) \le \inf \textstyle\sum_{i,j} c_i d_j = \big(\inf \sum_i c_i\big)\big(\inf \sum_j d_j\big) = H(f, \varphi)H(\varphi, \psi).$$

For (f), the fact that a continuous real-valued function on a compact set attains its maximum value allows us to choose y such that $f(y) = \|f\|_{\sup}$. Then $\|f\|_{\sup} = f(y) \leq \sum_j c_j\varphi(g_j y) \leq \sum_j c_j\|\varphi\|_{\sup}$ and hence $\|f\|_{\sup}\big/\|\varphi\|_{\sup} \leq \sum_j c_j$. Taking the infimum over systems of constants c_j gives $\|f\|_{\sup}\big/\|\varphi\|_{\sup} \leq H(f, \varphi)$. □

Following the outline above, we now perform the normalization. Fix a nonzero member f_0 of $C^+_{\text{com}}(G)$. If φ and f are nonzero members of $C^+_{\text{com}}(G)$, define

$$\ell_\varphi(f) = H(f, \varphi)\big/H(f_0, \varphi).$$

After listing some elementary properties of ℓ_φ, we shall prove in effect that ℓ_φ is close to being additive if the support of φ is small.

Lemma 6.11. $\ell_\varphi(f)$ has the properties that

(a) $0 < \frac{1}{H(f_0, f)} \leq \ell_\varphi(f) \leq H(f, f_0)$,
(b) $\ell_\varphi(gf) = \ell_\varphi(f)$ for g in G,
(c) $\ell_\varphi(f_1 + f_2) \leq \ell_\varphi(f_1) + \ell_\varphi(f_2)$,
(d) $\ell_\varphi(cf) = c\ell_\varphi(f)$ if $c > 0$ is a constant.

PROOF. Properties (b), (c), and (d) are immediate from (a), (b), and (c) of Lemma 6.10. For (a), we apply Lemma 6.10e with φ there equal to f_0 and with ψ there equal to φ, and the resulting inequality is $H(f, \varphi) \leq H(f, f_0)H(f_0, \varphi)$. Thus $\ell_\varphi(f) \leq H(f, f_0)$. Then we apply apply Lemma 6.10e with f there equal to f_0, φ there equal to f, and ψ there equal to φ. The resulting inequality is $H(f_0, \varphi) \leq H(f_0, f)H(f, \varphi)$. Thus $1/H(f_0, f) \leq \ell_\varphi(f)$. □

Lemma 6.12. If f_1 and f_2 are nonzero members of $C^+_{\text{com}}(G)$ and if $\epsilon > 0$ is given, then there exists an open neighborhood V of the identity in G such that

$$\ell_\varphi(f_1) + \ell_\varphi(f_2) \leq \ell_\varphi(f_1 + f_2) + \epsilon$$

for every nonzero φ in $C^+_{\text{com}}(G)$ whose support is contained in V.

PROOF. Let K be the support of $f_1 + f_2$, and let F be a member of $C_{\text{com}}(G)$ with values in [0, 1] such that F is 1 on K. The number $\epsilon > 0$ is given in the statement of the lemma, and we let δ be a positive number to be specified. Define $f = f_1 + f_2 + \delta F$, $h_1 = f_1/f$, and $h_2 = f_2/f$, with the convention that h_1 and h_2 are 0 on the set where f is 0.

The functions h_1 and h_2 are continuous: In fact, there is no problem on the open set where $f(x) \neq 0$. At a point x where $f(x) = 0$, the functions h_1 and h_2 are continuous unless x is a limit point of the set where $f_1 + f_2$ is not 0. This set is contained in K, and thus x must be in K. On the other hand, F is 1 on K, and hence f is $\geq \delta$ on K. Hence there are no points x where h_1 or h_2 fails to be continuous.

Let $\eta > 0$ be another number to be specified. By Proposition 6.6 let V be an open neighborhood of the identity such that $V = V^{-1}$ and also

$$|h_1(x) - h_1(y)| < \eta \qquad \text{and} \qquad |h_2(x) - h_2(y)| < \eta$$

whenever xy^{-1} is in V. If $\varphi \in C^+_{\text{com}}(G)$ has support in V and if positive constants c_j and group elements g_j are chosen such that $f(x) \leq \sum_j c_j \varphi(g_j x)$ for all x, then every x for which $\varphi(g_j x) > 0$ has the property that

$$|h_1(g_j^{-1}) - h_1(x)| < \eta \qquad \text{and} \qquad |h_2(g_j^{-1}) - h_2(x)| < \eta.$$

Hence

$$f_1(x) = f(x)h_1(x) \leq \sum_j c_j \varphi(g_j x) h_1(x) \leq \sum_j \big(c_j(h_1(g_j^{-1}) + \eta)\big)\varphi(g_j x).$$

Consequently

$$H(f_1, \varphi) \leq \sum_j \big(c_j(h_1(g_j^{-1}) + \eta)\big).$$

Similarly

$$H(f_2, \varphi) \leq \sum_j \big(c_j(h_2(g_j^{-1}) + \eta)\big).$$

Adding, we obtain

$$H(f_1, \varphi) + H(f_2, \varphi) \leq \sum_j \big(c_j(h_1(g_j^{-1}) + h_2(g_j^{-1}) + 2\eta)\big) \leq \sum_j c_j(1 + 2\eta)$$

since $h_1 + h_2 \leq 1$. Taking the infimum over the c_j's and the g_j's gives

$$H(f_1, \varphi) + H(f_2, \varphi) \leq H(f, \varphi)(1 + 2\eta).$$

Therefore

$$\begin{aligned}
&\ell_\varphi(f_1) + \ell_\varphi(f_2) \\
&\qquad \leq \ell_\varphi(f)(1 + 2\eta) \\
&\qquad \leq \big(\ell_\varphi(f_1 + f_2) + \delta \ell_\varphi(F)\big)(1 + 2\eta) \qquad \text{by (c) and (d) in Lemma 6.11} \\
&\qquad \leq \ell_\varphi(f_1 + f_2) + \big(\delta H(F, f_0) + 2\delta\eta H(F, f_0) + 2\eta H(f_1 + f_2, f_0)\big),
\end{aligned}$$

the last inequality holding by Lemma 6.11a. This proves the inequality of the lemma if δ and η are chosen small enough that

$$\delta H(F, f_0) + 2\delta\eta H(F, f_0) + 2\eta H(f_1 + f_2, f_0) < \epsilon. \qquad \square$$

Lemma 6.13. There exists a nonzero positive linear functional ℓ on $C_{\text{com}}(G)$ such that $\ell(f) = \ell(gf)$ for all $g \in G$ and $f \in C_{\text{com}}(G)$.

PROOF. For each nonzero f in $C^+_{\text{com}}(G)$, let S_f be the closed interval $[1/H(f_0, f),\ H(f, f_0)]$. Let S be the compact Hausdorff space

$$S = \bigtimes_{\substack{f \in C^+_{\text{com}}(G),\\ f \neq 0}} S_f.$$

A member of S is a function that assigns to each nonzero member f of $C^+_{\text{com}}(G)$ a real number in the closed interval S_f, and $\ell_\varphi(f)$ is such a function, according to Lemma 6.11a. For each open neighborhood V of the identity in G, define

$$E_V = \big\{\ell_\varphi \;\big|\; \varphi \in C^+_{\text{com}}(G),\ \varphi \neq 0,\ \text{support}(\varphi) \subseteq V\big\}$$

as a nonempty subset of S. If $V \subseteq V'$, then $E_V \subseteq E_{V'}$ and hence also $E_V^{\text{cl}} \subseteq E_{V'}^{\text{cl}}$. Thus if $V_1, \dots, V_n$ are open neighborhoods of the identity, then

$$E^{\text{cl}}_{V_1 \cap \dots \cap V_n} \subseteq E^{\text{cl}}_{V_1} \cap \dots \cap E^{\text{cl}}_{V_n}.$$

Consequently the closed sets E_V^{cl} have the finite-intersection property. Since S is compact, they have nonempty intersection. Let ℓ be a point of S lying in their intersection. For ℓ to be in E_V^{cl} for a particular V means that for each $\epsilon > 0$ and each finite set $f_1, \dots, f_n$ of nonzero members of $C^+_{\text{com}}(G)$, there is a nonzero φ in $C^+_{\text{com}}(G)$ with support in V such that

$$|\ell(f_j) - \ell_\varphi(f_j)| < \epsilon \qquad \text{for } 1 \leq j \leq n. \tag{$*$}$$

On the nonzero functions in $C^+_{\text{com}}(G)$, let us observe the following facts:

(i) $\ell(f) \geq 0$ and $\ell(f_0) = 1$, the latter because $\ell_\varphi(f_0) = 1$ for all φ.

(ii) $\ell(f) = \ell(gf)$ for $g \in G$, since for any $\epsilon > 0$, $|\ell(f) - \ell(gf)| \leq |\ell(f) - \ell_\varphi(f)| + |\ell_\varphi(f) - \ell_\varphi(gf)| + |\ell_\varphi(gf) - \ell(gf)| < 2\epsilon$ by Lemma 6.11b if V and φ are as in $(*)$ for the two functions f and gf.

(iii) $\ell(f_1 + f_2) = \ell(f_1) + \ell(f_2)$ because if $\epsilon > 0$ is given, if V is chosen for this ϵ according to Lemma 6.12, and if φ is chosen for f_1, f_2, and f as in $(*)$, then we have $\ell(f_1 + f_2) \leq \ell_\varphi(f_1 + f_2) + \epsilon \leq \ell_\varphi(f_1) + \ell_\varphi(f_2) + \epsilon \leq \ell(f_1) + \ell(f_2) + 3\epsilon$ and $\ell(f_1) + \ell(f_2) \leq \ell_\varphi(f_1) + \ell_\varphi(f_2) + 2\epsilon \leq \ell_\varphi(f_1 + f_2) + 3\epsilon \leq \ell(f_1 + f_2) + 4\epsilon$, the next-to-last inequality holding by Lemma 6.12.

(iv) $\ell(cf) = c\ell(f)$ for $c > 0$ because if V and φ are as in $(*)$ for $\epsilon > 0$ and the two functions f and cf, then we have $\ell(cf) \leq \ell_\varphi(cf) + \epsilon = c\ell_\varphi(f) + \epsilon \leq c\ell(f) + (c+1)\epsilon$ and $c\ell(f) \leq c\ell_\varphi(f) + c\epsilon = \ell_\varphi(cf) + c\epsilon \leq \ell(cf) + (c+1)\epsilon$.

Because of (iii) and (iv), ℓ extends to a linear functional on $C_{\text{com}}(G)$, and this linear functional is positive by (i) and satisfies the invariance condition $\ell(f) = \ell(gf)$ by (ii). $\square$

PROOF OF EXISTENCE IN THEOREM 6.8. Fix a nonzero function f_0 in $C^+_{\text{com}}(G)$, and let μ be the measure given by the Riesz Representation Theorem as corresponding to the positive linear functional ℓ in Lemma 6.13. If K_0 is a nonempty compact G_δ and if $\{f_n\}$ is a decreasing sequence in $C_{\text{com}}(G)$ with pointwise limit I_{K_0}, then we have $\int_G gf_n\,d\mu = \int_G f_n\,d\mu$ for all $g \in G$ and all n. Passing to the limit and applying dominated convergence gives $\int_G gI_{K_0}\,d\mu = \int_G I_{K_0}\,d\mu$. Now $gI_{K_0}(x) = I_{K_0}(g^{-1}x) = I_{gK_0}(x)$, and hence $\mu(gK_0) = \mu(K_0)$ for all g. In other words, the regular Borel measures $g^{-1}\mu$ and μ agree on compact G_δ's. This equality is enough[6] to force the equality $g^{-1}\mu = \mu$ for all g. Finally μ is not the 0 measure since $\int_G f_0\,d\mu = 1$. $\square$

3. Modular Function

We continue with G as a locally compact group. From now on, we shall often denote particular left and right Haar measures on G by $d_l x$ and $d_r x$, respectively.

An important property of left and right Haar measures is that

any nonempty open set has nonzero Haar measure.

In fact, in the case of a left Haar measure, if any compact set is given, finitely many left translates of the given open set together cover the compact set. If the open set had 0 measure, so would its left translates and so would every compact set. Then the measure would be identically 0 by regularity. A similar argument applies to any right Haar measure. We shall occasionally make use of this property without explicit mention.

Actually, left Haar measure and right Haar measure have the same sets of measure 0, as will follow from Proposition 6.15c below. Thus we are completely justified in using the expression "nonzero Haar measure" above.

Fix a left Haar measure $d_l x$. Since left translations on G commute with right translations, $d_l(\,\cdot\,g)$ is a left Haar measure for any $g \in G$. Left Haar measures are proportional, and we therefore define the **modular function** $\Delta : G \to \mathbb{R}^+$ of G by

$$d_l(\,\cdot\,g) = \Delta(g^{-1})\,d_l(\,\cdot\,).$$

Lemma 6.14. For any regular Borel measure μ on G, any g_0 in G, and any p with $1 \leq p < \infty$, the limit relations

$$\lim_{g\to g_0} \int_G |f(gx) - f(g_0x)|^p\,d\mu(x) = 0$$

and

$$\lim_{g\to g_0} \int_G |f(xg) - f(xg_0)|^p\,d\mu(x) = 0$$

[6] Propositions 11.19 and 11.18 of *Basic*.

hold for each f in $C_{\mathrm{com}}(G)$. In particular,

$$g \mapsto \int_G f(gx)\,d\mu(x) \qquad \text{and} \qquad g \mapsto \int_G f(xg)\,d\mu(x)$$

are continuous scalar-valued functions for such f.

PROOF. Corollary 6.7 shows that $g \mapsto f(g(\,\cdot\,))$ is continuous from G into $C(G)$. Let $\epsilon > 0$ be given, and choose a neighborhood N of g_0 such that $\sup_{x\in G} |f(gx) - f(g_0x)| \le \epsilon$ for g in N. If K is a compact neighborhood of g_0, then the set of products K support(f) is compact, being the continuous image of a compact subset of $G \times G$ under multiplication. It therefore has finite μ measure, say C. When g is in $K \cap N$, we have

$$\int_G |f(gx) - f(g_0x)|^p\,d\mu(x) \le \epsilon^p \mu(K\,\text{support}(f)) = C\epsilon^p,$$

and the first limit relation follows. Taking $p = 1$, we have

$$\Big|\int_G f(gx)\,d\mu(x) - \int_G f(g_0x)\,d\mu(x)\Big| \le \int_G |f(gx) - f(g_0x)|\,d\mu(x),$$

and we have just seen that the right side tends to 0 as g tends to g_0. This proves the first conclusion about continuity of scalar-valued functions.

For the other limit relation and continuity result, we replace f by the function $\widetilde{f}$ with $\widetilde{f}(x) = f(x^{-1})$, and we apply to $\widetilde{f}$ what has just been proved, taking into account the continuity of the inversion mapping on G. □

Proposition 6.15. The modular function Δ for G has the properties that

(a) $\Delta : G \to \mathbb{R}^+$ is a continuous group homomorphism,
(b) $\Delta(g) = 1$ for g in any compact subgroup of G,
(c) $d_l(x^{-1})$ and $\Delta(x)\,d_lx$ are right Haar measures and are equal,
(d) $d_r(x^{-1})$ and $\Delta(x)^{-1}\,d_rx$ are left Haar measures and are equal,
(e) $d_r(g\,\cdot\,) = \Delta(g)\,d_r(\,\cdot\,)$ for any right Haar measure on G.

PROOF. For (a), we take $d\mu(x) = d_lx$ in Lemma 6.14 and see that the function $g \mapsto \int_G f(xg)\,d_lx = \int_G f(x)\,d_l(xg^{-1}) = \Delta(g)\int_G f(x)\,d_lx$ is continuous if f is in $C_{\mathrm{com}}(G)$. Since there exist functions f in $C_{\mathrm{com}}(G)$ with $\int_G f(x)\,d_lx \ne 0$, $g \mapsto \Delta(g)$ is continuous. The homomorphism property follows from the fact that $\Delta(hg)\,d_lx = d_l(x(hg)^{-1}) = d_l((xg^{-1})h^{-1}) = \Delta(h)\,d_l(xg^{-1}) = \Delta(h)\Delta(g)\,d_lx$.

For (b), the image under Δ of any compact subgroup of G is a compact subgroup of $\mathbb{R}^+$ and hence is $\{1\}$.

In (c), put $d\mu(x) = \Delta(x)\,d_lx$. This is a regular Borel measure since Δ is continuous by (a). Since Δ is a homomorphism, we have

$$\begin{aligned}\int_G f(xg)\,d\mu(x) &= \int_G f(xg)\Delta(x)\,d_lx = \int_G f(x)\Delta(xg^{-1})\,d_l(xg^{-1})\\ &= \int_G f(x)\Delta(x)\Delta(g^{-1})\Delta(g)\,d_lx\\ &= \int_G f(x)\Delta(x)\,d_lx = \int_G f(x)\,d\mu(x).\end{aligned}$$

Hence $d\mu(x)$ is a right Haar measure. Meanwhile, $d_l(x^{-1})$ is a right Haar measure because

$$\begin{aligned}\textstyle\int_G f(xg)\,d_l(x^{-1}) &= \textstyle\int_G f(x^{-1}g)\,d_lx = \int_G f((g^{-1}x)^{-1})\,d_lx \\ &= \textstyle\int_G f(x^{-1})\,d_lx = \int_G f(x)\,d_l(x^{-1}).\end{aligned}$$

Thus Theorem 6.8 for right Haar measures implies that $d_l(x^{-1}) = c\Delta(x)\,d_lx$ for some constant $c > 0$. Changing x to x^{-1} in this formula, we obtain

$$d_lx = c\Delta(x^{-1})\,d_l(x^{-1}) = c^2\Delta(x^{-1})\Delta(x)\,d_lx = c^2\,d_lx.$$

Hence $c = 1$, and (c) is proved.

For (d) and (e) there is no loss of generality in assuming that $d_rx = d_l(x^{-1}) = \Delta(x)\,d_lx$, in view of (c). Conclusion (d) is immediate from this identity if we replace x by x^{-1}. For (e) we have

$$\begin{aligned}\textstyle\int_G f(x)\,d_r(gx) &= \textstyle\int_G f(g^{-1}x)\,d_rx = \int_G f(g^{-1}x)\Delta(x)\,d_lx = \int_G f(x)\Delta(gx)\,d_lx \\ &= \textstyle\Delta(g)\int_G f(x)\Delta(x)\,d_lx = \Delta(g)\int_G f(x)\,d_rx,\end{aligned}$$

and we conclude that $d_r(g\,\cdot\,) = \Delta(g)\,d_r(\,\cdot\,)$. □

The locally compact group G is said to be **unimodular** if every left Haar measure is a right Haar measure (and vice versa). In this case we can speak of **Haar measure** on G.

In view of Proposition 6.15e, G is unimodular if and only if $\Delta(t) = 1$ for all $t \in G$. Locally compact abelian groups are of course unimodular. Proposition 6.15b shows that compact groups are unimodular.

Any commutator $ghg^{-1}h^{-1}$ in G is carried to 1 by the modular function Δ. Consequently any group that is generated by commutators, such as $SL(N,\mathbb{R})$, is unimodular. More generally any group that is generated by commutators, elements of the center, and elements of finite order is unimodular; $GL(N,\mathbb{R})$ is an example.

Theorem 6.16. Let G be a separable locally compact group, and let S and T be closed subgroups such that $S \cap T$ is compact, multiplication $S \times T \to G$ is an open map, and the set of products ST exhausts G except possibly for a set of Haar measure 0. Let Δ_T and Δ_G denote the modular functions of T and G. Then the left Haar measures on G, S, and T can be normalized so that

$$\int_G f(x)\,d_lx = \int_{S\times T} f(st)\frac{\Delta_T(t)}{\Delta_G(t)}\,d_ls\,d_lt$$

for all Borel functions $f \geq 0$ on G.

REMARK. The assumption of separability avoids all potential problems with using Fubini's Theorem in the course of the proof. Problems 21–22 at the end of the chapter give a condition under which multiplication $S \times T \to G$ is an open map, and they provide examples.

PROOF. Let $\Omega \subseteq G$ be the set of products ST, and let $K = S \cap T$. The group $S \times T$ acts continuously on Ω by $(s, t)\omega = s\omega t^{-1}$, and the isotropy subgroup at 1 is $\operatorname{diag} K$. Thus the map $(s, t) \mapsto st^{-1}$ descends to a map $(S \times T)/\operatorname{diag} K \to \Omega$. This map is a homeomorphism since multiplication $S \times T \to G$ is assumed to be an open map.

Hence any Borel measure on Ω can be reinterpreted as a Borel measure on $(S \times T)/\operatorname{diag} K$. We apply this observation to the restriction of a left Haar measure $d_l x$ for G from G to Ω, obtaining a Borel measure $d\mu$ on $(S \times T)/\operatorname{diag} K$. On Ω, we have

$$d_l(s_0 x t_0^{-1}) = \Delta_G(t_0)\, d_l x,$$

and the action unwinds to

$$d\mu((s_0, t_0)(s, t)(\operatorname{diag} K)) = \Delta_G(t_0)\, d\mu((s, t)(\operatorname{diag} K)) \tag{$*$}$$

on $(S \times T)/\operatorname{diag} K$. Using the Riesz Representation Theorem, define a measure $d\widetilde{\mu}(s, t)$ on $S \times T$ in terms of a positive linear functional on $C_{\text{com}}(S \times T)$ by

$$\int_{S\times T} f(s, t)\, d\widetilde{\mu}(s, t) = \int_{(S\times T)/\operatorname{diag} K} \Big[\int_K f(sk, tk)\, dk\Big]\, d\mu((s, t)(\operatorname{diag} K)),$$

where dk is a Haar measure on K normalized to have total mass 1. From $(*)$ it follows that

$$d\widetilde{\mu}(s_0 s, t_0 t) = \Delta_G(t_0)\, d\widetilde{\mu}(s, t).$$

The same proof as for the uniqueness in Theorem 6.8 shows that any two Borel measures on $S \times T$ with this property are proportional, and $\Delta_G(t)\, d_l s\, d_l t$ is such a measure. Therefore

$$d\widetilde{\mu}(s, t) = \Delta_G(t)\, d_l s\, d_l t$$

for a suitable normalization of $d_l s\, d_l t$.

The resulting formula is

$$\int_\Omega f(x)\, d_l x = \int_{S\times T} f(st^{-1}) \Delta_G(t)\, d_l s\, d_l t$$

for all Borel functions $f \geq 0$ on Ω. On the right side the change of variables $t \mapsto t^{-1}$ makes the right side become

$$\int_{S\times T} f(st) \Delta_G(t)^{-1}\, d_l s\, \Delta_T(t)\, d_l t,$$

according to Proposition 6.15c, and we can replace Ω by G on the left side since the complement of Ω in G has measure 0 by assumption. This completes the proof. $\square$

4. Invariant Measures on Quotient Spaces

If H is a closed subgroup of G, then we can ask whether G/H has a nonzero G invariant Borel measure. Theorem 6.18 below will give a necessary and sufficient condition for this existence, but we need some preparation. Fix a left Haar measure $d_l h$ for H. If f is in $C_{\text{com}}(G)$, define

$$f^{\#}(g) = \int_H f(gh)\, d_l h.$$

This function is invariant under right translation by H, and we can define

$$f^{\#\#}(gH) = f^{\#}(g).$$

The function $f^{\#\#}$ has compact support on G/H.

Lemma 6.17. The map $f \mapsto f^{\#\#}$ carries $C_{\text{com}}(G)$ onto $C_{\text{com}}(G/H)$, and a nonnegative member of $C_{\text{com}}(G/H)$ has a nonnegative preimage in $C_{\text{com}}(G)$.

PROOF. Let $\pi : G \to G/H$ be the quotient map. Let $F \in C_{\text{com}}(G/H)$ be given, and let K be a compact set in G/H with $F = 0$ off K. We first produce a compact set $\widetilde{K}$ in G with $\pi(\widetilde{K}) = K$. For each coset in K, select an inverse image x and let N_x be a compact neighborhood of x in G. Since π is open, π of the interior of N_x is open. These open sets cover K, and a finite number of them suffices. Then we can take $\widetilde{K}$ to be the intersection of the closed set $\pi^{-1}(K)$ with the compact union of the finitely many N_x's.

Next let K_H be a compact neighborhood of 1 in H. Since nonempty open sets always have positive Haar measure, the left Haar measure on H is positive on K_H. Let $\widetilde{K}'$ be the compact set $\widetilde{K}' = \widetilde{K} K_H$, so that $\pi(\widetilde{K}') = \pi(\widetilde{K}) = K$. Choose $f_1 \in C_{\text{com}}(G)$ with $f_1 \geq 0$ everywhere and with $f_1 = 1$ on $\widetilde{K}'$. If g is in $\widetilde{K}'$, then $\int_H f_1(gh)\, d_l h$ is $\geq$ the H measure of K_H, and hence $f_1^{\#\#}$ is > 0 on K. Define

$$f(g) = \begin{cases} f_1(g)\, \dfrac{F(\pi(g))}{f_1^{\#\#}(\pi(g))} & \text{if } \pi(g) \in K, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f^{\#\#}$ equals F on K and equals 0 off K, and therefore $f^{\#\#} = F$ everywhere.

Certainly f has compact support. To see that f is continuous, it suffices to check that the two formulas for $f(g)$ fit together continuously at points g of the closed set $\pi^{-1}(K)$. It is enough to check points where $f(g) \neq 0$. Say $g_\alpha \to g$ for a net $\{g_\alpha\}$. We must have $F(\pi(g)) \neq 0$. Since F is continuous, $F(\pi(g_\alpha)) \neq 0$ eventually. Thus for all α sufficiently large, $f(g_\alpha)$ is given by the first of the two formulas. Thus f is continuous. $\square$

Theorem 6.18. Let G be a locally compact group, let H be a closed subgroup, and let Δ_G and Δ_H be the respective modular functions. Then a necessary and sufficient condition for G/H to have a nonzero G invariant regular Borel measure is that the restriction to H of Δ_G equal Δ_H. In this case such a measure $d\mu(gH)$ is unique up to a scalar, and it can be normalized so that

$$\boxed{\int_G f(g)\,d_l g = \int_{G/H} \Big[\int_H f(gh)\,d_l h\Big]\,d\mu(gH)}$$

for all $f \in C_{\text{com}}(G)$.

Proof. Let $d\mu(gH)$ be a nonzero invariant regular Borel measure on G/H. Using the function $f^{\#\#}$ defined above, we can define a measure $d\widetilde{\mu}(g)$ on G via a linear functional on $C_{\text{com}}(G)$ by

$$\int_G f(g)\,d\widetilde{\mu}(g) = \int_{G/H} f^{\#\#}(gH)\,d\mu(gH).$$

Since $f \mapsto f^{\#\#}$ commutes with left translation by G, $d\widetilde{\mu}$ is a left Haar measure on G. By Theorem 6.8, $d\widetilde{\mu}$ is unique up to a scalar; hence $d\mu(gH)$ is unique up to a scalar.

Under the assumption that G/H has a nonzero invariant Borel measure, we have just seen in essence that we can normalize the measure so that the boxed formula holds. If we replace f in the boxed formula by $f(\,\cdot\,h_0)$, then the left side is multiplied by $\Delta_G(h_0)$, and the right side is multiplied by $\Delta_H(h_0)$. Hence $\Delta_G\big|_H = \Delta_H$ is necessary for existence.

Let us prove that this condition is sufficient for existence. If h in $C_{\text{com}}(G/H)$ is given, we can choose f in $C_{\text{com}}(G)$ by Lemma 6.17 such that $f^{\#\#} = h$. Then we define $L(h) = \int_G f(g)\,d_l g$. If L is well defined, then it is a linear functional, Lemma 6.17 shows that it is positive, and L certainly is the same on a function as on its G translates. By the Riesz Representation Theorem, L defines a G invariant Borel measure $d\mu(gH)$ on G/H such that the boxed formula holds.

Thus all we need to do is see that L is well defined if $\Delta_G\big|_H = \Delta_H$. We are thus to prove that if $f \in C_{\text{com}}(G)$ has $f^{\#} = 0$, then $\int_G f(g)\,d_l g = 0$. Let ψ be in $C_{\text{com}}(G)$. Since Fubini's Theorem is applicable to continuous functions of compact support, we have

$$\begin{aligned} 0 &= \textstyle\int_G \psi(g) f^{\#}(g)\,d_l g \\ &= \textstyle\int_G \big[\int_H \psi(g) f(gh)\,d_l h\big]\,d_l g \end{aligned}$$

$$
\begin{aligned}
&= \int_H \big[\int_G \psi(g) f(gh)\, d_l g\big]\, d_l h \\
&= \int_H \big[\int_G \psi(gh^{-1}) f(g)\, d_l g\big]\, \Delta_G(h)\, d_l h && \text{by definition of } \Delta_G \\
&= \int_G f(g)\big[\int_H \psi(gh^{-1})\Delta_G(h)\, d_l h\big]\, d_l g \\
&= \int_G f(g)\big[\int_H \psi(gh)\Delta_G(h)^{-1}\Delta_H(h)\, d_l h\big]\, d_l g && \text{by Proposition 6.15c} \\
&= \int_G f(g)\psi^{\#}(g)\, d_l g && \text{since } \Delta_G\big|_H = \Delta_H.
\end{aligned}
$$

By Lemma 6.17 we can choose $\psi \in C_{\text{com}}(G)$ such that $\psi^{\#\#} = 1$ on the image in G/H of the support of f. Then the right side of the above display is $\int_G f(g)\, d_l g$, and the conclusion is that this is 0. Thus L is well defined, and existence is proved. □

EXAMPLE. Let $G = SL(2, \mathbb{R})$, and let $\mathcal{H}$ be the upper half plane in $\mathbb{C}$, namely $\{z \mid \operatorname{Im} z > 0\}$. The group G acts continuously on $\mathcal{H}$ by linear fractional transformations, the action being

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}.$$

This action is transitive since

$$\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix}(i) = x + iy \qquad \text{if } y > 0, \tag{$*$}$$

and the subgroup that leaves i fixed, by direct computation, is the rotation subgroup K, which consists of the matrices $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$. The mapping of G to $\mathcal{H}$ given by $g \mapsto g(i)$ therefore descends to a one-one continuous map of G/K onto $\mathcal{H}$, and Problem 3 at the end of the chapter shows that this map is a homeomorphism. The group G is generated by commutators and hence is unimodular, and the subgroup K is unimodular, being compact. Theorem 6.18 therefore says that $\mathcal{H}$ has a G-invariant Borel measure that is unique up to a scalar factor. Let us see for $p = -2$ that the measure $y^p\, dx\, dy$ is invariant under the subgroup acting in $(*)$. We have

$$\begin{pmatrix} y_0^{1/2} & x_0 y_0^{-1/2} \\ 0 & y_0^{-1/2} \end{pmatrix}(x+iy) = y_0(x+iy) + x_0 = (y_0 x + x_0) + i y_0 y. \tag{$**$}$$

If φ denotes left translation by the matrix on the left in $(**)$, then $(dx\, dy)_{\varphi^{-1}} = y_0^2\, dx\, dy$. Hence $(y^{-2}\, dx\, dy)_{\varphi^{-1}} = (y^{-2})^{\varphi}\, (dx\, dy)_{\varphi^{-1}} = (y_0^{-2} y^{-2})(y_0^2\, dx\, dy) = y^{-2}\, dx\, dy$, and $y^{-2}\, dx\, dy$ is preserved by every matrix in $(**)$. The group G is generated by the matrices in $(**)$ and the one additional matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Since

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(x+iy) = \frac{1}{(-1)(x+iy)} = \frac{-x+iy}{x^2+y^2},$$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ sends $y^{-2}\,dx\,dy$ to $\big(\frac{y}{x^2+y^2}\big)^{-2}|\det J|\,dx\,dy$, where J is the Jacobian matrix of $F(x,y) = \big(\frac{-x}{x^2+y^2}, \frac{y}{x^2+y^2}\big)$, namely $J = \begin{pmatrix} \frac{x^2-y^2}{(x^2+y^2)^2} & \frac{2xy}{(x^2+y^2)^2} \\ \frac{-2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2} \end{pmatrix}$. Calculation gives $|\det J| = (x^2+y^2)^{-2}$, and therefore $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ sends $y^{-2}\,dx\,dy$ to itself. Consequently $y^{-2}\,dx\,dy$ is, up to a multiplicative constant, the one and only G-invariant measure on $\mathcal{H}$.

5. Convolution and L^p Spaces

We turn our attention to the way that Haar measure arises in real analysis. This section will introduce convolution, and aspects of Fourier analysis in the setting of various kinds of locally compact groups will be touched upon in later sections and in the problems at the end of that chapter. In most such applications of Haar measure to Fourier analysis, one assumes that the group under study is unimodular, even if some of its closed subgroups are not.

Thus let G be a locally compact group. We assume throughout this section that G is unimodular. We can then write dx for a two-sided Haar measure on G. Proposition 6.15c shows that we have $\int_G f(x^{-1})\,dx = \int_G f(x)\,dx$ for all Borel functions $f \geq 0$. We abbreviate $L^p(G, dx)$ as $L^p(G)$.

Proposition 6.19. Let G be unimodular, let $1 \leq p < \infty$, and let f be a Borel function in L^p. Then $g \mapsto gf$ and $g \mapsto fg$ are continuous functions from G into L^p.

PROOF. Lemma 6.14 gives the result for f in $C_{\text{com}}(G)$. Proposition 11.21 of *Basic* shows that $C_{\text{com}}(G)$ is dense in $L^p(G)$. Given $g_0 \in G$ and $\epsilon > 0$, find h in $C_{\text{com}}(G)$ with $\|f-h\|_p \leq \epsilon$. Then

$$\begin{aligned} \|gf - g_0f\|_p &\leq \|gf - gh\|_p + \|gh - g_0h\|_p + \|g_0h - g_0f\|_p \\ &= 2\|f-h\|_p + \|gh - g_0h\|_p \qquad \text{by left invariance of } dx \\ &\leq 2\epsilon + \|gh - g_0h\|_p, \end{aligned}$$

and hence $\limsup_{g\to g_0} \|gf - g_0f\|_p \leq 2\epsilon$. Since ϵ is arbitrary, we see that gf tends to g_0f in $L^p(G)$ as g tends to g_0. Similarly fg tends to fg_0 in $L^p(G)$ as g tends to g_0. □

A key tool for real analysis on G is convolution, just as it was with $\mathbb{R}^N$. On a formal level the **convolution** $f * h$ of two functions f and h is

$$(f*h)(x) = \int_G f(xy^{-1})h(y)\,dy = \int_G f(y)h(y^{-1}x)\,dy.$$

The formal equality of the two integrals comes about by changing y into y^{-1} in the first integral and then replacing xy by y. If G is abelian, then $xy^{-1} = y^{-1}x$; thus the first integral for $f * h$ equals the second integral for $h * f$, and the conclusion is that convolution is commutative. However, convolution is not commutative if G is nonabelian.

To make mathematical sense out of $f * h$, we adapt the corresponding known discussion[7] for the special case $G = \mathbb{R}^N$. Let us begin with the case that f and h are nonnegative Borel functions on G. The question is whether $f*h$ is meaningful as a Borel function ≥ 0. In fact, $(x, y) \mapsto f(xy^{-1})$ is the composition of the continuous function $F : G \times G \to G$ given by $F(x, y) = xy^{-1}$, followed by the Borel function $f : G \to [0, +\infty]$. If U is open in $[0, +\infty]$, then $f^{-1}(U)$ is in $\mathcal{B}(G)$, and an argument like the one for Proposition 6.8 shows that $(f \circ F)^{-1}(U) = F^{-1}(f^{-1}(U))$ is in $\mathcal{B}(G \times G)$. Then the product $(x, y) \mapsto f(xy^{-1})g(y)$ is a Borel function, and we would like to use Fubini's Theorem to conclude that $x \mapsto (f * h)(x)$ is a Borel function ≥ 0. Unfortunately we do not know whether the σ-algebras match properly, specifically whether $\mathcal{B}(G \times G) = \mathcal{B}(G) \times \mathcal{B}(G)$.

On the other hand, this kind of product relation does hold for Baire sets. We therefore repeat the above argument with nonnegative Baire functions in place of nonnegative Borel functions. Now the only possible difficulty comes from the fact that Haar measure on G might not be σ-finite. This problem is easily handled by the same kind of localization argument as with the proof of uniqueness for Theorem 6.8: Suppose that G is not σ-compact and that $f \geq 0$ is a Baire function on G. If E is any subset of $[0, +\infty]$, then $f^{-1}(E)$ and $f^{-1}(E^c)$ are disjoint Baire sets. Since any two Baire sets that fail to be σ-bounded have nonempty intersection, only one of $f^{-1}(E)$ and $f^{-1}(E^c)$ can fail to be σ-bounded. It follows that there is exactly one member c of $[0, +\infty]$ for which $f^{-1}(c)$ is not σ-bounded. So as to avoid unimportant technicalities, let us assume for all Baire functions under discussion that this value is 0, i.e., that each Baire function considered in some convolution vanishes off some σ-bounded set. Any σ-bounded set is contained in some σ-compact open subgroup G_0 of G, and thus the convolution effectively takes place on the σ-compact open subgroup G_0; the convolution is 0 outside G_0.

Proposition 6.20. Suppose that f and h are nonnegative Baire functions on G, each vanishing off a σ-bounded subset of G. Let $1 \leq p \leq \infty$, and let p' be the dual index. Then convolution is finite almost everywhere in the following cases, and then the indicated inequalities of norms are satisfied:

(a) for f in $L^1(G)$ and h in $L^p(G)$, and then $\|f * h\|_p \leq \|f\|_1 \|h\|_p$,
for f in $L^p(G)$ and h in $L^1(G)$, and then $\|f * h\|_p \leq \|f\|_p \|h\|_1$,

[7] The discussion in question appears in Section VI.2 of *Basic*.

(b) for f in $L^p(G)$ and h in $L^{p'}(G)$, and then $\|f * h\|_{\sup} \leq \|f\|_p \|h\|_{p'}$,
for f in $L^{p'}(G)$ and h in $L^p(G)$, and then $\|f * h\|_{\sup} \leq \|f\|_{p'} \|h\|_p$.

Consequently $f * h$ is defined in the above situations even if the scalar-valued functions f and h are not necessarily ≥ 0, and the estimates on the norm of $f * h$ are still valid. In case (b), the function $f * h$ is actually continuous.

REMARK. The proof of the continuity in (b) will show actually that $f * h$ is uniformly continuous in a certain sense.

PROOF. The argument for measurability has been given above. The argument for the norm inequalities is proved in the same way[8] as in the special case that $G = \mathbb{R}^N$. Namely, we use Minkowski's inequality for integrals to handle (a), and we use Hölder's inequality to handle (b).

Now consider the question of continuity in (b). At least one of the indices p and p' is finite. First suppose that p is finite. We observe for $g \in G$ that $g(f*h)(x) = (f*h)(g^{-1}x) = \int_G f(g^{-1}xy^{-1})h(y)\,dy = \int_G (gf)(xy^{-1})h(y)\,dy = (gf) * h(x)$. Then we use the bound $\|f * h\|_{\sup} \leq \|f\|_p \|h\|_{p'}$ to make the estimate, for $g \in G$, that

$$\begin{aligned}\|g(f * h) - (f * h)\|_{\sup} &= \|(gf) * h - f * h\|_{\sup} \\ &= \|(gf - f) * h\|_{\sup} \leq \|gf - f\|_p \|h\|_{p'}.\end{aligned}$$

Proposition 6.19 shows that the right side tends to 0 as g tends to 1, and hence $\lim_{g \to 1}(f * h)(g^{-1}x) = (f * h)x$. If instead p' is finite, we argue similarly with right translations of h, finding first that $(f * h)g = f * (hg)$ and then that $\|(f * h)g - (f * h)\|_{\sup} \leq \|f\|_p \|hg - h\|_{p'}$. Application of Proposition 6.19 therefore shows that $\lim_{g \to 1}(f * h)(xg^{-1}) = (f * h)(x)$. $\square$

Corollary 6.21. Convolution makes $L^1(G)$ into an associative algebra (possibly without identity) in such a way that the norm satisfies $\|f * h\|_1 \leq \|f\|_1 \|h\|_1$ for all f and h in $L^1(G)$.

PROOF. The norm inequality was proved in Proposition 6.20a, and it justifies the interchange of integrals in the calculation

$$\begin{aligned}((f_1 * f_2) * f_3)(x) &= \int_G \int_G f_1(y) f_2(y^{-1}z) f_3(z^{-1}x)\,dy\,dz \\ &= \int_G \int_G f_1(y) f_2(y^{-1}z) f_3(z^{-1}x)\,dz\,dy \\ &= \int_G \int_G f_1(y) f_2(z) f_3(z^{-1}y^{-1}x)\,dz\,dy \quad \text{under } z \mapsto yz \\ &= (f_1 * (f_2 * f_3))(x),\end{aligned}$$

which in turn proves associativity. $\square$

[8] Propositions 6.14 and 9.10 of *Basic*.

We shall need the following result in proving the Peter–Weyl Theorem in Section 7.

Proposition 6.22. Let G be a compact group, let f be in $L^1(G)$, and let h be in $L^2(G)$. Put $F(x) = \int_G f(y)h(y^{-1}x)\,dy$. Then F is the limit in $L^2(G)$ of a sequence of functions, each of which is a finite linear combination of left translates of h.

REMARK. For a comparable result in $\mathbb{R}^N$, see Corollary 6.17 of *Basic*. We know from Proposition 6.15b that compact groups are unimodular.

For the proof we require a lemma.

Lemma 6.23. Let G be a compact group, and let h be in $L^2(G)$. For any $\epsilon > 0$, there exist finitely many $y_i \in G$ and Borel sets $E_i \subseteq G$ such that the E_i disjointly cover G and

$$\|h(y^{-1}x) - h(y_i^{-1}x)\|_{2,x} < \epsilon \qquad \text{for all } i \text{ and for all } y \in E_i.$$

PROOF. By Proposition 6.19 choose an open neighborhood U of 1 such that $\|h(gx) - h(x)\|_{2,x} < \epsilon$ whenever g is in U. For each $z_0 \in G$, we have $\|h(gz_0x) - h(z_0x)\|_{2,x} < \epsilon$ whenever g is in U. The set Uz_0 is an open neighborhood of z_0, and such sets cover G as z_0 varies. Find a finite subcover, say $Uz_1, \dots, Uz_n$, and let $U_i = Uz_i$. Define $F_j = U_j - \bigcup_{i=1}^{j-1} U_i$ for $1 \le j \le n$. Then the lemma follows with $y_i = z_i^{-1}$ and $E_i = F_i^{-1}$. $\square$

PROOF OF PROPOSITION 6.22. Given $\epsilon > 0$, choose y_i and E_i as in Lemma 6.23, and put $c_i = \int_{E_i} f(y)\,dy$. Then

$$\begin{aligned}
\Big\| \int_G f(y)h(y^{-1}x)\,dy - \textstyle\sum_i c_i h(y_i^{-1}x)\Big\|_{2,x} \\
&\le \Big\| \textstyle\sum_i \int_{E_i} |f(y)||h(y^{-1}x) - h(y_i^{-1}x)|\,dy\Big\|_{2,x} \\
&\le \textstyle\sum_i \int_{E_i} |f(y)|\,\|h(y^{-1}x) - h(y_i^{-1}x)\|_{2,x}\,dy \\
&\le \textstyle\sum_i \int_{E_i} |f(y)|\epsilon\,dy = \epsilon\|f\|_1. \qquad \square
\end{aligned}$$

6. Representations of Compact Groups

The subject of functional analysis always suggests trying to replace a mathematical problem about functions by a problem about a space of functions and working at solving the latter. By way of example, this point of view is what lay behind our approach in Section I.2 to certain kinds of boundary-value problems by using the method of separation of variables. In some of the cases of separation of variables we considered, as well as in other situations arising in nature, the problem has some symmetry to it, and that symmetry gets passed along to the space of functions under study. Mathematically the symmetry is captured by a group, since the set of symmetries is associative and is closed under composition and inversion. The subject of representation theory deals with exploiting such symmetry, at least in cases for which the problem about functions is linear.

We shall begin with a definition and some examples of finite-dimensional representations of an arbitrary topological group, and then we shall develop a certain amount of theory of finite-dimensional representations under the assumption that the group is compact. The main theorem in this situation is the Peter–Weyl Theorem, which we take up in the next section. In Section 8 we introduce infinite-dimensional representations because vector spaces of functions that arise in analysis problems are frequently infinite-dimensional; in that section we study what happens when the group is compact, but a considerable body of mathematics beyond the scope of this book investigates what can happen for a noncompact group.

Historically the original representations that were studied were matrix representations. An N-by-N **matrix representation** of a topological group G is a continuous homomorphism Φ of G into the group $GL(N,\mathbb{C})$ of invertible complex matrices. In other words, $\Phi(g)$ is an N-by-N invertible complex matrix for each g in G, the matrices are related by the condition that $\Phi(gh)_{ij} = \sum_{k=1}^N \Phi(g)_{ik}\Phi(h)_{kj}$, and the functions $g \mapsto \Phi(g)_{ij}$ are continuous.

Eventually it was realized that sticking to matrices obscures what is really happening. For one thing the group $GL(N,\mathbb{C})$ is being applied to the space $\mathbb{C}^N$ of column vectors, and some vector subspaces of $\mathbb{C}^N$ seem more important than others when they are really not. Instead, it is better to replace $\mathbb{C}^N$ by a finite-dimensional complex vector space V and consider continuous homomorphisms of G into the group $GL_{\mathbb{C}}(V)$ of invertible linear transformations on V. Specifying an ordered basis of V allows one to identify $GL_{\mathbb{C}}(V)$ with $GL(N,\mathbb{C})$, and then the homomorphism gets identified with a matrix representation. In the special case that $V = \mathbb{C}^N$, this identification can be taken to be the usual identification of linear functions and matrices. The point, however, is that it is unwise to emphasize one particular ordered basis in advance, and it is better to work with a general finite-dimensional complex vector space.

Thus we define a **finite-dimensional representation** of a topological group G on a finite-dimensional complex vector space V to be a continuous homomorphism Φ of G into $GL_{\mathbb{C}}(V)$. The continuity condition means that in any basis of V the matrix entries of $\Phi(g)$ are continuous for $g \in G$. It is equivalent to say that $g \mapsto \Phi(g)v$ is a continuous function from G into V for each v in V, i.e., that for each v in V, if $\Phi(g)v$ is expanded in terms of a basis of V, then each entry is a continuous function of g. The vector space V is allowed to be $\mathbb{C}^N$ in the definition, and thus matrix representations are part of the theory.

Before coming to a list examples, let us dispose of two easy kinds of examples that immediately suggest themselves.

For any G the **trivial representation** of G on V is the representation Φ of G for which $\Phi(g) = 1$ for all $g \in G$. Sometimes when the term "trivial representation" is used, it is understood that $V = \mathbb{C}$; sometimes the case $V = \mathbb{C}$ is indicated by referring to the "trivial 1-dimensional representation."

If G is a group of real or complex invertible N-by-N matrices, then G is a subgroup of $GL(N, \mathbb{C})$, and the relative topology from $GL(N, \mathbb{C})$ makes G into a topological group. The inclusion mapping Φ of G into $GL(N, \mathbb{C})$ is a representation known as the **standard representation** of G. The following question then arises: If G is such a group, why consider representations of G when we already have one? The answer, from an analyst's point of view, is that representations are thrust on us by some mathematical problem that we want to solve, and we have to work with what we are given; other representations than the standard one may occur in the process.

EXAMPLES OF FINITE-DIMENSIONAL REPRESENTATIONS.

(1) One-dimensional representations. A continuous homomorphism of a topological group G into the multiplicative group $\mathbb{C}^\times$ of nonzero complex numbers is a representation because we can regard $\mathbb{C}^\times$ as $GL(1, \mathbb{C})$. Of special interest are the representations of this kind that take values in the unit circle $\{e^{i\theta}\}$. These are called **multiplicative characters**.

(a) The exponential functions that arise in Fourier series are examples; the group G in this case is the circle group S^1, namely the quotient of $\mathbb{R}$ modulo the subgroup $2\pi\mathbb{Z}$ of multiples of 2π, and for each integer n, the function $x \mapsto e^{inx}$ is a multiplicative character of $\mathbb{R}$ that descends to a well-defined multiplicative character of S^1.

(b) The exponential functions that arise in the definition of the Fourier transform on $\mathbb{R}^N$, namely $x \mapsto e^{ix\cdot y}$, are multiplicative characters of the additive group $\mathbb{R}^N$.

(c) Let J_m be the cyclic group $\{0, 1, 2, \dots, m-1\}$ of integers modulo m under addition, and let $\zeta_m = e^{2\pi i/m}$. For each integer n and for k in J_m, the formula $\chi_n(k) = (\zeta_m^n)^k$ defines a multiplicative character χ_n of J_m. These multiplicative characters are distinct for $0 \le n \le m-1$.

(d) If G is the symmetric group $\mathfrak{S}_n$ on n letters, then the sign mapping $\sigma \mapsto \operatorname{sgn}\sigma$ is a multiplicative character.

(e) The integer powers of the determinant are multiplicative characters of the unitary group $U(N)$.

(2) Some representations of the symmetric group $\mathfrak{S}_3$ on three letters.

(a) The trivial character and the sign character defined in Example 1d above are the only multiplicative characters.

(b) For each permutation σ, let $\Phi(\sigma)$ be the 3-by-3 matrix of the linear transformation carrying the standard ordered basis (e_1, e_2, e_3) of $\mathbb{C}^3$ to the ordered basis $(e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)})$. To check that Φ is indeed a representation, we start from $\Phi(\sigma)e_j = e_{\sigma(j)}$; applying $\Phi(\tau)$ to both sides, we obtain $\Phi(\tau)\Phi(\sigma)e_j = \Phi(\tau)e_{\sigma(j)} = e_{\tau(\sigma(j))} = e_{(\tau\sigma)(j)} = \Phi(\tau\sigma)e_j$, and we conclude that $\Phi(\tau)\Phi(\sigma) = \Phi(\tau\sigma)$. The vector $e_1 + e_2 + e_3$ is fixed by each $\Phi(\sigma)$, and therefore the 1-dimensional vector subspace $\mathbb{C}(e_1 + e_2 + e_3)$ is "invariant" in the sense of being carried to itself under $\Phi(\mathfrak{S}_3)$.

(c) Place an equilateral triangle in the plane $\mathbb{R}^2$ with its center at the origin and with vertices given in polar coordinates by $(r, \theta) = (1, 0)$, $(1, 2\pi/3)$, and $(1, 4\pi/3)$. Let the vertices be numbered 1, 2, 3, and let $\Phi(\sigma)$ be the matrix of the linear transformation carrying vertex j to vertex $\sigma(j)$ for each j. Then Φ is given on the transpositions $(\,1\quad 2\,)$ and $(\,2\quad 3\,)$ by

$$\Phi((\,1\quad 2\,)) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \qquad \text{and} \qquad \Phi((\,2\quad 3\,)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and is given on any product of these two transpositions by the corresponding product of the above two matrices. The eigenspaces for $\Phi((\,2\quad 3\,))$ are $\mathbb{C}e_1$ and $\mathbb{C}e_2$, and these subspaces are not eigenspaces for $\Phi((\,1\quad 2\,))$. Consequently the only vector subspaces carried to themselves by $\Phi(\mathfrak{S}_3)$ are the trivial ones, namely 0 and $\mathbb{C}^2$. The functions on $\mathfrak{S}_3$ of the form $\sigma \mapsto \Phi(\sigma)_{ij}$ will play a role similar to the role of the functions $x \mapsto e^{inx}$ in Fourier series, and we record their values here:

σ	$\Phi(\sigma)_{11}$	$\Phi(\sigma)_{12}$	$\Phi(\sigma)_{21}$	$\Phi(\sigma)_{22}$
(1)	1	0	0	1
(123)	$-1/2$	$-\sqrt{3}/2$	$\sqrt{3}/2$	$-1/2$
(132)	$-1/2$	$\sqrt{3}/2$	$-\sqrt{3}/2$	$-1/2$
(12)	$-1/2$	$\sqrt{3}/2$	$\sqrt{3}/2$	$1/2$
(23)	1	0	0	-1
(13)	$-1/2$	$-\sqrt{3}/2$	$-\sqrt{3}/2$	$1/2$

(3) A family of representations of the unitary group $G = U(N)$. Let V consist of all polynomials in $z_1, \ldots, z_N, \bar{z}_1, \ldots, \bar{z}_N$ homogeneous of degree k, i.e., having every monomial of total degree k, and let

$$\Phi(g)P\left(\begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix}, \begin{pmatrix} \bar{z}_1 \\ \vdots \\ \bar{z}_N \end{pmatrix}\right) = P\left(g^{-1}\begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix}, \bar{g}^{-1}\begin{pmatrix} \bar{z}_1 \\ \vdots \\ \bar{z}_N \end{pmatrix}\right).$$

The vector subspace V' of **holomorphic polynomials** (those with no $\bar{z}$'s) is carried to itself by all $\Phi(g)$, and therefore V' is an **invariant subspace** in the sense of being carried to itself by $\Phi(G)$. The restriction of the $\Phi(g)$'s to V' is thus itself a representation. When $k=1$, this representation on V' may at first seem to be the standard representation of $U(N)$, but it is not. In fact, V' for $k=1$ consists of all linear combinations of the N linear functionals

$$\begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix} \mapsto z_1 \qquad \text{through} \qquad \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix} \mapsto z_N.$$

In other words, V' is actually the space of all linear functionals on $\mathbb{C}^N$. The definition of Φ by $\Phi(g)\ell(z)=\ell(g^{-1}z)$ for $z\in\mathbb{C}^N$ and for ℓ in the space of linear functionals involves no choice of basis. The representation on V' when $N=1$ is the "contragredient" of the standard representation, in a sense that will be defined for any representation in Example 6 below.

(4) A family of representations of the special unitary group $G=SU(2)$ of all 2-by-2 unitary matrices of determinant 1, namely all matrices $\left(\begin{smallmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{smallmatrix}\right)$ with $|\alpha|^2+|\beta|^2=1$. Let V be the space of homogeneous holomorphic polynomials of degree n in z_1 and z_2, let Φ be the representation defined in the same way as in Example 3, and let V' be the space of all holomorphic polynomials in z of degree n with

$$\Phi'\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} Q(z) = (\bar\beta z+\alpha)^n Q\left(\frac{\bar\alpha z-\beta}{\bar\beta z+\alpha}\right).$$

Define $E: V\to V'$ by $(EP)(z)=P\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right)$. Then E is an invertible linear mapping and satisfies $E\Phi(g)=\Phi'(g)E$ for all g, and we say that E exhibits Φ and Φ' as **equivalent** (i.e., isomorphic).

(5) A family of representations for G equal to the orthogonal group $O(N)$ or the rotation subgroup $SO(N)$. Let V consist of all polynomials in $x_1,\dots,x_N$ homogeneous of degree k, and let

$$\Phi(g)P\left(\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}\right) = P\left(g^{-1}\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}\right).$$

Then Φ is a representation. When we want to emphasize the degree, let us write Φ_k and V_k. Define the Laplacian operator as usual by

$$\Delta = \frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_N^2}.$$

This carries V_k to V_{k-2}, and one checks easily that it satisfies $\Delta\Phi_k(g)=\Phi_{k-2}(g)\Delta$. This commutativity property implies that the kernel of Δ is an invariant subspace of V_k, the space of homogeneous **harmonic polynomials** of degree k.

(6) Contragredient representation. Let G be any topological group, and let Φ be a finite-dimensional representation of G on the complex vector space V. The **contragredient** of Φ is the representation Φ^c of G on the space of all linear functionals on V defined by $(\Phi^c(g)\ell)(v) = \ell(\Phi(g^{-1})v)$ for any linear functional ℓ and any v in V.

Having given a number of examples, let us return to a general topological group G. An important equivalent definition of finite-dimensional representation is that Φ is a continuous group action of G on a finite-dimensional complex vector space V by linear transformations. In this case the assertion about continuity is that the map $G \times V \to V$ is continuous jointly, rather than continuous only as a function of the first variable.

Let us deduce the joint continuity from continuity in the first variable. To do so, it is enough to verify continuity of $G \times V \to V$ at $g = 1$ and $v = 0$. Let $\dim_{\mathbb{C}} V = N$. The topology on V is obtained, as was spelled out above, by choosing an ordered basis and identifying V with $\mathbb{C}^N$. The resulting topology makes V into a topological vector space, and the topology does not depend on the choice of ordered basis; the independence of basis follows from the fact that every linear mapping on $\mathbb{C}^N$ is continuous. Thus we fix an ordered basis $(v_1, \dots, v_N)$ and regard the map $\{c_i\}_{i=1}^N \mapsto \sum_{i=1}^N c_i v_i$ as a homeomorphism of $\mathbb{C}^N$ onto V. Put $\big\| \sum_{i=1}^N c_i v_i \big\| = \big(\sum_{i=1}^N |c_i|^2 \big)^{1/2}$. Given $\epsilon > 0$, choose for each i between 1 and N a neighborhood U_i of 1 in G such that $\|\Phi(g)v_i - v_i\| < 1$ for $g \in U_i$. If g is in $\bigcap_{i=1}^N U_i$ and if $v = \sum_i c_i v_i$ has $\|v\| < \epsilon$, then

$$\begin{aligned}
\|\Phi(g)v\| &\le \big\| \Phi(g)\big(\textstyle\sum c_i v_i\big) - \big(\textstyle\sum c_i v_i\big) \big\| + \|v\| \\
&\le \textstyle\sum |c_i| \|\Phi(g)v_i - v_i\| + \|v\| \\
&\le \big(\textstyle\sum |c_i|^2\big)^{1/2} N^{1/2} + \|v\| \qquad \text{by the Schwarz inequality} \\
&\le (N^{1/2} + 1)\epsilon.
\end{aligned}$$

This proves the joint continuity at $(g, v) = (1, 0)$, and the joint continuity everywhere follows by translation in the two variables separately.

A representation on a *nonzero* finite-dimensional complex vector space V is **irreducible** if it has no invariant subspaces other than 0 and V. Every 1-dimensional representation is irreducible, and we observed that Example 2c is irreducible. We observed also that Examples 2b and 3 are not irreducible.

A representation Φ on the finite-dimensional complex vector space V is called **unitary** if an inner product, always assumed Hermitian, has been specified for V and if each $\Phi(g)$ is unitary relative to that inner product (i.e., has $\Phi(g)^*\Phi(g) = 1$ and hence $\Phi(g)^* = \Phi(g)^{-1}$ for all $g \in G$). On the level of the inner product for V, a unitary representation has the property that $(\Phi(g)u, v) = (u, \Phi(g)^*v) = (u, \Phi(g)^{-1}v) = (u, \Phi(g^{-1})v)$.

The question of whether a representation is unitary is important for analysis because it gets at the notion of exploiting symmetries by using representation theory. Specifically for a unitary representation the orthogonal complement $U^\perp$ of an invariant vector subspace U is an invariant subspace because

$$(\Phi(g)u^\perp, u) = (u^\perp, \Phi(g^{-1})u) \in (u^\perp, U) = 0 \qquad \text{for } u^\perp \in U^\perp,\ u \in U.$$

Thus when an analysis problem leads us to a unitary representation and we locate an invariant vector subspace, the orthogonal complement will be an invariant vector subspace also. In this way the analysis problem may have been subdivided into two simpler problems.

Now let us suppose that the topological group G is compact. One of the critical properties of such a group for representation theory is that G has, up to a scalar multiple, a unique two-sided Haar measure, i.e., a nonzero regular Borel measure that is invariant under all left and right translations. This result was proved in Theorem 6.8 and Proposition 6.15b. Let us normalize this Haar measure so that it has total measure 1. Since the normalized measure is unambiguous, we usually write integrals with respect to normalized Haar measure by expressions like $\int_G f(x)\,dx$, dropping any name like μ from the notation. Also, we write $L^1(G)$ and $L^2(G)$ in place of $L^1(G, dx)$ and $L^2(G, dx)$.

We shall want to use convolution of functions on G, and we therefore need to confront the technical problem that the measurability in Fubini's Theorem can break down with Borel measurable functions if G is not separable. For this reason we shall stick to Baire measurable functions, where no such difficulty occurs.[9] In particular the spaces $L^1(G)$ and $L^2(G)$ will be understood to have the Baire sets as the relevant σ-algebras.[10]

The prototypes for the theory with G compact are the cases that G is the circle group S^1 and that G is a finite group, such as the symmetric group $\mathfrak{S}_3$. The Haar measure is $\frac{1}{2\pi}\,dx$ in the first case, where this time we retain the convention that dx is Lebesgue measure. The Haar measure is $\frac{1}{6}$ times the counting measure in the second case, the $\frac{1}{6}$ having the effect of making the total measure be 1.

Proposition 6.24. If Φ is a representation of a compact group G on a finite-dimensional complex vector space V, then V admits an inner product such that Φ is unitary.

[9] Corollary 11.16 of *Basic* shows that every continuous function of compact support on a locally compact Hausdorff space is Baire measurable.

[10] Problem 3 at the end of Chapter XI of *Basic* shows for any regular Borel measure on a compact Hausdorff space that every Borel measurable function can be adjusted on a Borel set of measure 0 to be Baire measurable. Consequently the spaces $L^1(G)$ and $L^2(G)$ as Banach spaces are unaffected by specifying Baire measurability rather than Borel measurability if the Borel measure is regular.

PROOF. Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be any Hermitian inner product on V, and define

$$(u, v) = \int_G \langle \Phi(x)u, \Phi(x)v\rangle\, dx.$$

It is straightforward to see that $(\,\cdot\,,\,\cdot\,)$ has the required properties. $\square$

Corollary 6.25. If Φ is a representation of a compact group G on a finite-dimensional complex vector space V, then Φ is the direct sum of irreducible representations. In other words, $V = V_1 \oplus \cdots \oplus V_k$, with each V_j an invariant vector subspace on which Φ acts irreducibly.

REMARK. The "direct-sum" notation $V = V_1 \oplus \cdots \oplus V_k$ means that each element of V has a unique expansion as a linear combination of k vectors, one from each V_j. If G is the *noncompact* group of all complex matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, then the standard representation of G on $\mathbb{C}^2$ has $\mathbb{C}e_1$ as an invariant subspace, but there is no other invariant subspace V' such that $\mathbb{C}^2 = \mathbb{C}e_1 \oplus V'$. Thus the corollary breaks down if the hypothesis of compactness is dropped completely.

PROOF. Form $(\,\cdot\,,\,\cdot\,)$ as in Proposition 6.24. Find an invariant subspace $U \neq 0$ of minimal dimension and take its orthogonal complement $U^\perp$. Since the representation is unitary relative to $(\,\cdot\,,\,\cdot\,)$, $U^\perp$ is an invariant subspace. Repeating the argument with $U^\perp$ and iterating, we obtain the required decomposition. $\square$

Proposition 6.26 (Schur's Lemma, part 1). Suppose that Φ and Φ' are irreducible representations of a compact group G on finite-dimensional complex vector spaces V and V', respectively. If $L : V \to V'$ is a linear map such that $\Phi'(g)L = L\Phi(g)$ for all $g \in G$, then L is one-one onto or $L = 0$.

PROOF. We see easily that $\ker L$ and image L are invariant subspaces of V and V', respectively, and then the only possibilities are the ones listed. $\square$

Corollary 6.27 (Schur's Lemma, part 2). Suppose Φ is an irreducible representation of a compact group G on a finite-dimensional complex vector space V. If $L : V \to V$ is a linear map such that $\Phi(g)L = L\Phi(g)$ for all $g \in G$, then L is scalar.

REMARK. This is the first place where we make use of the fact that the scalars are complex, not real.

PROOF. Let λ be an eigenvalue of L. Then $L - \lambda I$ is not one-one onto, but it does commute with $\Phi(g)$ for all $g \in G$. By Proposition 6.26, $L - \lambda I = 0$. $\square$

Corollary 6.28. Every irreducible finite-dimensional representation of a compact abelian group G is given, up to equivalence, by a multiplicative character.

PROOF. If G is abelian and Φ is irreducible, we apply Corollary 6.27 with $L = \Phi(g_0)$ and see that $\Phi(g_0)$ is scalar. All the members of $\Phi(G)$ are therefore scalar, and every vector subspace is invariant. For irreducibility the representation must then be 1-dimensional. Fixing a basis $\{v\}$ of the 1-dimensional vector space and forming the corresponding 1-by-1 matrices, we obtain a multiplicative character. $\square$

EXAMPLE 1a, CONTINUED. For the circle group $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, we observed that we obtain a family of multiplicative characters parametrized by the integers, the n^{th} such character being

$$x \mapsto e^{inx}.$$

The corresponding 1-dimensional representation is $x \mapsto$ multiplication by e^{inx}. In the next corollary we shall prove that the multiplicative characters are orthogonal in $L^2(S^1)$ in the same sense that the exponential functions are orthogonal. The known completeness of the orthonormal system of exponential functions therefore gives a proof, though not the simplest proof, that the exponential functions are the only multiplicative characters of S^1. A simpler proof can be constructed via real-variable theory by making direct use of the multiplicative property and the continuity.

EXAMPLES 2a AND 2c, CONTINUED. We noted that the trivial character and the sign character are the only multiplicative characters of $\mathfrak{S}_3$. These are the following two functions of $\sigma \in \mathfrak{S}_3$:

σ	$\Phi = 1$	$\Phi = \text{sign}$
(1)	1	1
(123)	1	1
(132)	1	1
(12)	1	-1
(23)	1	-1
(13)	1	-1

For this example the corollary below will say that these two functions on $\mathfrak{S}_3$, together with the four functions listed earlier for Example 2c, form an orthogonal set of six functions. They are not quite orthonormal since the four functions f listed earlier have $\|f\|_2 = \sqrt{\frac{1}{2}}$ relative to the *normalized* counting measure. The interpretation of $\sqrt{\frac{1}{2}}$ is that its square is the reciprocal of the dimension of the underlying vector space.

Corollary 6.29 (Schur orthogonality relations).

(a) Let Φ and Φ' be inequivalent irreducible unitary representations of a compact group G on finite-dimensional complex vector spaces V and V', respectively, and let the understood inner products be denoted by $(\,\cdot\,,\,\cdot\,)$. Then

$$\int_G (\Phi(x)u, v)\overline{(\Phi'(x)u', v')}\,dx = 0 \qquad \text{for all } u, v \in V \text{ and } u', v' \in V.$$

(b) Let Φ be an irreducible unitary representation on a finite-dimensional complex vector space V, and let the understood inner product be denoted by $(\,\cdot\,,\,\cdot\,)$. Then

$$\int_G (\Phi(x)u_1, v_1)\overline{(\Phi(x)u_2, v_2)}\,dx = \frac{(u_1, u_2)\overline{(v_1, v_2)}}{\dim V} \quad \text{for } u_1, v_1, u_2, v_2 \in V.$$

REMARK. The proof of (b) will make use of the notion of the "trace" of a square matrix or of a linear map from a finite-dimensional vector space V to itself. For an n-by-n square matrix A the **trace** is the sum of the diagonal entries. This is $(-1)^{n-1}$ times the coefficient of λ^{n-1} in the polynomial $\det(A - \lambda 1)$. Because of the multiplicative property of the determinant, this polynomial is the same for A as for BAB^{-1} if B is invertible. Hence A and BAB^{-1} have the same trace. Then it follows that the **trace** $\operatorname{Tr} L$ of a linear map L from V to itself is well defined as the trace of the matrix of the linear map relative to any basis. For further background about the trace, see Section II.5.

PROOF. (a) Let $l : V' \to V$ be any linear map, and form the linear map

$$L = \textstyle\int_G \Phi(x) l \Phi'(x^{-1})\,dx.$$

(This integration can be regarded as occurring for matrix-valued functions and is to be handled entry-by-entry.) Because of the left invariance of dx, we obtain $\Phi(y)L\Phi'(y^{-1}) = L$, so that $\Phi(y)L = L\Phi'(y)$ for all $y \in G$. By Proposition 6.26 and the assumed inequivalence, $L = 0$. Thus $(Lv', v) = 0$. For the particular choice of l as $l(w') = (w', u')u$, we have

$$\begin{aligned} 0 = (Lv', v) &= \textstyle\int_G (\Phi(x) l \Phi'(x^{-1})v', v)\,dx \\ &= \textstyle\int_G \big(\Phi(x)(\Phi'(x^{-1})v', u')u, v\big)\,dx = \int_G (\Phi(x)u, v)(\Phi'(x^{-1})v', u')\,dx, \end{aligned}$$

and (a) results since $(\Phi'(x^{-1})v', u') = \overline{(\Phi'(x)u', v')}$.

(b) We proceed in the same way, starting from $l : V \to V$, and obtain $L = \lambda I$ from Corollary 6.27. Taking the trace of both sides, we find that

$$\lambda \dim V = \operatorname{Tr} L = \operatorname{Tr} l,$$

so that $\lambda = (\mathrm{Tr}\ l)\big/\dim V$. Thus

$$(Lv_2, v_1) = \frac{\mathrm{Tr}\ l}{\dim V}\,\overline{(v_1, v_2)}.$$

Choose $l(w) = (w, u_2)u_1$, so that $\mathrm{Tr}\ l = (u_1, u_2)$. Then

$$\begin{aligned}\frac{(u_1, u_2)\overline{(v_1, v_2)}}{\dim V} &= \frac{\mathrm{Tr}\ l}{\dim V}\,\overline{(v_1, v_2)} = (Lv_2, v_1) = \int_G (\Phi(x) l \Phi(x^{-1})v_2, v_1)\, dx \\ &= \int_G \big(\Phi(x)(\Phi(x^{-1})v_2, u_2)u_1, v_1\big)\, dx = \int_G (\Phi(x)u_1, v_1)(\Phi(x^{-1})v_2, u_2)\, dx,\end{aligned}$$

and (b) results since $(\Phi(x^{-1})v_2, u_2) = \overline{(\Phi(x)u_2, v_2)}$. □

We can interpret Corollary 6.29 as follows. Let $\{\Phi^{(\alpha)}\}$ be a maximal set of mutually inequivalent finite-dimensional irreducible unitary representations of the compact group G. For each $\Phi^{(\alpha)}$, choose an orthonormal basis for the underlying vector space, and let $\Phi^{(\alpha)}_{ij}(x)$ be the matrix of $\Phi^{(\alpha)}(x)$ in this basis. Then the functions $\{\Phi^{(\alpha)}_{ij}(x)\}_{i,j,\alpha}$ form an orthogonal set in the space $L^2(G)$ of square integrable functions on G. In fact, if $d^{(\alpha)}$ denotes the **degree** of $\Phi^{(\alpha)}$ (i.e., the dimension of the underlying vector space), then $\{(d^{(\alpha)})^{1/2}\Phi^{(\alpha)}_{ij}(x)\}_{i,j,\alpha}$ is an orthonormal set in $L^2(G)$. The Peter–Weyl Theorem in the next section will generalize Parseval's Theorem in the subject of Fourier series by showing that this orthonormal set is an orthonormal basis.

We can use Schur orthogonality to get a qualitative idea of the decomposition into irreducible representations in Corollary 6.25 when Φ is a given finite-dimensional representation of the compact group G. By Proposition 6.24 there is no loss of generality in assuming that Φ is unitary. If Φ is a unitary finite-dimensional representation of G, a **matrix coefficient** of Φ is any function on G of the form $(\Phi(x)u, v)$. The **character** or **group character** of Φ is the function

$$\chi_\Phi(x) = \mathrm{Tr}\ \Phi(x) = \sum\nolimits_j (\Phi(x)u_j, u_j),$$

where $\{u_i\}$ is an orthonormal basis. This function depends only on the equivalence class of Φ and satisfies

$$\chi_\Phi(gxg^{-1}) = \chi_\Phi(x) \qquad \text{for all } g, x \in G.$$

If Φ is the direct sum of representations $\Phi_1, \ldots, \Phi_n$, then

$$\chi_\Phi = \chi_{\Phi_1} + \cdots + \chi_{\Phi_n}.$$

Any multiplicative character is the group character of the corresponding 1-dimensional representation.

EXAMPLE 4, CONTINUED. Characters for $SU(2)$. Let Φ_n be the representation of $SU(2)$ on the homogeneous holomorphic polynomials of degree n in z_1 and z_2. A basis for V consists of the monomials $z_1^k z_2^{n-k}$ for $0 \leq k \leq n$, and we easily check that Φ of the diagonal matrix $t_\theta = \operatorname{diag}\big(e^{i\theta}, e^{-i\theta}\big)$ has $z_1^k z_2^{n-k}$ as an eigenvector with eigenvalue $e^{i(n-2k)\theta}$. Therefore

$$\chi_{\Phi_n}(t_\theta) = \operatorname{Tr}\Phi_n(t_\theta) = e^{in\theta} + e^{i(n-2)\theta} + \cdots + e^{-in\theta}.$$

Every element of $SU(2)$ is conjugate to some matrix t_θ, and therefore this formula determines χ_{Φ_n} on all of $SU(2)$.

Corollary 6.30. If G is a compact group, then the character χ of an irreducible finite-dimensional representation has L^2 norm satisfying $\|\chi\|_2 = 1$. If χ and χ' are characters of inequivalent irreducible finite-dimensional representations, then $\int_G \chi(x)\overline{\chi'(x)}\,dx = 0$.

PROOF. These formulas are immediate from Corollary 6.29 since characters are sums of matrix coefficients. $\square$

Now let Φ be a given finite-dimensional representation of G, and write Φ as the direct sum of irreducible representations $\Phi_1, \dots, \Phi_n$. If τ is an irreducible finite-dimensional representation of G, then the sum formula for characters, together with Corollary 6.30, shows that $\int_G \chi_\Phi(x)\overline{\chi_\tau(x)}\,dx$ is the number of summands Φ_i equivalent to τ. Evidently this integer is independent of the decomposition of Φ into irreducible representations. It is called the **multiplicity** of τ in Φ.

7. Peter–Weyl Theorem

The goal of this section is to extend Parseval's Theorem for the circle group $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ to a theorem valid for all compact groups. The extension is the Peter–Weyl Theorem. We continue with the notation of the previous section, letting G be the group, dx be a two-sided Haar measure normalized to have total measure one, and, in cases when G is not separable, working with Baire measurable functions rather than Borel measurable functions.

For S^1, we observed in Corollary 6.28 that the irreducible finite-dimensional representations are 1-dimensional, hence are given by multiplicative characters. The exponential functions $x \mapsto e^{inx}$ are examples of multiplicative characters, and it is an exercise in real-variable theory, not hard, to prove that there are no other examples. The matrix coefficients of the 1-dimensional representations are just the same exponential functions $x \mapsto e^{inx}$. The Peter–Weyl Theorem specialized to this group says that the vector space of finite linear combinations of exponential functions is dense in $L^2(S^1)$; the statement is a version of Fejér's Theorem for L^2 but without the precise detail of Fejér's Theorem. In view of the known orthogonality of the exponential functions, an equivalent formulation of the result for S^1 is that $\{e^{inx}\}_{n=-\infty}^{\infty}$ is a maximal orthonormal set in $L^2(S^1)$. By Hilbert-space theory, $\{e^{inx}\}_{n=-\infty}^{\infty}$ is an orthonormal basis of $L^2(S^1)$. For general compact G, the Peter–Weyl Theorem asserts that the vector space of finite linear combinations of all matrix coefficients of all irreducible finite-dimensional representations is again dense in $L^2(G)$. The new ingredient is that we must allow irreducible representations of dimension > 1; indeed, examination of the group $\mathfrak{S}_3$ shows that the 1-dimensional representations are not enough. An equivalent formulation in terms of orthonormal bases will be given in Corollary 6.32 below and will use Schur orthogonality (Corollary 6.29).

Theorem 6.31 (Peter–Weyl Theorem). If G is a compact group, then the linear span of all matrix coefficients for all finite-dimensional irreducible unitary representations of G is dense in $L^2(G)$.

PROOF. If $h(x) = (\Phi(x)u, v)$ is such a matrix coefficient, then the following functions of x are also matrix coefficients for the same representation:

$$\begin{aligned}\overline{h(x^{-1})} &= (\Phi(x)v, u),\\ h(gx) &= (\Phi(x)u, \Phi(g^{-1})v),\\ h(xg) &= (\Phi(x)\Phi(g)u, v).\end{aligned}$$

Then the closure U in $L^2(G)$ of the linear span of all matrix coefficients of all finite-dimensional irreducible unitary representations is stable under the map $h(x) \mapsto \overline{h(x^{-1})}$ and under left and right translation. Arguing by contradiction, suppose that $U \neq L^2(G)$. Then $U^\perp \neq 0$, and $U^\perp$ is closed under $h(x) \mapsto \overline{h(x^{-1})}$ and under left and right translation.

We first prove that there is a nonzero continuous function in $U^\perp$. Thus let $H \neq 0$ be in $U^\perp$. For each open neighborhood N of 1 that is a G_δ, we define

$$f_N(x) = \tfrac{1}{|N|}(I_N * H)(x) = \tfrac{1}{|N|}\int_G I_N(y)H(y^{-1}x)\,dy,$$

where I_N is the indicator function of N and $|N|$ is the Haar measure of N. Since I_N and H are in $L^2(G)$, Proposition 6.20 shows that f_N is continuous. As N shrinks to $\{1\}$, the functions f_N tend to H in L^2 by the usual approximate-identity argument; hence some f_N is not 0. Finally each linear combination of left translates of H is in $U^\perp$, and f_N is therefore in $U^\perp$ by Proposition 6.22.

Thus $U^\perp$ contains a nonzero continuous function. Using translations and scalar multiplications, we can adjust this function so that it becomes a continuous function F_1 in $U^\perp$ with $F_1(1)$ real and nonzero. Set

$$F_2(x) = \int_G F_1(yxy^{-1})\,dy.$$

Then F_2 is continuous, $F_2(gxg^{-1}) = F_2(x)$ for all $g \in G$, and $F_2(1) = F_1(1)$ is real and nonzero. To see that F_2 is in $U^\perp$, we argue as follows: Corollary 6.7 shows that the map $(g, g') \mapsto F_1(g(\cdot)g')$ is continuous from $G \times G$ into $C(G)$, and hence the restriction $y \mapsto F_1(y(\cdot)y^{-1})$ is continuous from G into $C(G)$. The domain is compact, and therefore the image is compact, hence totally bounded. Consequently if $\epsilon > 0$ is given, then there exist $y_1, \dots, y_n$ such that each $y \in G$ has some y_j such that $\|F_1(y(\cdot)y^{-1}) - F_1(y_j(\cdot)y_j^{-1})\|_{\sup} < \epsilon$. Let E_j be the subset of y's such that j is the first index for which this happens, and let $|E_j|$ be its Haar measure. Then

$$\begin{aligned}
&\Big| \textstyle\int_G F_1(yxy^{-1})\,dy - \sum_j |E_j|\, F_1(y_jxy_j^{-1}) \Big| \\
&\qquad\qquad = \Big| \textstyle\sum_j \int_{E_j} [F_1(yxy^{-1}) - F_1(y_jxy_j^{-1})]\,dy \Big| \\
&\qquad\qquad \le \textstyle\sum_j \int_{E_j} |F_1(yxy^{-1}) - F_1(y_jxy_j^{-1})|\,dy \le \sum_j \epsilon \int_{E_j} dy = \epsilon,
\end{aligned}$$

and we see that F_2 is the uniform limit of finite linear combinations of group conjugates of F_1. Each such finite linear combination is in $U^\perp$, and hence F_2 is in $U^\perp$.

Finally put

$$F(x) = F_2(x) + \overline{F_2(x^{-1})}.$$

Then F is continuous and is in $U^\perp$, $F(gxg^{-1}) = F(x)$ for all $g \in G$, $F(1) = 2F_2(1)$ is real and nonzero, and $F(x) = \overline{F(x^{-1})}$. In particular, F is not the 0 function in $L^2(G)$.

Form the continuous function $K(x, y) = F(x^{-1}y)$ and the integral operator

$$Tf(x) = \int_G K(x, y) f(y)\,dy = \int_G F(x^{-1}y) f(y)\,dy \qquad \text{for } f \in L^2(G).$$

Then $K(x, y) = \overline{K(y, x)}$ and $\int_{G\times G} |K(x, y)|^2\,dx\,dy < \infty$. Also, T is not 0 since $F \ne 0$. The Hilbert–Schmidt Theorem (Theorem 2.4) applies to T as a linear operator from $L^2(G)$ to itself, and there must be a real nonzero eigenvalue λ, the corresponding eigenspace $V_\lambda \subseteq L^2(G)$ being finite dimensional.

Let us see that the subspace V_λ is invariant under left translation by g, which we write as $(L(g)f)(x) = f(g^{-1}x)$. In fact, f in V_λ implies

$$\begin{aligned}
TL(g)f(x) &= \int_G F(x^{-1}y) f(g^{-1}y)\,dy = \int_G F(x^{-1}gy) f(y)\,dy \\
&= Tf(g^{-1}x) = \lambda f(g^{-1}x) = \lambda L(g)f(x).
\end{aligned}$$

By Proposition 6.19, $g \mapsto L(g)f$ is continuous from G into $L^2(G)$, and therefore L is a representation of G in the finite-dimensional space V_λ. By dimensionality, V_λ contains an irreducible invariant subspace $W_\lambda \ne 0$.

Let $(f_1, \dots, f_n)$ be an ordered orthonormal basis of W_λ. The matrix coefficients for W_λ are the functions

$$h_{ij}(x) = (L(x)f_j, f_i) = \int_G f_j(x^{-1}y)\overline{f_i(y)}\,dy$$

and by definition are in U. Since F is in $U^\perp$, we have

$$\begin{aligned}
0 &= \int_G F(x)\overline{h_{ii}(x)}\,dx = \int_G\int_G F(x)\overline{f_i(x^{-1}y)}f_i(y)\,dy\,dx \\
&= \int_G\int_G F(x)\overline{f_i(x^{-1}y)}f_i(y)\,dx\,dy \\
&= \int_G\int_G F(yx^{-1})\overline{f_i(x)}f_i(y)\,dx\,dy \\
&= \int_G\Big[\int_G F(x^{-1}y)f_i(y)\,dy\Big]\,\overline{f_i(x)}\,dx \qquad\qquad \text{since } F(gxg^{-1}) = F(x) \\
&= \int_G [Tf_i(x)]\overline{f_i(x)}\,dx = \lambda\int_G |f_i(x)|^2\,dx
\end{aligned}$$

for all i, in contradiction to the fact that $W_\lambda \neq 0$. We conclude that $U^\perp = 0$ and therefore that $U = L^2(G)$. $\square$

Corollary 6.32. If $\{\Phi^{(\alpha)}\}$ is a maximal set of mutually inequivalent finite-dimensional irreducible unitary representations of a compact group G and if $\{(d^{(\alpha)})^{1/2}\Phi^{(\alpha)}_{ij}(x)\}_{i,j,\alpha}$ is a corresponding orthonormal set of matrix coefficients, then $\{(d^{(\alpha)})^{1/2}\Phi^{(\alpha)}_{ij}(x)\}_{i,j,\alpha}$ is an orthonormal basis of $L^2(G)$. Consequently any f in $L^2(G)$ has the property that

$$\|f\|_2^2 = \sum_\alpha\sum_{i,j} d_\alpha\,|(f, \Phi^{(\alpha)}_{ij})|^2,$$

where $(\cdot\,,\,\cdot)$ is the L^2 inner product.

REMARK. The displayed formula, which extends Parseval's Theorem from S^1 to the compact group G, is called the **Plancherel formula** for G.

PROOF. The linear span of the orthonormal set in question equals the linear span of all matrix coefficients for all finite-dimensional irreducible unitary representations of G. Theorem 6.31 implies that the orthonormal set is maximal. Hilbert-space theory then shows that the orthonormal set is an orthonormal basis and that Parseval's equality holds, and the latter fact yields the corollary. $\square$

As is implicit in the proof of Corollary 6.32, the partial sums in the expansion of f in terms of the orthonormal set of normalized matrix coefficients are converging to f in $L^2(G)$. The next result along these lines gives an analog of Fejér's Theorem for Fourier series of continuous functions. Taking a cue from the theory of Fourier series, let us refer to any finite linear combination of the functions $\Phi^{(\alpha)}_{ij}(x)$ in the above corollary as a **trigonometric polynomial**.

Corollary 6.33 (Approximation Theorem). There exists a net $T^{(\beta)}$ of uniformly bounded linear operators from $C(G)$ into itself such that for every f in $C(G)$, $T^{(\beta)}f$ is a trigonometric polynomial for each β and $\lim_\beta T^{(\beta)}f = f$ uniformly on G.

PROOF. The directed set will consist of pairs $\beta = (N, \epsilon)$, where N is an open G_δ containing the identity of G and where $1 \geq \epsilon > 0$, and the partial ordering is that $(N, \epsilon) \leq (N', \epsilon')$ if $N \supseteq N'$ and $\epsilon \geq \epsilon'$. If $\beta = (N, \epsilon)$ is given, let $|N|$ be the Haar measure of N, and let $\psi_N = |N|^{-1} I_N$ be the positive multiple of the indicator function of N that makes ψ_N have $\|\psi_N\|_1 = 1$. Since ψ_N is in $L^2(G)$, Theorem 6.31 shows that we can find a trigonometric polynomial φ_β such that $\|\psi_N - \varphi_\beta\|_2 \leq \epsilon$. The operator $T^{(\beta)}$ will be given by convolution: $T^{(\beta)}f = \varphi_\beta * f$.

Since $\|\psi_N - \varphi_\beta\|_1 \leq \|\psi_N - \varphi_\beta\|_2 \leq \epsilon \leq 1$, we have $\|\varphi_\beta\|_1 \leq 2$. Therefore the operator norm of $T^{(\beta)}$ on $C(G)$ is ≤ 2.

To see that $T^{(\beta)}f$ converges uniformly to f, we use a variant of a familiar argument with approximate identities. We write

$$\|T^{(\beta)}f - f\|_{\sup} \leq \|(\varphi_\beta - \psi_N) * f\|_{\sup} + \|\psi_N * f - f\|_{\sup}.$$

The first term on the right is $\leq \|\varphi_\beta - \psi_N\|_1 \|f\|_{\sup} \leq \|\varphi_\beta - \psi_N\|_2 \|f\|_{\sup} \leq \epsilon \|f\|_{\sup}$. For the second term we have

$$\begin{aligned}
|\psi_N * f(x) - f(x)| &= \Big| \textstyle\int_G \psi_N(y)[f(y^{-1}x) - f(x)]\,dy \Big| \\
&\leq \textstyle\int_G \psi_N(y)|f(y^{-1}x) - f(x)|\,dy \\
&= |N|^{-1} \textstyle\int_N |f(y^{-1}x) - f(x)|\,dy \\
&\leq \sup_{y \in N} |f(y^{-1}x) - f(x)|,
\end{aligned}$$

and Proposition 6.6 shows that this expression tends to 0 as N shrinks to $\{1\}$.

Finally we show that $T^{(\beta)}f$ is a trigonometric polynomial, i.e., that there are only finitely many irreducible representations Φ, up to equivalence, such that the L^2 inner product $(T^{(\beta)}f, \Phi_{ij})$ can be nonzero. This inner product is equal to

$$\begin{aligned}
\textstyle\int_G (\varphi_\beta * f)(x)\overline{\Phi_{ij}(x)}\,dx &= \textstyle\iint_{G\times G} \varphi_\beta(xy^{-1}) f(y) \overline{\Phi_{ij}(x)}\,dx\,dy \\
&= \textstyle\iint_{G\times G} \varphi_\beta(x) f(y) \overline{\Phi_{ij}(xy)}\,dx\,dy \\
&= \textstyle\sum_k \iint_{G\times G} \varphi_\beta(x) f(y) \overline{\Phi_{ik}(x)}\ \overline{\Phi_{kj}(y)}\,dx\,dy \\
&= \textstyle\sum_k \int_G f(y)\overline{\Phi_{kj}(y)} \big[\int_G \varphi_\beta(x) \overline{\Phi_{ik}(x)}\,dx \big]\,dy,
\end{aligned}$$

and Schur orthogonality (Corollary 6.29) shows that the expression in brackets is 0 unless Φ is equivalent to one of the irreducible representations whose matrix coefficients contribute to φ_β. □

8. Fourier Analysis Using Compact Groups

In the discussion of the representation theory of compact groups in the previous two sections, all the representations were finite dimensional. A number of applications of compact groups to analysis, however, involve naturally arising infinite-dimensional representations, and a theory of such representations is needed. We address this problem now, and we illustrate how the theory of infinite-dimensional representations can be used to simplify analysis problems having a compact group of symmetries.

We continue with the notation of the previous two sections, letting G be the compact group and dx be a two-sided Haar measure normalized to have total measure one. In cases in which G is not separable, we work with Baire measurable functions rather than Borel measurable functions.

Recall from Section II.4 and Proposition 2.6 that if V is a complex Hilbert space with inner product $(\,\cdot\,,\,\cdot\,)$ and norm $\|\cdot\|$, then a unitary operator U on V is a bounded linear operator from V into itself such that U^* is a two-sided inverse of U, or equivalently is a linear operator from V to itself that preserves norms and is onto V, or equivalently is a linear operator from V to itself that preserves inner products and is onto V.

From the definition the unitary operators on V form a group. Unlike what happens with the N-by-N unitary group $U(N)$, this group is not compact if V is infinite-dimensional. A **unitary representation** of G on the complex Hilbert space V is a homomorphism of G into the group of unitary operators on V such that a certain continuity property holds. Continuity is a more subtle matter in the present context than it was in the finite-dimensional case because not all possible definitions of continuity are equivalent here. The continuity property we choose is that the group action $G \times V \to V$, given by $g \times v \mapsto \Phi(g)v$, is continuous. When Φ is unitary, this property is equivalent to **strong continuity**, namely that $g \mapsto \Phi(g)v$ is continuous for every v in V.

Let us see this equivalence. Strong continuity results from fixing the V variable in the definition of continuity of the group action, and therefore continuity of the group action implies strong continuity. In the reverse direction the triangle inequality and the equality $\|\Phi(g)\| = 1$ give

$$\begin{aligned}\|\Phi(g)v - \Phi(g_0)v_0\| &\le \|\Phi(g)(v - v_0)\| + \|\Phi(g)v_0 - \Phi(g_0)v_0\| \\ &= \|v - v_0\| + \|\Phi(g)v_0 - \Phi(g_0)v_0\|,\end{aligned}$$

and it follows that strong continuity implies continuity of the group action.

With this definition of continuity in place, an example of a unitary representation is the **left-regular representation** of G on the complex Hilbert space $L^2(G)$, given by $(l(g)f)(x) = f(g^{-1}x)$. Strong continuity is satisfied according to Proposition 6.19. The **right-regular representation** of G on $L^2(G)$, given by $(r(g)f)(x) = f(xg)$, also satisfies this continuity property.

In working with a unitary representation Φ of G on V, it is helpful to define $\Phi(f)$ for f in $L^1(G)$ as a smeared-out version of the various $\Phi(x)$'s for x in G. Formally $\Phi(f)$ is to be $\int_G f(x)\Phi(x)\,dx$. But to avoid integrating functions whose values are in an infinite-dimensional space, we define $\Phi(f)$ as follows: The function $\int_G f(x)(\Phi(x)v, v')\,dx$ of v and v' is linear in v, conjugate linear in v', and bounded in the sense that $\left|\int_G f(x)(\Phi(x)v, v')\,dx\right| \le \|f\|_1\|v\|\|v'\|$. Hilbert-space theory shows as a consequence[11] that there exists a unique linear operator $\Phi(f)$ such that

$$(\Phi(f)v, v') = \int_G f(x)(\Phi(x)v, v')\,dx \qquad \text{for all } v \text{ and } v' \text{ in } V$$

and that this operator is bounded with

$$\|\Phi(f)\| \le \|f\|_1.$$

From the existence and uniqueness of $\Phi(f)$, it follows that $\Phi(f)$ depends linearly on f.

Let us digress for a moment to consider $\Phi(f)$ if Φ happens to be finite-dimensional. If $\{u_i\}$ is an ordered orthonormal basis of the underlying finite-dimensional vector space, then the matrix corresponding to $\Phi(f)$ in this basis has $(i, j)^{\text{th}}$ entry $(\Phi(f)u_i, u_j) = \int_G f(x)(\Phi(x)u_i, u_j)\,dx$. The expression

$$\textstyle\sum_{i,j} |(\Phi(f)u_i, u_j)|^2 = \sum_{i,j} \left|\int_G f(x)(\Phi(x)u_i, u_j)\,dx\right|^2$$

is, on the one hand, the kind of term that appears in the Plancherel formula in Corollary 6.32 and, on the other hand, is what in Section II.5 was called the Hilbert–Schmidt norm squared $\|\Phi(f)\|^2_{\text{HS}}$ of $\Phi(f)$. It has to be independent of the basis here in order to yield consistent formulas as we change orthonormal bases, and that independence of basis was proved in Section II.5. Using the Hilbert–Schmidt norm, we can rewrite the **Plancherel formula** in Corollary 6.32 as

$$\boxed{\|f\|^2 = \sum_\alpha d_\alpha\, \|\Phi^{(\alpha)}(f)\|^2_{\text{HS}}.}$$

Unlike the formula in Corollary 6.32, this formula is canonical, not depending on any choice of bases.

[11] See the remarks near the beginning of Section XII.3 of *Basic*.

Returning from our digression, let us again allow Φ to be infinite-dimensional. The mapping $f \mapsto \Phi(f)$ for f in $L^1(G)$ has two other properties of note. The first is that

$$\Phi(f)^* = \Phi(f^*),$$

where $f^*(x) = \overline{f(x^{-1})}$. To prove this formula, we simply write everything out:

$$\begin{aligned}(\Phi(f)^*v, v') &= (v, \Phi(f)v') = \int_G (v, f(x)\Phi(x)v')\,dx \\ &= \int_G \overline{f(x)}(v, \Phi(x)v')\,dx = \int_G \overline{f(x^{-1})}(v, \Phi(x^{-1})v')\,dx \\ &= \int_G f^*(x)(\Phi(x)v, v')\,dx = (\Phi(f^*)v, v').\end{aligned}$$

The other property concerns convolution and is that

$$\Phi(f * h) = \Phi(f)\Phi(h).$$

The formal computation to prove this is

$$\begin{aligned}\Phi(f * h) &= \int_G\int_G f(xy^{-1})h(y)\Phi(x)\,dy\,dx = \int_G\int_G f(xy^{-1})h(y)\Phi(x)\,dx\,dy \\ &= \int_G\int_G f(x)h(y)\Phi(xy)\,dx\,dy = \int_G\int_G f(x)h(y)\Phi(x)\Phi(y)\,dx\,dy \\ &= \Phi(f)\Phi(h).\end{aligned}$$

To make this computation rigorous, we put the appropriate inner products in place and use Fubini's Theorem to justify the interchange of order of integration:

$$\begin{aligned}&(\Phi(f * h)v, v') \\ &= \int_G\int_G f(xy^{-1})h(y)(\Phi(x)v, v')\,dy\,dx = \int_G\int_G f(xy^{-1})h(y)(\Phi(x)v, v')\,dx\,dy \\ &= \int_G\int_G f(x)h(y)(\Phi(xy)v, v')\,dx\,dy = \int_G\int_G f(x)h(y)(\Phi(x)\Phi(y)v, v')\,dx\,dy \\ &= \int_G\int_G f(x)h(y)(\Phi(y)v, \Phi(x)^*v')\,dx\,dy \\ &= \int_G\int_G f(x)h(y)(\Phi(y)v, \Phi(x)^*v')\,dy\,dx = \int_G f(x)(\Phi(h)v, \Phi(x)^*v')\,dx \\ &= \int_G f(x)(\Phi(x)\Phi(h)v, v')\,dx = (\Phi(f)\Phi(h)v, v').\end{aligned}$$

This kind of computation translating a formal argument about $\Phi(f)$ into a rigorous argument is one that we shall normally omit from now on.

An important instance of a convolution $f * h$ is the case that f and h are characters of irreducible finite-dimensional representations. The formula in this case is

$$\chi_\tau * \chi_{\tau'} = \begin{cases} d_\tau^{-1}\chi_\tau & \text{if } \tau \cong \tau' \text{ and } d_\tau \text{ is the degree of } \tau, \\ 0 & \text{if } \tau \text{ and } \tau' \text{ are inequivalent.}\end{cases}$$

This follows by expanding the characters in terms of matrix coefficients and computing the integrals using Schur orthogonality (Corollary 6.29).

If $f \geq 0$ vanishes outside an open neighborhood N of 1 that is a G_δ in G and if $\int_G f(x)\,dx = 1$, then $(\Phi(f)v - v, v') = \int_G f(x)(\Phi(x)v - v, v')\,dx$. When $\|v'\| \leq 1$, the Schwarz inequality therefore gives

$$|(\Phi(f)v - v, v')| \leq \int_N f(x)\|\Phi(x)v - v\|\|v'\|\,dx \leq \sup_{x \in N} \|\Phi(x)v - v\|.$$

Taking the supremum over v' with $\|v'\| \leq 1$ allows us to conclude that

$$\|\Phi(f)v - v\| \leq \sup_{x \in N} \|\Phi(x)v - v\|.$$

We shall make use of this inequality shortly.

An **invariant subspace** for a unitary representation Φ on V is, just as in the finite-dimensional case, a vector subspace U such that $\Phi(g)U \subseteq U$ for all $g \in G$. This notion is useful mainly when U is a closed subspace. In any event if U is invariant, so is the closed orthogonal complement $U^\perp$ since $u^\perp \in U^\perp$ and $u \in U$ imply that

$$(\Phi(g)u^\perp, u) = (u^\perp, \Phi(g)^* u) = (u^\perp, \Phi(g)^{-1} u) = (u^\perp, \Phi(g^{-1})u)$$

is in $(u^\perp, U) = 0$. If $V \neq 0$, the representation is **irreducible** if its only *closed* invariant subspaces are 0 and V.

Two unitary representations of G, Φ on V and Φ' on V', are said to be **equivalent** if there is a bounded linear $E : V \to V'$ with a bounded inverse such that $\Phi'(g)E = E\Phi(g)$ for all $g \in G$.

Theorem 6.34. If Φ is a unitary representation of the compact group G on a complex Hilbert space V, then V is the orthogonal sum of finite-dimensional irreducible invariant subspaces.

REMARK. The new content of the theorem is for the case that V is infinite dimensional. The theorem says that if one takes the union of orthonormal bases for each of certain finite-dimensional irreducible invariant subspaces, then the result is an orthonormal basis of V.

PROOF. By Zorn's Lemma, choose a maximal orthogonal set of finite-dimensional irreducible invariant subspaces, and let U be the closure of the sum. Arguing by contradiction, suppose that U is not all of V. Then $U^\perp$ is a nonzero closed invariant subspace. Fix $v \neq 0$ in $U^\perp$. For each open neighborhood N of 1 that is a G_δ in G, let f_N be the indicator function of N divided by the measure of N. Then f_N is an integrable function ≥ 0 with integral 1. It is immediate from the definition of $(\Phi(f_N)v, u)$ that $\Phi(f_N)v$ is in $U^\perp$ for every N and every $u \in U$. The inequality $\|\Phi(f_N)v - v\| \le \sup_{x\in N} \|\Phi(x)v - v\|$ and strong continuity of Φ show that $\Phi(f_N)v$ tends to v as N shrinks to $\{1\}$. Hence some $\Phi(f_N)v$ is not 0. Fix such an N.

Choose by the Peter–Weyl Theorem (Theorem 6.31) a function h in the linear span of all matrix coefficients for all finite-dimensional irreducible unitary representations such that $\|f_N - h\|_2 \le \frac{1}{2}\|\Phi(f_N)v\| \big/ \|v\|$. Then

$$\begin{aligned}\|\Phi(f_N)v - \Phi(h)v\| &= \|\Phi(f_N - h)v\| \le \|f_N - h\|_1 \|v\| \\ &\le \|f_N - h\|_2 \|v\| \le \tfrac{1}{2}\|\Phi(f_N)v\|.\end{aligned}$$

Hence

$$\|\Phi(h)v\| \ge \|\Phi(f_N)v\| - \|\Phi(f_N)v - \Phi(h)v\| \ge \tfrac{1}{2}\|\Phi(f_N)v\| > 0,$$

and $\Phi(h)v$ is not 0.

The function h lies in some finite-dimensional vector subspace S of $L^2(G)$ that is invariant under left translation. Let $h_1, \dots, h_n$ be a basis of S, and write $h_j(g^{-1}x) = \sum_{i=1}^n c_{ij}(g)h_i(x)$. The formal computation

$$\begin{aligned}\Phi(g)\Phi(h_j)v &= \Phi(g)\textstyle\int_G h_j(x)\Phi(x)v\,dx = \int_G h_j(x)\Phi(gx)v\,dx \\ &= \textstyle\int_G h_j(g^{-1}x)\Phi(x)v\,dx = \sum_{i=1}^n c_{ij}(g)\int_G h_i(x)\Phi(x)v\,dx \\ &= \textstyle\sum_{i=1}^n c_{ij}(g)\Phi(h_i)v\end{aligned}$$

suggests that the vector subspace $\sum_{j=1}^n \mathbb{C}\Phi(h_j)v$, which is finite dimensional and lies in $U^\perp$, is an invariant subspace for Φ containing the nonzero vector $\Phi(h)v$. To justify the formal computation, we argue as in the proof of the formula $\Phi(f * h) = \Phi(f)\Phi(h)$, redoing the calculation with an inner product with v' in place throughout. The existence of this subspace of $U^\perp$ contradicts the maximality of U and proves the theorem. □

Corollary 6.35. Every irreducible unitary representation of a compact group is finite dimensional.

PROOF. This is immediate from Theorem 6.34. □

Corollary 6.36. Let Φ be a unitary representation of the compact group G on a complex Hilbert space V. For each irreducible unitary representation τ of G, let E_τ be the orthogonal projection on the sum of all irreducible invariant subspaces of V that are equivalent to τ. Then E_τ is given by $d_\tau\Phi(\overline{\chi_\tau})$, where d_τ is the degree of τ and χ_τ is the character of τ, and the image of E_τ is the orthogonal sum of irreducible invariant subspaces that are equivalent to τ. Moreover, if τ and τ' are inequivalent, then $E_\tau E_{\tau'} = E_{\tau'} E_\tau = 0$. Finally every v in V satisfies

$$v = \sum_\tau E_\tau v,$$

with the sum an infinite sum over a set of representatives τ of all equivalence classes of irreducible unitary representations of G and taken in the sense of convergence in the Hilbert space.

REMARK. For each τ, the projection E_τ is called the orthogonal projection on the **isotypic subspace** of type τ.

PROOF. Let τ be irreducible with degree d_τ, and put $E'_\tau = d_\tau \Phi(\overline{\chi_\tau})$. Our formulas for characters and for operators $\Phi(f)$ give us the two formulas

$$\begin{aligned} E'_\tau E'_{\tau'} &= d_\tau d_{\tau'} \Phi(\overline{\chi_\tau}) \Phi(\overline{\chi_{\tau'}}) = d_\tau d_{\tau'} \Phi(\overline{\chi_\tau} * \overline{\chi_{\tau'}}) = 0 \quad \text{if } \tau \ncong \tau', \\ {E'_\tau}^2 &= d_\tau^2 \Phi(\overline{\chi_\tau} * \overline{\chi_\tau}) = d_\tau \Phi(\overline{\chi_\tau}) = E'_\tau. \end{aligned}$$

The first of these says that $E'_\tau E'_{\tau'} = E'_{\tau'} E'_\tau = 0$ if τ and τ' are inequivalent, and the second says that E'_τ is a projection. In fact, E'_τ is self adjoint and is therefore an orthogonal projection. To see the self-adjointness, we let $\{u_i\}$ be an orthonormal basis of the vector space on which τ operates by unitary transformations. Then $\overline{\chi_\tau}^*(x) = \chi_\tau(x^{-1}) = \sum_i (\tau(x^{-1})u_i, u_i) = \sum_i \overline{(u_i, \tau(x^{-1})u_i)} = \sum_i \overline{(\tau(x)u_i, u_i)} = \overline{\chi_\tau}(x)$. Therefore

$${E'_\tau}^* = d_\tau \Phi(\overline{\chi_\tau})^* = d_\tau \Phi(\overline{\chi_\tau}^*) = d_\tau \Phi(\overline{\chi_\tau}) = E'_\tau,$$

and the projection $E_{\tau'}$ is an orthogonal projection.

Let U be an irreducible finite-dimensional subspace of V on which $\Phi\big|_U$ is equivalent to τ, and let $u_1, \dots, u_n$ be an orthonormal basis of U. If we write $\Phi(x)u_j = \sum_{i=1}^n \Phi_{ij}(x)u_i$, then $\Phi_{ij}(x) = (\Phi(x)u_j, u_i)$ and $\chi_\tau(x) = \sum_{i=1}^n \Phi_{ii}(x)$. Thus a formal computation with Schur orthogonality gives

$$E'_\tau u_j = d_\tau \textstyle\int_G \overline{\chi_\tau(x)} \Phi(x)u_j \, dx = d_\tau \int_G \sum_{i,k} \overline{\Phi_{kk}(x)} \Phi_{ij}(x) u_i \, dx = u_j,$$

and we can justify this computation by using inner products with v' throughout. As a result, we see that E'_τ is the identity on every irreducible subspace of type τ.

Now let us apply E'_τ to a Hilbert space orthogonal sum $V = \sum V_\alpha$ of the kind in Theorem 6.34. We have just seen that E'_τ is the identity on V_α if V_α is of type τ. If V_α is of type τ' with τ' not equivalent to τ, then $E'_{\tau'}$ is the identity on V_α, and we have $E'_\tau u = E'_\tau E'_{\tau'} u = 0$ for all $u \in V_\alpha$. Consequently E'_τ is 0 on V_α, and we conclude that $E'_\tau = E_\tau$. This completes the proof. □

EXAMPLE. The right-regular representation r of G on $L^2(G)$. Let τ be an abstract irreducible unitary representation of G, let $(u_1, \dots, u_n)$ be an ordered orthonormal basis of the space on which τ acts, and form matrices relative to this basis that realize each $\tau(x)$. The formula is $\tau_{ij}(x) = (\tau(x)u_j, u_i)$. The computation $(r(g)\tau_{ij})(x) = \tau_{ij}(xg) = \sum_k \tau_{ik}(x)\tau_{kj}(g) = \sum_{i'} \tau_{i'j}(g)\tau_{ii'}(x)$ shows that the matrix coefficients corresponding to a fixed row, those with i fixed and j varying, form an invariant subspace for r. The matrix of this representation is $[\tau_{i'j}(g)]$, and thus the representation is irreducible of type τ. Since these spaces are orthogonal to one another by Schur orthogonality, the dimension of the image of E_τ is at least d_τ^2. On the other hand, Corollary 6.32 says that such matrix coefficients relative to an orthonormal basis, as τ varies through representatives of all equivalence classes of irreducible representations, form a maximal orthogonal system in $L^2(G)$. The coefficients corresponding to any τ' not equivalent to τ are in the image of $E_{\tau'}$ and are not of type τ. Therefore the orthogonal sum of the spaces of matrix coefficients for each fixed row equals the image of E_τ, and the dimension of the image equals d_τ^2. The corollary tells us that the formula for the projection is $E_\tau f = r(d_\tau \overline{\chi_\tau}) f$. To see what this is concretely, we use the definitions to compute that $(E_\tau f, h) = (r(\overline{d_\tau \chi_\tau}) f, h) = \int_G d_\tau \overline{\chi_\tau}(x)(r(x)f, h)\,dx = \int_G \int_G d_\tau \overline{\chi_\tau}(x)(r(x)f)(y)\overline{h(y)}\,dy\,dx = \int_G \int_G d_\tau \overline{\chi_\tau}(x) f(yx)\overline{h(y)}\,dy\,dx = \int_G \int_G d_\tau \chi_\tau(x^{-1}) f(yx)\overline{h(y)}\,dx\,dy = (f * d_\tau \chi_\tau, h)$. Therefore the orthogonal projection is given by $E_\tau f = f * d_\tau \chi_\tau$.

Corollary 6.36 is a useful result in taking advantage of symmetries in analysis problems. Imagine that the problem is to understand some linear operator on the space in question, and suppose that the space carries a representation of a compact group that commutes with the operator. This is exactly the situation with some of the examples of separation of variables in partial differential equations as in Section I.2. The idea is that under mild assumptions, the operator carries each isotypic subspace to itself. Hence the problem gets reduced to an understanding of the linear operator on each of the isotypic subspaces.

In order to have a concrete situation for purposes of illustration, let us assume that the linear operator is bounded, has domain the whole Hilbert space, and carries the space into itself. The following proposition then applies.

Proposition 6.37. Let $T : V \to V$ be a bounded linear operator on the Hilbert space V, and suppose that Φ is a unitary representation of the compact group G on V such that $T\Phi(g) = \Phi(g)T$ for all g in G. Let τ be an abstract irreducible unitary representation of G, and let E_τ be the orthogonal projection of V on the isotypic subspace of type τ. Then $TE_\tau = E_\tau T$.

PROOF. For v and v' in V, $(TE_\tau v, v')$ is equal to

$$\begin{aligned}(E_\tau v, T^*v') &= d_\tau \int_G \chi_\tau(x)(\Phi(x)v, T^*v')\,dx = d_\tau \int_G \chi_\tau(x)(T\Phi(x)v, v')\,dx \\ &= d_\tau \int_G \chi_\tau(x)(\Phi(x)Tv, v')\,dx = (E_\tau Tv, v')\,dx,\end{aligned}$$

and the result follows. □

EXAMPLE. The Fourier transform on $L^2(\mathbb{R}^N)$ commutes with each member ρ of the orthogonal group $O(N)$ because if f has Fourier transform $\widehat{f}$, then $\widehat{f}(\rho y) = \int_{\mathbb{R}^N} f(x)e^{-2\pi i x\cdot\rho y}\,dx = \int_{\mathbb{R}^N} f(x)e^{-2\pi i\rho^{-1}x\cdot y}\,dx = \int_{\mathbb{R}^N} f(\rho x)e^{-2\pi i x\cdot y}\,dx$ says that $x \mapsto f(\rho x)$ has Fourier transform $y \mapsto \widehat{f}(\rho y)$. Proposition 6.37 says that the Fourier transform carries each isotypic subspace of $L^2(\mathbb{R}^N)$ under $O(N)$ into itself. Let us return to Example 5 in Section 6, in which we dealt with the vector space V_k of all polynomials on $\mathbb{R}^N$ homogeneous of degree k. We saw that the vector subspace H_k of harmonic polynomials homogeneous of degree k is an invariant subspace under $O(N)$. In fact, more is true. One can show that H_k is irreducible and that the Laplacian Δ carries V_k *onto* $|x|^2 V_{k-2}$. It follows from the latter fact that the space of restrictions to the unit sphere S^{N-1} of all polynomials is the same as the space of restrictions to S^{N-1} of all harmonic polynomials, with each irreducible representation H_k of $O(N)$ occurring with multiplicity 1. Applying the Stone–Weierstrass Theorem on S^{N-1} and untangling matters, we find for $L^2(S^{N-1})$ that the isotypic subspaces under $O(N)$ are the restrictions of the members of H_k, each having multiplicity 1. Passing to $L^2(\mathbb{R}^N)$ and thinking in terms of spherical coordinates, we see that each relevant τ for $L^2(\mathbb{R}^N)$ is the representation on some H_k and that the image of E_τ is the space of L^2 functions that are finite linear combinations $\sum_j h_j f_j(|x|)$ of products of a member of H_k and a function of $|x|$, the members of H_k being linearly independent. According to the proposition, this image is carried to itself by the Fourier transform. The restriction of the Fourier transform to this image still commutes with members of $O(N)$, and the idea is to use Schur's Lemma (Corollary 6.27) to show that the Fourier transform has to send any $h_j(x)f(|x|)$ to $h_j(x)g(|x|)$; the details are carried out in Problem 14 at the end of the chapter. Thus we can see on the basis of general principles that the Fourier transform formula reduces to a single 1-dimensional integral on each space corresponding to some H_k. Armed with this information, one can look for a specific integral formula, and the actual formula turns out to involve an integration and classical Bessel functions.[12]

CONCLUDING REMARKS. Proposition 6.37 and the above example are concerned with understanding a particular bounded linear operator, but realistic applications are more concerned with linear operators that are unbounded. For example, when the domain of a linear partial differential operator can be arranged in such a way that the operator is self adjoint and a compact group of symmetries operates, then one wants to exploit the symmetry group in order to express the space of all functions annihilated by the operator as the limit of the sum of those functions in an isotypic subspace. In mathematical physics the very hope that this kind of reduction is possible has itself been useful, even without knowing in advance the differential operator and the group of symmetries. The reason

[12]Bessel functions were defined in Section IV.8 of *Basic*.

is that numerical invariants of the compact group, such as the dimensions of some of the irreducible representations, appear in physical data. One can look for an appropriate group yielding those numerical invariants. This approach worked long ago in analyzing spin, it worked more recently in attempts to classify elementary particles, and it has been used still more recently in order to guess at the role of group theory in string theory.

9. Problems

1. Let G be a topological group.
   (a) Prove that the connected component of the identity element of G, i.e., the union of all connected sets containing the identity, is a closed subgroup that is group-theoretically normal. This subgroup is called the **identity component** of G.
   (b) Give an example of a topological group whose identity component is not open.
2. The rotation group $SO(N)$ acts continuously on the the unit sphere S^{N-1} in $\mathbb{R}^N$ by matrix multiplication.
   (a) Prove that the subgroup fixing the first standard basis vector is isomorphic to $SO(N-1)$.
   (b) Prove that the action by $SO(N)$ is transitive on S^{N-1} for $N \geq 2$.
   (c) Deduce that there is a homeomorphism $SO(N)/SO(N-1) \to S^{N-1}$ for $N \geq 2$ that respects the action by $SO(N)$.
3. Let G be a separable locally compact group, and suppose that G has a continuous transitive group action on a locally compact Hausdorff space X. Suppose that x_0 is in X and that H is the (closed) subgroup of G fixing x_0, so that there is a one-one continuous map π of G/H onto X. Using the Baire Category Theorem for locally compact Hausdorff spaces (Problem 3 of Chapter X of *Basic*), prove that π is an open map and that π is therefore a homeomorphism.
4. Let G_1 and G_2 be separable locally compact groups, and let $\pi : G_1 \to G_2$ be a continuous one-one homomorphism onto. Prove that π is a homeomorphism.
5. Let $T^2 = \{(e^{i\theta}, e^{i\varphi})\}$. The line $\mathbb{R}^1$ acts on T^2 by
$$\big(x, (e^{i\theta}, e^{i\varphi})\big) \mapsto (e^{i\theta+ix}, e^{i\varphi+ix\sqrt{2}}).$$
   Let p be the point $(1, 1)$ of T^2 corresponding to $\theta = \varphi = 0$. The mapping of $\mathbb{R}^1$ into T^2 given by $x \mapsto xp$ is one-one. Is it a homeomorphism? Explain.
6. Let G be a noncompact locally compact group, and let V be a bounded open set. By using the fact that G cannot be covered by finitely many left translates of V, prove that G must have infinite left Haar measure, i.e., that a Haar measure for a locally compact group can be finite only if the group is compact.

7. (a) Suppose that G is a compact group, λ is a left Haar measure, ρ is a right Haar measure, and E is a Baire set. By evaluating $\int_{G\times G} I_E(xy)\,d(\rho\times\lambda)(x,y)$ as an iterated integral in each order, prove that $\lambda(E)\rho(G)=\lambda(G)\rho(E)$.
   (b) Deduce the uniqueness of Haar measure for compact groups, together with the unimodularity, from (a) and the existence of left and right Haar measures for the group.

8. Suppose that $\{G_n\}_{n=1}^{\infty}$ is a sequence of separable compact groups. Let $G^{(n)} = G_1\times\cdots\times G_n$, and let G be the direct product of all G_n. Let μ_n, $\mu^{(n)}$, and μ be Haar measures on G_n, $G^{(n)}$, and G, all normalized to have total measure 1.
   (a) Why is $\mu^{(n)}$ equal to the product measure $\mu_1\times\cdots\times\mu_n$?
   (b) Show that $\mu^{(n)}$ defines a measure on a certain σ-algebra of Borel sets of G that is consistent with μ.
   (c) Show that the smallest σ-algebra containing, for every n, the "certain σ-algebra of Borel sets of G" as in (b), is the σ-algebra of all Borel sets of G, so that μ can be regarded as the infinite product of $\mu_1, \mu_2, \ldots$.

9. Let G be a locally compact topological group with a left Haar measure d_lx, and let Φ be an automorphism of G as a topological group, i.e., an automorphism of the group structure that is also a homeomorphism of G. Prove that there is a positive constant $a(\Phi)$ such that $d_l(\Phi(x)) = a(\Phi)\,d_lx$.

10. Let G be a locally compact group with two closed unimodular subgroups S and T such that $G = S\times T$ topologically and such that T is group-theoretically normal. Write elements of G as st with $s\in S$ and $t\in T$. Let ds and dt be Haar measures on S and T. Since $t\mapsto sts^{-1}$ is an automorphism of T for each $s\in S$, the previous problem produces a constant $\delta(s)$ such that $d(sts^{-1}) = \delta(s)\,dt$.
    (a) Prove that $ds\,dt$ is a left Haar measure for G.
    (b) Prove that $\delta(s)\,ds\,dt$ is a right Haar measure for G.

11. This problem leads to the same conclusion as Proposition 4.8, that any locally compact topological vector space over $\mathbb{R}$ is finite-dimensional, but it gives a more conceptual proof than the one in Chapter IV. Let V be such a space. For each real $c\neq 0$, let $|c|_V$ be the constant $a(\Phi)$ from Problem 9 when the measure is an additive Haar measure for V and Φ is multiplication by c. Define $|0|_V = 0$.
    (a) Prove that $c\mapsto |c|_V$ is a continuous function from $\mathbb{R}$ into $[0,+\infty)$ such that $|c_1c_2|_V = |c_1|_V|c_2|_V$ and such that $|c_1|\le|c_2|$ implies $|c_1|_V\le|c_2|_V$.
    (b) If W is a closed vector subspace of V, use Theorem 6.18 to prove that $|c|_V = |c|_W|c|_{V/W}$.
    (c) Using (b), Proposition 4.5, Corollary 4.6, and the formula $|c|_{\mathbb{R}^N} = |c|^N$, prove that V has to be finite-dimensional.

12. Let Φ be a finite-dimensional unitary representation of a compact group G on a finite-dimensional inner-product space V. The members of the dual V^* are of the form $\ell_v = (\,\cdot\,, v)$ with v in V, by virtue of the Riesz Representation Theorem

for Hilbert spaces. Define $(\ell_{v_1}, \ell_{v_2}) = (v_2, v_1)$. Prove that the result is the inner product on V^* giving rise to the Banach-space norm on V^*, and prove that the contragredient representation Φ^c has $\Phi^c(x)\ell_v = \ell_{\Phi(x)v}$ and is unitary in this inner product.

13. Let Φ and Φ' be two irreducible unitary representations of a compact group G on the same finite-dimensional vector space V, and suppose that they are equivalent in the sense that there is some linear invertible $E : V \to V$ with $E\Phi(g) = \Phi'(g)E$ for all $g \in G$. Prove that Φ and Φ' are unitarily equivalent in the sense that this equality for some invertible E implies this equality for some unitary E.

14. This problem seeks to fill in the argument concerning Schur's Lemma in the example near the end of Section 8. Introduce an inner product in the space H_k of harmonic polynomials on $\mathbb{R}^N$ homogeneous of degree k to make the representation of $O(N)$ on H_k be unitary, and let $\{h_j\}$ be an orthonormal basis. The representation Φ on H_k and its corresponding matrices $[\Phi(\rho)_{ij}]$ are given by $(\Phi(\rho)h_j)(x) = h_j(\rho^{-1}x) = \sum_i \Phi(\rho)_{ij}h_i(x)$. Let $\mathcal{F}$ be the Fourier transform on $\mathbb{R}^N$, and fix a function $f(|x|)$ such that $|x|^k f(|x|)$ is in $L^2(\mathbb{R}^N)$. Define a matrix $F(|y|) = [f_{ij}(|y|)]$ for each $|y|$ by $\mathcal{F}(h_j(x)f(|x|))(y) = \sum_i h_i(y)f_{ij}(|y|)$.
    (a) Assuming that the functions f and F are continuous functions of $|x|$, prove that $F(|y|)[\Phi(\rho)_{ij}] = [\Phi(\rho)_{ij}]F(|y|)$ for all ρ.
    (b) Deduce from (a) and Corollary 6.27 that $\mathcal{F}(h(x)f(|x|))$ is of the form $h(y)g(|y|)$ if h is in H_k and the continuity hypothesis is satisfied.
    (c) Show how the continuity hypothesis can be dropped in the above argument.

15. Making use of the result of Problem 12, show that the matrix coefficients of the contragredient Φ^c of a finite-dimensional representation Φ of a compact group are the complex conjugates of those of Φ and the characters satisfy $\chi_{\Phi^c} = \overline{\chi_\Phi}$.

16. An example in Section 8 examined the right-regular representation r of a compact group G, given by $(r(g)f)(x) = f(xg)$, and showed that the linear span of the matrix coefficients of an irreducible τ equals the whole isotypic space of type τ, a decomposition of this space into irreducible representations being given by the decomposition into rows. Show similarly for the *left-regular* representation l, given by $(l(g)f)(x) = f(g^{-1}x)$, that the linear span of the matrix coefficients of the irreducible τ equals the whole isotypic space of type τ^c, a decomposition of this space into irreducible representations being given by the decomposition into columns.

17. Let G be a compact group, and let V be a complex Hilbert space.
    (a) For $G = S^1$, prove that the left-regular representation l of G on $L^2(G)$ is not continuous in the operator norm topology, i.e., that $g \mapsto l(g)$ is not continuous from G into the Banach space of bounded linear operators on $L^2(G)$.

(b) Suppose that $g \mapsto \Phi(g)$ is a homomorphism of G into unitary operators on V that is **weakly continuous**, i.e., that has the property that $g \mapsto (\Phi(g)u, v)$ is continuous for each u and v in V. Prove that $g \mapsto \Phi(g)$ is strongly continuous in the sense that $g \mapsto \Phi(g)v$ is continuous for each v in V, i.e., that Φ is a unitary representation.

18. Let G be a compact group.
    (a) Let Φ be an irreducible unitary representation of G, and let f be a linear combination of matrix coefficients of the contragredient Φ^c of Φ. Prove that $f(1) = d \operatorname{Tr} \Phi(f)$, where d is the degree of f.
    (b) Let $\{\Phi^{(\alpha)}\}$ be a maximal set of mutually inequivalent irreducible unitary representations of G, and let $d^{(\alpha)}$ be the degree of $\Phi^{(\alpha)}$. Prove that each trigonometric polynomial f on G satisfies the **Fourier inversion formula** $f(1) = \sum_\alpha d^{(\alpha)} \operatorname{Tr} \Phi^{(\alpha)}(f)$, the sum being a finite sum in the case of a trigonometric polynomial.
    (c) Deduce the Plancherel formula for trigonometric polynomials on G from (b).
    (d) If G is a finite group, prove that every complex-valued function on G is a trigonometric polynomial.

19. Let G be a compact group.
    (a) Prove that if h is any member of $C(G)$ such that $h(gxg^{-1}) = h(x)$ for every g and x in G, then $h * f = f * h$ for every f in $L^1(G)$.
    (b) Prove that if f is a trigonometric polynomial, then $x \mapsto \int_G f(gxg^{-1})\,dg$ is a linear combination of characters of irreducible representations.
    (c) Using the Approximation Theorem, prove that any member of $C(G)$ such that $h(gxg^{-1}) = h(x)$ for every g and x in G is the uniform limit of a sequence of linear combinations of irreducible characters.
    (d) Prove that the irreducible characters form an orthonormal basis of the closed vector subspace of all members h of $L^2(G)$ satisfying $h(x) = \int_G h(gxg^{-1})\,dg$ almost everywhere.

20. Let G be a finite group, let $\{\Phi^{(\alpha)}\}$ be a maximal set of inequivalent irreducible representations of G, and let $d^{(\alpha)}$ be the degree of $\Phi^{(\alpha)}$.
    (a) Prove that $\sum_\alpha (d^{(\alpha)})^2$ equals the number of elements in G.
    (b) Using (d) in the previous problem, prove that the number of $\Phi^{(\alpha)}$'s equals the number of conjugacy classes of G, i.e., the number of equivalence classes of G under the equivalence relation that $x \sim y$ if $x = gyg^{-1}$ for some $g \in G$.
    (c) In a symmetric group $\mathfrak{S}_n$, two elements are conjugate if and only if they have the same cycle structure. In $\mathfrak{S}_4$, two of the irreducible representations are 1-dimensional. Using this information and the above facts, determine how many $\Phi^{(\alpha)}$'s there are for $\mathfrak{S}_4$ and what degrees they have.

Problems 21–22 concern Theorem 6.16, its hypotheses, and related ideas. In the theory of (separable) "Lie groups," if S and T are closed subgroups of a Lie group G

whose intersection is discrete and the sum of whose dimensions equals the dimension of G, then multiplication $S \times T \to G$ is an open map. These problems deduce this open mapping property in a different way without any knowledge of Lie groups, and then they apply the result to give two explicit formulas for the Haar measure of $SL(2,\mathbb{R})$ in terms of measures on subgroups.

21. Let G be a separable locally compact group, and let S and T be closed subgroups such that the image of multiplication as a map $S \times T \to G$ is an open set in G. Using the result of Problem 3, prove that $S \times T \to G$ is an open map.

22. For the group $G = SL(2,\mathbb{R})$, let $K = \left\{k_\theta = \left(\begin{smallmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{smallmatrix}\right)\right\}$, $M = \{m_\pm = \pm 1\}$, $A = \left\{a_x = \left(\begin{smallmatrix} e^x & 0 \\ 0 & e^{-x}\end{smallmatrix}\right)\right\}$, $N = \left\{n_y = \left(\begin{smallmatrix}1 & y \\ 0 & 1\end{smallmatrix}\right)\right\}$, and $V = \left\{v_t = \left(\begin{smallmatrix}1 & 0 \\ t & 1\end{smallmatrix}\right)\right\}$.
    (a) Prove that AN is a closed subgroup and that every element of G is uniquely the product of an element of K and an element of AN. Using Theorem 6.16, show that the formula
    $$\ell(f) = \int_{\theta=0}^{2\pi}\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f(k_\theta a_x n_y)e^{2x}\,dy\,dx\,d\theta$$
    defines a translation-invariant linear functional on $C_{\text{com}}(G)$.
    (b) Prove that MAN is a closed subgroup and that every element $\left(\begin{smallmatrix}a & b \\ c & d\end{smallmatrix}\right)$ of G with $a \neq 0$, and no other element of G, is a product of an element of V and an element of MAN. Assume that the subset of elements $\left(\begin{smallmatrix}a & b \\ c & d\end{smallmatrix}\right)$ of G with $a = 0$ has Haar measure 0. Using Theorem 6.16, show that the formula
    $$\ell(f) = \sum_{m_\pm \in M}\int_{t=-\infty}^{\infty}\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f(v_t m_\pm a_x n_y)e^{2x}\,dy\,dx\,dv$$
    defines a translation-invariant linear functional on $C_{\text{com}}(G)$.

Problems 23–27 do some analysis on the group $G = SU(2)$ of 2-by-2 unitary matrices of determinant 1. Following the notation introduced in Example 4 in Section 6 and in its continuation later in that section, let Φ_n be the representation of G on the homogeneous holomorphic polynomials of degree n in z_1 and z_2 given by $(\Phi_n(g)P)\left(\begin{smallmatrix}z_1 \\ z_2\end{smallmatrix}\right) = P\left(g^{-1}\left(\begin{smallmatrix}z_1 \\ z_2\end{smallmatrix}\right)\right)$. Let $T = \{t_\theta\}$, with $t_\theta = \operatorname{diag}\big(e^{i\theta}, e^{-i\theta}\big)$, be the diagonal subgroup. The text calculated that the character χ_n of Φ_n is given on T by
$$\chi_n(t_\theta) = \operatorname{Tr}\Phi_n(t_\theta) = e^{in\theta} + e^{i(n-2)\theta} + \cdots + e^{-in\theta} = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}}.$$
Take for granted that Φ_n is irreducible for each $n \geq 0$.

23. Take as known from linear algebra that every member of $SU(2)$ is of the form $gt_\theta g^{-1}$ for some $g \in SU(2)$ and some θ. Show that the only ambiguity in t_θ is between θ and $-\theta$. Prove that the linear mapping of $C(G)$ to $C(T)$ carrying f in $C(G)$ to the function $t_\theta \mapsto \int_G f(gt_\theta g^{-1})\,dg$ has image all functions $\varphi \in C(T)$ with $\varphi(t_{-\theta}) = \varphi(t_\theta)$.

24. Reinterpret the image in the previous problem as all continuous functions on the quotient space $T/\{1,\psi\}$, where $\psi : T \to T$ interchanges $t_{-\theta}$ and t_θ. Why is this space compact Hausdorff? Why then can it be identified with $[0,\pi]$?

25. Prove that there is a Borel measure μ on $[0,\pi]$ such that
$$\int_G f(x)\,dx = \int_{[0,\pi]} \int_G f(gt_\theta g^{-1})\,dg\,d\mu(\theta)$$
for all f in $C(G)$.

26. Follow these steps to identify $d\mu(\theta)$ in the previous problem and thereby have a formula for integrating over $G = SU(2)$ by first integrating over conjugacy classes. Such a formula can be obtained by computations with coordinates and use of the change-of-variables formula for multiple integrals, but the method here is shorter.
   (a) Using the orthogonality relations $\int_G \chi_n(x)\overline{\chi_0(x)}\,dx = \delta_{n0}$, prove that $\int_{[0,\pi]} d\mu(\theta) = 1$ and that $\int_{[0,\pi]} (e^{ik\theta} + e^{-ik\theta})\,d\mu(\theta)$ is -1 for $k = 2$ but is 0 for $k = 1$ and $k \geq 3$.
   (b) Extend μ to $[-\pi,\pi]$ by setting it equal to 0 on $[-\pi,0)$, define μ' on $[-\pi,\pi]$ by $\mu'(E) = \tfrac{1}{2}\big(\mu(E) + \mu(-E)\big)$, observe that μ' is even, and check that $\int_{[-\pi,\pi]} \cos n\theta\,d\mu'(\theta)$ is equal to 1 for $n = 0$, to -1 for $n = 2$, and to 0 for $n = 1$ and $n \geq 3$.
   (c) Deduce that the periodic extension of μ' from $(-\pi,\pi]$ to $\mathbb{R}$ is given by its Fourier–Stieltjes series $d\mu'(\theta) = \frac{1}{2\pi}(1 - \cos 2\theta)\,d\theta$.
   (d) (Special case of **Weyl integration formula**) Conclude that
$$\int_G f(x)\,dx = \frac{1}{\pi}\int_\pi^\pi \Big[\int_G f(gt_{\pm\theta}g^{-1})\,dg\Big] \sin^2\theta\,d\theta.$$

27. Prove that every irreducible unitary representation of $SU(2)$ is equivalent to some Φ_n.

Problems 28–32 concern locally compact topological fields. Each such is of interest from the point of view of the present chapter because its additive group is a locally compact abelian group and its nonzero elements form another locally compact abelian group under multiplication. A **topological field** is a field with a Hausdorff topology such that addition, negation, multiplication, and inversion are continuous. The fields $\mathbb{R}$ and $\mathbb{C}$ are examples. Another example is the field $\mathbb{Q}_p$ of p-adic numbers, where p is a prime. To construct this field, one defines on the rationals $\mathbb{Q}$ a function $|\cdot|_p$ by setting $|0|_p = 0$ and taking $|p^n r/s|_p$ equal to p^{-n} if r and s are relatively prime integers. Then $d(x,y) = |x-y|_p$ is a metric on $\mathbb{Q}$, and the metric space completion is $\mathbb{Q}_p$. The function $|\cdot|_p$ extends continuously to $\mathbb{Q}_p$ and is called the p-**adic norm**. It satisfies something better than the triangle inequality, namely $|x+y|_p \leq \max\{|x|_p, |y|_p\}$; this is called the **ultrametric inequality**. Problems 27–31 of Chapter II of *Basic* show that the arithmetic operations on $\mathbb{Q}$ extend continuously to $\mathbb{Q}_p$ and that $\mathbb{Q}_p$ becomes a topological field such that $|xy|_p = |x|_p|y|_p$. Because of the ultrametric inequality the subset $\mathbb{Z}_p$ of $\mathbb{Q}_p$ with $|x|_p \le 1$ is a commutative ring with identity; it is called the ring of p-**adic integers**. It is a **topological ring** in that its addition, negation, and multiplication are continuous. Moreover, it is compact because every closed bounded subset of $\mathbb{Q}_p$ can be shown to be compact. The subset I of $\mathbb{Z}_p$ with $|x|_p \le p^{-1}$ is the unique maximal ideal of $\mathbb{Z}_p$, and the quotient $\mathbb{Z}_p/I$ is a field of p elements.

28. Prove that every compact topological field is finite.

29. Let F be a locally compact topological field, and let $F^\times$ be the group of nonzero elements, the group operation being multiplication.
    (a) Let c be in $F^\times$, and define $|c|_F$ to be the constant $a(\Phi)$ from Problem 9 when the measure is an additive Haar measure and Φ is multiplication by c. Define $|0|_F = 0$. Prove that $c \mapsto |c|_F$ is a continuous function from F into $[0, +\infty)$ such that $|c_1c_2|_F = |c_1|_F|c_2|_F$.
    (b) If dx is a Haar measure for F as an additive locally compact group, prove that $dx/|x|_F$ is a Haar measure for $F^\times$ as a multiplicative locally compact group.
    (c) Let $F = \mathbb{R}$ be the locally compact field of real numbers. Compute the function $x \mapsto |x|_F$. Do the same thing for the locally compact field $F = \mathbb{C}$ of complex numbers.
    (d) Let $F = \mathbb{Q}_p$ be the locally compact field of p-adic numbers, where p is a prime. Compute the function $x \mapsto |x|_F$.
    (e) For the field $F = \mathbb{Q}_p$ of p-adic numbers, suppose that the ring $\mathbb{Z}_p$ of p-adic integers has additive Haar measure 1. What is the additive Haar measure of the maximal ideal I of $\mathbb{Z}_p$?

30. Consider $\mathbb{Q}_p$ as a locally compact abelian group under addition.
    (a) Prove from the continuity that any multiplicative character of the additive group $\mathbb{Q}_p$ is trivial on some subgroup $p^n\mathbb{Z}_p$ for sufficiently large n.
    (b) Tell how to define a multiplicative character φ_0 of the additive group $\mathbb{Q}_p$ in such a way that φ_0 is 1 on $\mathbb{Z}_p$ and $\varphi_0(p^{-1}) = e^{2\pi i/p}$.
    (c) If φ is any multiplicative character of the additive group $\mathbb{Q}_p$, prove that there exists a unique element k of $\mathbb{Q}_p$ such that $\varphi(x) = \varphi_0(kx)$ for all x in $\mathbb{Q}_p$.

31. Let $P = \{\infty\} \cup \{\text{primes}\}$. For v in P, let $\mathbb{Q}_v$ be the field of p-adic numbers if v is a prime p, or $\mathbb{R}$ if $v = \infty$. For v in P, define $|\cdot|_v$ on $\mathbb{Q}_v$ as follows: this is to be the p-adic norm on $\mathbb{Q}_p$ if v is a prime p, and it is to be the ordinary absolute value on $\mathbb{R}$ if $v = \infty$. Each member of the rationals $\mathbb{Q}$ can be regarded as a member of $\mathbb{Q}_v$ for each v in P. Prove that each rational number x has $|x|_v \ne 1$ for only finitely many v.

32. **(Artin product formula)** For each nonzero rational number x, the fact that $|x|_v \ne 1$ for only finitely many v in P shows that $\prod_v |x|_v$ is a well-defined rational number. Prove that actually $\prod_v |x|_v = 1$.

Problems 33–38 concern the ring $\mathbb{A}_\mathbb{Q}$ of adeles of the rationals $\mathbb{Q}$ and the group of ideles defined in terms of it. These objects are important tools in algebraic number theory, and they provide interesting examples of locally compact abelian groups. Part of the idea behind them is to study number-theoretic questions about the integers, such as the solving of Diophantine equations or the factorization of monic polynomials with integer coefficients, by first studying congruences. One studies a congruence modulo each power of any prime, as well as any limitations imposed by treating the coefficients as real. The ring $\mathbb{A}_\mathbb{Q}$ of **adeles** of $\mathbb{Q}$ is a structure that incorporates simultaneously information about all congruences modulo each prime power, together with information about $\mathbb{R}$. Its definition makes use of the construction of direct limits of topological spaces as in Problems 26–30 in Chapter IV, as well as the material concerning p-adic numbers in Problems 29–32 above.

33. The construction of restricted direct products in Problem 30 at the end of Chapter IV assumed that I is a nonempty index set, S_0 is a finite subset, X_i is a locally compact Hausdorff space X_i for each $i \in I$, and K_i is a compact open subset of X_i for each $i \notin S_0$. As in that problem, for each finite subset S of I containing S_0, let
$$X(S) = \big(\bigtimes_{i\in S} X_i\big) \times \big(\bigtimes_{i\notin S} K_i\big),$$
giving it the product topology. Suppose that each X_i, for $i \in I$, is in fact a locally compact group and K_i, for $i \notin S_0$, is a compact open subgroup of X_i. Prove that each $X(S)$, with coordinate-by-coordinate operations, is a locally compact group and that the direct limit X acquires the structure of a locally compact group. Prove also that if each X_i is a locally compact topological ring and each K_i is a compact subring, then each $X(S)$ is a locally compact topological ring and so is the direct limit X.

34. In the construction of the previous problem, let $I = P = \{\infty\} \cup \{\text{primes}\}$ and $S_0 = \{\infty\}$, and form the restricted direct product of the various topological fields $\mathbb{Q}_v$ for $v \in P$ with respect to the compact open subrings $\mathbb{Z}_v$. The above constructions lead to locally compact commutative rings $\mathbb{A}_\mathbb{Q}(S)$ for each finite subset S of P containing S_0, and the direct limit $\mathbb{A}_\mathbb{Q}$ is the locally compact commutative topological ring of **adeles** for $\mathbb{Q}$. Show that each $\mathbb{A}_\mathbb{Q}(S)$ is an open subring of $\mathbb{A}_\mathbb{Q}$. Show that we can regard elements of $\mathbb{A}_\mathbb{Q}$ as tuples $x = (x_\infty, x_2, x_3, x_5, \dots, x_v, \dots) = (x_v)_{v\in P}$ in which all but finitely many coordinates x_p are in $\mathbb{Z}_p$.

35. For each rational number x, the fact that $|x|_v \le 1$ for all but finitely many v allows us to regard the tuple $(x, x, x, \dots)$ as a member of $\mathbb{A}_\mathbb{Q}$. Thus we may regard $\mathbb{Q}$, embedded "diagonally," as a subfield of the ring $\mathbb{A}_\mathbb{Q}$. Prove that $\mathbb{Q}$ is discrete, hence closed.

36. In the setting of the previous problem, prove that $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ is compact.

37. For the rings $\mathbb{Q}_v$, $\mathbb{Z}_v$, and $\mathbb{A}_{\mathbb{Q}}$, let $\mathbb{Q}_v^\times$, $\mathbb{Z}_v^\times$, and $\mathbb{A}_{\mathbb{Q}}^\times$ be the groups consisting of the members of the rings whose multiplicative inverses are in the rings. Give $\mathbb{Q}_v^\times$ and $\mathbb{Z}_v^\times$ the relative topology. In the case of $\mathbb{A}_{\mathbb{Q}}^\times$, define the topology as a restricted direct product of the locally compact groups $\mathbb{Q}_v^\times$ for $v \in P$ with respect to the compact open subgroups $\mathbb{Z}_v^\times$. The locally compact group $\mathbb{A}_{\mathbb{Q}}^\times$ is called the group of **ideles** of $\mathbb{Q}$. Show that the set-theoretic inclusion of $\mathbb{A}_{\mathbb{Q}}^\times$ into $\mathbb{A}_{\mathbb{Q}}$ is continuous but is not a homeomorphism of $\mathbb{A}_{\mathbb{Q}}^\times$ with its image.

38. This problem constructs Haar measure on the ring $\mathbb{A}_{\mathbb{Q}}$ considered as an additive group. As in Problem 34, S denotes any finite subset of P containing $\{\infty\}$.
    (a) Fix S. This part of the problem constructs Haar measure on $\mathbb{A}_{\mathbb{Q}}(S)$. For each prime p in S, define Haar measure μ_p on $\mathbb{Q}_p$ to be normalized so that $\mu_p(\mathbb{Z}_p) = 1$. Form a measure μ_S on $\mathbb{A}_{\mathbb{Q}}(S)$ as follows: On the product $X(S)$ of $\mathbb{R}$ and the $\mathbb{Q}_p$ for p prime in S, use the product of Lebesgue measure and μ_p. On the product $Y(S)$ of all $\mathbb{Z}_p$ for $p \notin S$, use the Haar measure on the infinite product of the $\mathbb{Z}_p$'s obtained as in Problem 8. Then $\mathbb{A}_{\mathbb{Q}}(S) = X(S) \times Y(S)$. Show that Haar measure μ_S on $\mathbb{A}_{\mathbb{Q}}(S)$ may be taken as the product of these measures on $X(S)$ and $Y(S)$ and that the resulting measures are consistent as S varies.
    (b) Show that each measure μ_S defines a set function on a certain σ-subalgebra $\mathcal{B}(S)$ of Borel sets of $\mathbb{A}_{\mathbb{Q}}$ that is the restriction to $\mathcal{B}(S)$ of a Haar measure on all Borel subsets of $\mathbb{A}_{\mathbb{Q}}$.
    (c) Show that the smallest σ-algebra for $\mathbb{A}_{\mathbb{Q}}$ containing, for every finite S containing $\{\infty\}$, the σ-algebra $\mathcal{B}(S)$ as in (b) is the σ-algebra of all Borel sets of $\mathbb{A}_{\mathbb{Q}}$.

Problems 39–47 concern almost periodic functions on topological groups. Let G be any topological group. Define a bounded continuous function $f : G \to \mathbb{C}$ to be **left almost periodic** if every sequence of left translates of f, i.e., every sequence of the form $\{g_n f\}$ with $(g_n f)(x) = f(g_n^{-1}x)$, has a uniformly convergent subsequence; equivalently the condition is that the closure in the uniform norm of the set of left translates of f is compact. Define **right almost periodic** functions similarly; it will turn out that left almost periodic and right almost periodic imply each other. Take for granted that the set of left almost periodic functions, call it $LAP(G)$, is a uniformly closed algebra stable under conjugation and containing the constants. Application of the Stone Representation Theorem (Theorem 4.15) to $LAP(G)$ produces a compact Hausdorff space S_1, a continuous map $p : G \mapsto S_1$ with dense image, and a norm-preserving algebra isomorphism of $LAP(G)$ onto $C(S_1)$. The space S_1 is called the **Bohr compactification** of G. These problems show that S_1 has the structure of a compact group and that the map of G into S_1 is a continuous group homomorphism. Application of the Peter–Weyl Theorem to S_1 will give a Fourier analysis of $LAP(G)$ and an approximation property for its members in terms of finite-dimensional unitary representations of G.

39. Suppose that K is a compact group and that $\iota : G \to K$ is a continuous homomorphism.
   (a) Prove that every member of $C(K)$ is left almost periodic and right almost periodic on K.
   (b) If F is in $C(K)$, let f be the function on G defined by $f(x) = F(\iota(x))$ for $x \in G$. Prove that f is left almost periodic and right almost periodic on G.

40. Let Φ be a finite-dimensional unitary representation of G, and let f be a matrix coefficient of Φ. Prove that f is left almost periodic and right almost periodic.

41. Let f be left almost periodic on G, let L_f be the subset of $C(G)$ consisting of the left translates of f, and let K_f be the closure in $C(G)$ of L_f. The set K_f is compact by definition of left almost periodicity.
   (a) Prove that f is **left uniformly continuous** in the sense that for any $\epsilon > 0$, there is a neighborhood U of $\{1\}$ such that $\|gf - f\|_{\sup} < \epsilon$ for all g in U.
   (b) Each member of the group G acts on L_f with $g_0(gf) = (g_0 g)f$. Prove that this operation of g_0 on L_f is an isometry of L_f onto itself.
   (c) Prove that the operation of each g_0 on L_f extends uniquely to an isometry $\iota_f(g_0)$ of K_f onto itself.

42. Let X be a compact metric space with metric d, and let Γ be the group of isometries of X onto itself. Make Γ into a metric space (Γ, ρ) by defining $\rho(\varphi_1, \varphi_2) = \sup_{x \in X} d(\varphi_1(x), \varphi_2(x))$.
   (a) Prove that Γ is compact as a metric space.
   (b) Prove that Γ is a topological group in this topology, hence a compact group.
   (c) Prove that the group action $\Gamma \times X \to X$ given by $(\gamma, x) \mapsto \gamma(x)$ is continuous.

43. Let Γ_f be the isometry group of K_f, and consider Γ_f as a compact metric space with metric as in the previous problem.
   (a) Prove that the mapping $\iota_f : G \to \Gamma_f$ defined in Problem 41c is continuous.
   (b) Prove that if h is in K_f, then the definition $F_f(h)(\gamma) = (\gamma^{-1}h)(1)$ for $\gamma \in \Gamma_f$ yields a continuous function on Γ such that $h(g_0) = F_f(h)(\iota_f(g_0))$.
   (c) Conclude from the foregoing that f is right almost periodic and hence that left almost periodic functions can now be considered as simply **almost periodic**.

44. For each almost periodic function f on G, let $\iota_f : G \to \Gamma_f$ be the continuous homomorphism discussed in Problems 41c and 43a. Let $\Gamma = \prod_f \Gamma_f$ be the product of the compact groups Γ_f, and define $\iota(g) = \prod_f \iota_f(g)$, so that $\iota : G \to \Gamma$ is a continuous homomorphism. Problem 39b shows that if F is in $C(\Gamma)$, then the function h defined on G by $h(x) = F(\iota(x))$ is almost periodic. Prove that every almost periodic function on G arises in this way from some continuous F on this particular Γ.

45. Let K be the closure of $\iota(G)$ in the compact group Γ in the previous problem, let S_1 be the Bohr compactification of G, and let $p : G \to S_1$ be the continuous map defined by evaluations at the points of G. Prove that there is a homeomorphism $\Phi : S_1 \to K$ such that $\Phi \circ p = \iota$, so that the construction of K can be regarded as imposing a compatible group structure on the Bohr compactification of G.

46. Apply the Approximation Theorem to prove that every almost periodic function on G can be approximated uniformly by linear combinations of matrix coefficients of finite-dimensional unitary representations of G.

47. Suppose that G is abelian, and let $p : G \to K$ be the continuous homomorphism of G into its Bohr compactification. Prove that the continuous multiplicative characters of G coincide with the continuous multiplicative characters of K under an identification by p. (Educational note: It is known from "Pontryagin duality" that if the group $\widehat{K}$ of continuous multiplicative characters of the compact abelian group K is given the discrete topology, then K is isomorphic to the compact group of multiplicative characters of $\widehat{K}$, the topology on this character group being the relative topology as a subset of the unit ball of the dual of $C(\widehat{K})$ in the weak-star topology. Thus K may be obtained by forming the group of continuous multiplicative characters of G, imposing the discrete topology, and forming the group of multiplicative characters of the result.)