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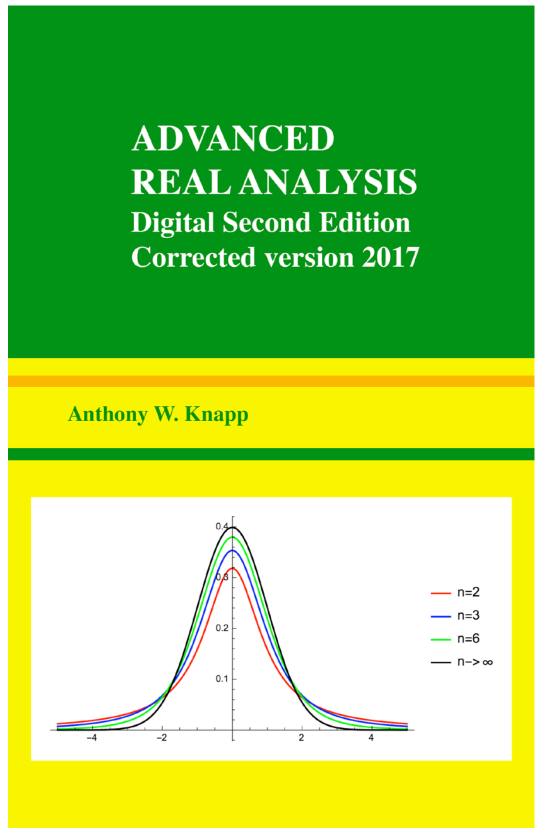

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CHAPTER III

Topics in Euclidean Fourier Analysis

Abstract. This chapter takes up several independent topics in Euclidean Fourier analysis, all having some bearing on the subject of partial differential equations.

Section 1 elaborates on the relationship between the Fourier transform and the Schwartz space, the subspace of $L^1(\mathbb{R}^N)$ consisting of smooth functions with the property that the product of any iterated partial derivative of the function with any polynomial is bounded. It is possible to make the Schwartz space into a metric space, and then one can consider the space of continuous linear functionals; these continuous linear functionals are called "tempered distributions." The Fourier transform carries the space of tempered distributions in one-one fashion onto itself.

Section 2 concerns weak derivatives, and the main result is Sobolev's Theorem, which tells how to recover information about ordinary derivatives from information about weak derivatives. Weak derivatives are easy to manipulate, and Sobolev's Theorem is therefore a helpful tool for handling derivatives without continually having to check the validity of interchanges of limits.

Sections 3–4 concern harmonic functions, those functions on open sets in Euclidean space that are annihilated by the Laplacian. The main results of Section 3 are a characterization of harmonic functions in terms of a mean-value property, a reflection principle that allows the extension to all of Euclidean space of any harmonic function in a half space that vanishes at the boundary, and a result of Liouville that the only bounded harmonic functions in all of Euclidean space are the constants. The main result of Section 4 is a converse to properties of Poisson integrals for half spaces, showing that harmonic functions in a half space are given as Poisson integrals of functions or of finite complex measures if their L^p norms over translates of the bounding Euclidean space are bounded.

Sections 5–6 concern the Calderón–Zygmund Theorem, a far-reaching generalization of the theorem concerning the boundedness of the Hilbert transform. Section 5 gives the statement and proof, and two applications are the subject of Section 6. One of the applications is to Riesz transforms, and the other is to the Beltrami equation, whose solutions are "quasiconformal mappings."

Sections 7–8 concern multiple Fourier series for smooth periodic functions. The theory is established in Section 7, and an application to traces of integral operators is given in Section 8.

1. Tempered Distributions

We fix normalizations for the Euclidean Fourier transform as in *Basic*: For f in $L^1(\mathbb{R}^N)$, the definition is

$$\widehat{f}(y) = (\mathcal{F}f)(y) = \int_{\mathbb{R}^N} f(x) e^{-2\pi i x\cdot y}\,dx,$$

with $x \cdot y$ referring to the dot product and with the 2π in the exponent. The inversion formula is valid whenever $\widehat{f}$ is in L^1; it says that f is recovered as

$$f(x) = (\mathcal{F}^{-1}\widehat{f})(x) = \int_{\mathbb{R}^N} \widehat{f}(y)e^{2\pi i x\cdot y}\,dy$$

almost everywhere, including at all points of continuity of f. The operator $\mathcal{F}$ carries $L^1 \cap L^2$ into L^2 and extends to a linear map $\mathcal{F}$ of L^2 onto L^2 such that $\|\mathcal{F}f\|_2 = \|f\|_2$. This is the Plancherel formula.

The Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R}^N)$ is the vector space of all functions f in $C^\infty(\mathbb{R}^N)$ such that the product of any polynomial by any iterated partial derivative of f is bounded. This is a vector subspace of $L^1 \cap L^2$, and it was shown in *Basic* that $\mathcal{F}$ carries $\mathcal{S}$ one-one onto itself. It will be handy sometimes to use a notation for partial derivatives and their iterates that is different from that in Chapter I. Namely,[1] let $D_j = \dfrac{\partial}{\partial x_j}$. If $\alpha = (\alpha_1, \dots, \alpha_N)$ is an N-tuple of nonnegative integers, we write $|\alpha| = \sum_{j=1}^N \alpha_j$, $\alpha! = \alpha_1! \cdots \alpha_N!$, $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$, and $D^\alpha = D_1^{\alpha_1} \cdots D_N^{\alpha_N}$. Addition of such tuples α is defined component by component, and we say that $\alpha \le \beta$ if $\alpha_j \le \beta_j$ for $1 \le j \le N$. We write $|\alpha|$ for the total order $\alpha_1 + \cdots + \alpha_N$, and we call α a **multi-index**. If $Q(x) = \sum_\alpha a_\alpha x^\alpha$ is a complex-valued polynomial on $\mathbb{R}^N$, define $Q(D)$ to be the partial differential operator $\sum_\alpha a_\alpha D^\alpha$ with constant coefficients obtained by substituting, for each j with $1 \le j \le N$, the operator $D_j = \frac{\partial}{\partial x_j}$ for x_j. The Schwartz functions are then the smooth functions f on $\mathbb{R}^N$ such that $P(x)Q(D)f$ is bounded for each pair of polynomials P and Q.

The Schwartz space is directly usable in connection with certain linear partial differential equations with constant coefficients. A really simple example concerns the Laplacian operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_N^2}$, which we can write as $\Delta = |D|^2$ in the new notation for differential operators. Specifically the equation

$$(1 - \Delta)u = f$$

has a unique solution u in $\mathcal{S}$ for each f in $\mathcal{S}$. To see this, we take the Fourier transform of both sides, obtaining $\mathcal{F}u - \mathcal{F}(\Delta u) = \mathcal{F}f$ or $\mathcal{F}u - \mathcal{F}(|D|^2(u)) = \mathcal{F}f$. Using the formulas relating the Fourier transform, multiplication by polynomials, and differentiation,[2] we can rewrite this equation as $(1 + 4\pi^2|y|^2)\mathcal{F}(u) = \mathcal{F}(f)$. Problem 1 at the end of the chapter asks one to check that $(1+4\pi^2|y|^2)^{-1}g$ is in $\mathcal{S}$ if

[1] Some authors prefer to abbreviate $\frac{\partial}{\partial x_j}$ as ∂_j, reserving the notation D_j for the product of ∂_j and a certain imaginary scalar that depends on the definition of the Fourier transform.

[2] These, with hypotheses in place, appear as Proposition 8.1 of *Basic*.

g is in $\mathcal{S}$, and then existence of a solution in $\mathcal{S}$ to the differential equation is proved by the formula $u = \mathcal{F}^{-1}\big((1+4\pi^2|y|^2)^{-1}\mathcal{F}(f)\big)$. For uniqueness let u_1 and u_2 be two solutions in $\mathcal{S}$ corresponding to the same f. Then $(1-\Delta)(u_1-u_2) = 0$, and hence $(1+4\pi^2|y|^2)\mathcal{F}(u_1-u_2)(y) = 0$ for all y. Therefore $\mathcal{F}(u_1-u_2)(y) = 0$ everywhere. Since $\mathcal{F}$ is one-one on $\mathcal{S}$, we conclude that $u_1 = u_2$.

A deeper use of the Schwartz space in connection with linear partial differential equations comes about because of the relationship between the Schwartz space and the theory of "distributions." Distributions are continuous linear functionals on vector spaces of smooth functions, i.e., continuous linear maps from such a space to the scalars, and they will be considered more extensively in Chapter V. For now, we shall be content with discussing "tempered distributions," the distributions associated with the Schwartz space. In order to obtain a well-defined notion of continuity, we shall describe how to make $\mathcal{S}(\mathbb{R}^N)$ into a metric space.

For each pair of polynomials P and Q, we define

$$\|f\|_{P,Q} = \sup_{x\in\mathbb{R}^N} |P(x)(Q(D)f)(x)|.$$

Each function $\|\cdot\|_{P,Q}$ on $\mathcal{S}$ is a **seminorm** on $\mathcal{S}$ in the sense that[3]

(i) $\|f\|_{P,Q} \geq 0$ for all f in $\mathcal{S}$,

(ii) $\|cf\|_{P,Q} = |c|\|f\|_{P,Q}$ for all f in $\mathcal{S}$ and all scalars c,

(iii) $\|f+g\|_{P,Q} \leq \|f\|_{P,Q} + \|g\|_{P,Q}$ for all f and g in $\mathcal{S}$.

Collectively these seminorms have a property that goes in the converse direction to (i), namely

(iv) $\|f\|_{P,Q} = 0$ for all P and Q implies $f = 0$.

In fact, f will already be 0 if the seminorm for $P = Q = 1$ is 0 on f.

Each seminorm gives rise to a pseudometric $d_{P,Q}(f, g) = \|f-g\|_{P,Q}$ in the usual way, and the topology on $\mathcal{S}$ is the weakest topology making all the functions $d_{P,Q}(\,\cdot\,, g)$ continuous. That is, a base for the topology consists of all sets $U_{g,P,Q,n} = \{f \mid \|f-g\|_{P,Q} < 1/n\}$.

A feature of $\mathcal{S}$ is that only countably many of the seminorms are relevant for obtaining the open sets, and a consequence is that the topology of $\mathcal{S}$ is defined by a metric. The important seminorms are the ones in which P and Q are monomials, each with coefficient 1. In fact, if $P(x) = \sum_\alpha a_\alpha x^\alpha$ and $Q(x) = \sum_\beta b_\beta x^\beta$, then it is easy to check that $d_{P,Q}(f,g) \leq \sum_{\alpha,\beta} |a_\alpha b_\beta| d_{x^\alpha, x^\beta}(f,g)$. Hence any open set that $d_{P,Q}$ defines is a union of finite intersections of the open sets defined by the finitely many d_{x^α, y^β}'s.

[3]The reader may notice that the definition of "seminorm" is the same as the definition of "pseudonorm" in *Basic*. The only distinction is that the word "seminorm" is often used in the context of a whole family of such objects, while the word "pseudonorm" is often used when there is only one such object under consideration.

Let us digress and consider the situation more abstractly because it will arise again later. Suppose we have a real or complex vector space V on which are defined countably many seminorms $\|\cdot\|_n$ satisfying (i), (ii), and (iii) above.

Each seminorm $\|\cdot\|_n$ gives rise to a pseudometric $\widetilde{d}_n$ on V and then to open sets defined relative to $\widetilde{d}_n$. For any pseudometric $\widetilde{\rho}$, the function $\rho = \min\{1, \widetilde{\rho}\}$ is easily checked to be a pseudometric, and ρ defines the same open sets on V as $\widetilde{\rho}$ does. We shall use the following abstract result about pseudometrics; this was proved as Proposition 10.28 of *Basic*, and we therefore omit the proof here.

Proposition 3.1. Suppose that V is a nonempty set and $\{d_n\}_{n\geq 1}$ is a sequence of pseudometrics on V such that $d_n(x, y) \leq 1$ for all n and for all x and y in V. Then $d(x, y) = \sum_{n=1}^{\infty} 2^{-n} d_n(x, y)$ is a pseudometric. If the open balls relative to d_n are denoted by $B_n(r; x)$ and the open balls relative to d are denoted by $B(r; x)$, then the B_n's and B's are related as follows:

(a) whenever some $B_n(r_n; x)$ is given with $r_n > 0$, there exists some $B(r; x)$ with $r > 0$ such that $B(r; x) \subseteq B_n(r_n; x)$,

(b) whenever $B(r; x)$ is given with $r > 0$, there exist finitely many $r_n > 0$, say for $n \leq K$, such that $\bigcap_{n=1}^{K} B_n(r_n; x) \subseteq B(r; x)$.

In the situation with countably many seminorms $\|\cdot\|_n$ for the vector space V, we see that we can introduce a pseudometric d such that three conditions hold:

- $d(x, y) = d(0, y - x)$ for all x and y,
- whenever some x in V is given and an index n and corresponding number $r_n > 0$ are given, then there is a number $r > 0$ such that $d(x, y) < r$ implies $\|y - x\|_n < r_n$,
- whenever some x in V is given and some $r > 0$ is given, then there exist finitely many $r_n > 0$, say for $n \leq K$, such that any y with $\|y - x\|_n < r_n$ for $n \leq K$ implies $d(x, y) < r$.

If the seminorms collectively have the property that $\|x\|_n = 0$ for all n only for $x = 0$, then d is a metric, and we say that the family of seminorms is a **separating family**. The specific form of d is not important: in the case of $\mathcal{S}$, the metric d depended on the choice of the countable subfamily of pseudometrics and the order in which they were enumerated, and these choices do not affect any results about $\mathcal{S}$. The important thing about this construction is that it shows that the topology is given by *some* metric.

The three conditions marked with bullets enable us to detect continuity of linear functions with domain V and range another such space W by using the seminorms directly.

Proposition 3.2. Let $L : V \to W$ be a linear function between vector spaces that are both real or both complex. Suppose that V is topologized by means of countably many seminorms $\|\cdot\|_{V,m}$ and W is topologized by means of countably many seminorms $\|\cdot\|_{W,n}$. Then L is continuous if and only if for each n, there is a finite set $F = F(n)$ of m's and there are corresponding positive numbers δ_m such that $\|v\|_{V,m} \le \delta_m$ for all $m \in F$ implies $\|L(v)\|_{W,n} \le 1$.

PROOF. Let d_V and d_W be the distance functions in V and W. When n is given, the second item in the bulleted list shows that there is some $r > 0$ such that $d_W(0, w) \le r$ implies $\|w\|_{W,n} \le 1$. If L is continuous at 0, then there is a $\delta > 0$ such that $d_V(0, v) \le \delta$ implies $d_W(0, L(v)) \le r$. From the third item in the bulleted list, we know that there is a finite set F of indices m and there are corresponding numbers $\delta_m > 0$ such that $\|v\|_{V,m} \le \delta_m$ implies $d_V(0, v) \le \delta$. Then $\|v\|_{V,m} \le \delta_m$ for all m in F implies $\|L(v)\|_{W,n} \le 1$.

Conversely suppose for each n that there is a finite set F and there are numbers $\delta_m > 0$ for m in F such that the stated condition holds. To see that L is continuous at 0, let $\epsilon > 0$ be given. Choose K and numbers $\epsilon_n > 0$ for $n \le K$ such that $\|w\|_{W,n} \le \epsilon_n$ for $n \le K$ implies $d_W(0, w) \le \epsilon$. For each $n \le K$, the given condition on L allows us to find a finite set F_n of indices m and numbers $\delta_m > 0$ such that $\|v\|_{V,m} \le \delta_m$ implies $\|L(v)\|_{W,n} \le 1$. If $\|v\|_{V,m} \le \delta_m\epsilon_n$ for all m in $F = \bigcup_{n \le K} F_n$, then $\|L(v)\|_{W,n} \le \epsilon_n$ for all $n \le K$ and hence $d_W(0, L(v)) \le \epsilon$. We know that there is a number $\delta > 0$ such that $d_V(0, v) \le \delta$ implies $\|v\|_{V,m} \le \delta_m\epsilon_n$ for all m in F, and then $d_W(0, L(v)) \le \epsilon$. Hence L is continuous at 0.

Once L is continuous at 0, it is continuous everywhere because of the translation invariance of d_V and d_W: $d_V(v_1, v_2) = d_V(0, v_2 - v_1)$ and $d_W(L(v_1), L(v_2)) = d_W(0, L(v_2) - L(v_1)) = d_W(0, L(v_2 - v_1))$. $\square$

Now we return to the Schwartz space $\mathcal{S}$ to apply our construction and Proposition 3.2. The bulleted items above make it clear that it does not matter which countable set of generating seminorms we use nor what order we put them in; the open sets and the criterion for continuity are still the same. The following corollary is immediate from Proposition 3.2, the definition of $\mathcal{S}$, and the behavior of the Fourier transform under multiplication by polynomials and under differentiation.

Corollary 3.3. For the Schwartz space $\mathcal{S}$ on $\mathbb{R}^N$,

(a) a linear functional ℓ is continuous if and only if there is a finite set F of pairs (P, Q) of polynomials and there are corresponding numbers $\delta_{P,Q} > 0$ such that $\|f\|_{P,Q} \le \delta_{P,Q}$ for all (P, Q) in F implies $|\ell(f)| \le 1$.

(b) the Fourier transform mapping $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is continuous, and so is its inverse.

A continuous linear functional on the Schwartz space is called a **tempered distribution**, and the space of all tempered distributions is denoted by $\mathcal{S}' =$

$\mathcal{S}'(\mathbb{R}^N)$. It will be convenient to write $\langle T, \varphi\rangle$ for the value of the tempered distribution T on the Schwartz function φ. The space of tempered distributions is huge. A few examples will give an indication just how huge it is.

EXAMPLES.

(1) Any function f on $\mathbb{R}^N$ with $|f(x)| \le (1+|x|^2)^n|g(x)|$ for some integer n and some integrable function g defines a tempered distribution T by integration: $\langle T, \varphi\rangle = \int_{\mathbb{R}^N} f(x)\varphi(x)\,dx$ when φ is in $\mathcal{S}$. In view of Corollary 3.3a, the continuity follows from the chain of inequalities

$$\begin{aligned}
|\langle T, \varphi\rangle| &\le \textstyle\int_{\mathbb{R}^N} \big(|f(x)|(1+|x|^2)^{-n}\big)\big((1+|x|^2)^n|\varphi(x)|\big)\,dx \\
&\le \big(\textstyle\int_{\mathbb{R}^N} |g(x)|\,dx\big)\big(\sup_x\{(1+|x|^2)^n|\varphi(x)|\}\big) \\
&= \|g\|_1\|\varphi\|_{P,1} \qquad \text{for } P(x) = (1+|x|^2)^n.
\end{aligned}$$

(2) Any function f with $|f(x)| \le (1+|x|^2)^n|g(x)|$ for some integer n and some function g in $L^\infty(\mathbb{R}^N)$ defines a tempered distribution T by integration: $\langle T, \varphi\rangle = \int_{\mathbb{R}^N} f(x)\varphi(x)\,dx$. In fact, $|f(x)| \le (1+|x|^2)^{n+N}\big((1+|x|^2)^{-N}|g(x)|\big)$, and $(1+|x|^2)^{-N}|g(x)|$ is integrable; hence this example is an instance of Example 1.

(3) Any function f with $|f(x)| \le (1+|x|^2)^n|g(x)|$ for some integer n and some function g in $L^p(\mathbb{R}^N)$, where $1 \le p \le \infty$, defines a tempered distribution T by integration because such a distribution is the sum of one as in Example 1 and one as in Example 2.

(4) Suppose that f is as in Example 3 and that $Q(D)$ is a constant-coefficients partial differential operator. Then the formula $\langle T, \varphi\rangle = \int_{\mathbb{R}^N} f(x)(Q(D)\varphi)(x)\,dx$ defines a tempered distribution.

(5) In the above examples, Lebesgue measure dx may be replaced by any Borel measure $d\mu(x)$ on $\mathbb{R}^N$ such that $\int_{\mathbb{R}^N}(1+|x|^2)^{n_0}\,d\mu(x) < \infty$ for some n_0. A particular case of interest is that $d\mu(x)$ is a point mass at a point x_0; in this case, the tempered distributions $\varphi \mapsto \langle T, \varphi\rangle$ that are obtained by combining the above constructions are the linear combinations of iterated partial derivatives of φ at the point x_0.

(6) Any finite linear combination of tempered distributions as in Example 5 is again a tempered distribution.

Two especially useful operations on tempered distributions are multiplication by a Schwartz function and differentiation. Both of these definitions are arranged to be extensions of the corresponding operations on Schwartz functions. The definitions are $\langle \psi T, \varphi\rangle = \langle T, \psi\varphi\rangle$ and $\langle D^\alpha T, \varphi\rangle = (-1)^{|\alpha|}\langle T, D^\alpha\varphi\rangle$; in the latter case the factor $(-1)^{|\alpha|}$ is included because integration by parts requires its presence when T is given by a Schwartz function.

A useful feature of distributions in connection with differential equations, as we shall see in more detail in later chapters, is that we can first look for solutions of a given differential equation that are distributions and then consider how close those distributions are to being functions. The special feature of *tempered* distributions is that the Fourier transform makes sense on them, as follows.

As with the operations of multiplication by a Schwartz function and differentiation, the definition of Fourier transform of a tempered distribution is arranged to be an extension of the definition of the Fourier transform of a member ψ of $\mathcal{S}$ when we identify the function ψ with the distribution $\psi(x)\,dx$. If φ is in $\mathcal{S}$, then $\int \widehat{\psi}\varphi\,dx = \int \psi\widehat{\varphi}\,dx$ by the multiplication formula,[4] which we reinterpret as $\langle \mathcal{F}(\psi\,dx), \varphi\rangle = \langle \psi\,dx, \widehat{\varphi}\rangle$. The definition is

$$\langle \mathcal{F}(T), \varphi\rangle = \langle T, \widehat{\varphi}\rangle$$

for $T \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$. To see that $\mathcal{F}(T)$ is in $\mathcal{S}'$, we have to check that $\mathcal{F}(T)$ is continuous. The definition is $\mathcal{F}(T) = T \circ \mathcal{F}$, and $\mathcal{F}$ is continuous on $\mathcal{S}$ by Corollary 3.3b. Thus the Fourier transform carries tempered distributions to tempered distributions.

Proposition 3.4. The Fourier transform $\mathcal{F}$ is one-one from $\mathcal{S}'(\mathbb{R}^N)$ onto $\mathcal{S}'(\mathbb{R}^N)$.

PROOF. If T is in $\mathcal{S}'$ and $\mathcal{F}(T) = 0$, then $\langle T, \mathcal{F}(\varphi)\rangle = 0$ for all φ in $\mathcal{S}$. Since $\mathcal{F}$ carries $\mathcal{S}$ onto $\mathcal{S}$, $\langle T, \psi\rangle = 0$ for all ψ in $\mathcal{S}$, and thus $T = 0$. Therefore $\mathcal{F}$ is one-one on $\mathcal{S}'$.

If T' is given in $\mathcal{S}'$, put $T = T' \circ \mathcal{F}^{-1}$, where $\mathcal{F}^{-1}$ is the inverse Fourier transform as a map of $\mathcal{S}$ to itself. Then $T' = T \circ \mathcal{F}$ and $\mathcal{F}(T) = T \circ \mathcal{F} = T'$. Therefore $\mathcal{F}$ is onto $\mathcal{S}'$. $\square$

2. Weak Derivatives and Sobolev Spaces

A careful study of a linear partial differential equation often requires attention to the domain of the operator, and it is helpful to be able to work with partial derivatives without investigating a problem of interchange of limits at each step. Sobolev spaces are one kind of space of functions that are used for this purpose, and their definition involves "weak derivatives." At the end one wants to be able to deduce results about ordinary partial derivatives from results about weak derivatives, and Sobolev's Theorem does exactly that.

We shall make extensive use in this book of techniques for regularizing functions that have been developed in *Basic*. Let us assemble a number of these in one place for convenient reference.

[4] Proposition 8.1e of *Basic*.

Proposition 3.5.

(a) (Theorems 6.20 and 9.13) Let φ be in $L^1(\mathbb{R}^N, dx)$, define $\varphi_\varepsilon(x) = \varepsilon^{-N}\varphi(\varepsilon^{-1}x)$ for $\varepsilon > 0$, and put $c = \int_{\mathbb{R}^N} \varphi(x)\,dx$.

(i) If f is in $L^p(\mathbb{R}^N, dx)$ with $1 \le p < \infty$, then

$$\lim_{\varepsilon \downarrow 0} \|\varphi_\varepsilon * f - cf\|_p = 0.$$

(ii) If f is bounded on $\mathbb{R}^N$ and is continuous at x, then $\lim_{\varepsilon\downarrow 0}(\varphi_\varepsilon * f)(x) = cf(x)$, and the convergence is uniform for any set E of x's such that f is uniformly continuous at the points of E.

(b) (Proposition 9.9) If μ is a Borel measure on a nonempty open set U in $\mathbb{R}^N$ and if $1 \le p < \infty$, then $L^p(U, \mu)$ is separable, and $C_{\text{com}}(U)$ is dense in $L^p(U, \mu)$.

(c) (Corollary 6.19) Suppose that φ is a compactly supported function of class C^n on $\mathbb{R}^N$ and that f is in $L^p(\mathbb{R}^N, dx)$ with $1 \le p \le \infty$. Then $\varphi * f$ is of class C^n, and $D^\alpha(\varphi * f) = (D^\alpha\varphi) * f$ for any iterated partial derivative D^α of order $\le n$.

(d) (Lemma 8.11) If δ_1 and δ_2 are given positive numbers with $\delta_1 < \delta_2$, then there exists ψ in $C^\infty_{\text{com}}(\mathbb{R}^N)$ with values in [0, 1] such that $\psi(x) = \psi_0(|x|)$, ψ_0 is nonincreasing, $\psi(x) = 1$ for $|x| \le \delta_1$, and $\psi(x) = 0$ for $|x| \ge \delta_2$.

(e) (Consequence of (d)) If $\delta > 0$, then there exists $\varphi \ge 0$ in $C^\infty_{\text{com}}(\mathbb{R}^N)$ such that $\varphi(x) = \varphi_0(|x|)$ with φ_0 nonincreasing, $\varphi(x) = 0$ for $|x| \ge 1$, and $\int_{\mathbb{R}^N} \varphi(x)\,dx = 1$.

(f) (Proposition 8.12) If K and U are subsets of $\mathbb{R}^N$ with K compact, U open, and $K \subseteq U$, then there exists $\varphi \in C^\infty_{\text{com}}(U)$ with values in [0, 1] such that φ is identically 1 on K.

In this section we work with a nonempty open subset U of $\mathbb{R}^N$, an index p satisfying $1 \le p < \infty$, and the spaces $L^p(U) = L^p(U, dx)$, the underlying measure being understood to be Lebesgue measure. Let $p' = p/(p-1)$ be the dual index. For Sobolev's Theorem, we shall impose two additional conditions on U, namely boundedness for U and a certain regularity condition for the **boundary** $\partial U = U^{\text{cl}} - U$ of the open set U, but we do not impose those additional conditions yet.

Corollary 3.6. If U is a nonempty open subset of $\mathbb{R}^N$, then $C^\infty_{\text{com}}(U)$ is

(a) uniformly dense in $C_{\text{com}}(U)$,

(b) dense in $L^p(U)$ for every p with $1 \le p < \infty$.

In (a), any member of $C_{\text{com}}(U)$ is the uniform limit of members of $C^\infty_{\text{com}}(U)$.

PROOF. Let f in $C_{\text{com}}(U)$ be given. Choose by Proposition 3.5e a function φ in $C^\infty_{\text{com}}(\mathbb{R}^N)$ that is ≥ 0, vanishes outside the unit ball about the origin, and has total integral 1. For $\varepsilon > 0$, define $\varphi_\varepsilon(x) = \varepsilon^{-N}\varphi(\varepsilon^{-1}x)$. The function $\varphi_\varepsilon * f$ is of class C^∞ by (c). If $U = \mathbb{R}^N$, let $\varepsilon_0 = 1$; otherwise let ε_0 be the distance from the support of f to the complement of U. For $\varepsilon < \varepsilon_0$, $\varphi_\varepsilon * f$ has compact support contained in U. As ε decreases to 0, Proposition 3.5a shows that $\|\varphi_\varepsilon * f - f\|_{\text{sup}}$ tends to 0 and so does $\|\varphi_\varepsilon * f - f\|_p$. This proves the first conclusion of the corollary and proves also that $C^\infty_{\text{com}}(U)$ is L^p dense in $C_{\text{com}}(U)$ if $1 \leq p < \infty$. Since Proposition 3.5b shows that $C_{\text{com}}(U)$ is dense in $L^p(U)$, the second conclusion of the corollary follows. □

Suppose that f and g are two complex-valued functions that are **locally integrable** on U in the sense of being integrable on each compact subset of U. If α is a differentiation index, we say that $D^\alpha f = g$ in the sense of **weak derivatives** if

$$\int_U f(x) D^\alpha \varphi(x)\, dx = (-1)^{|\alpha|} \int_U g(x)\varphi(x)\, dx \qquad \text{for all } \varphi \in C^\infty_{\text{com}}(U).$$

The definition is arranged so that g gives the result that one would expect for iterated partial differentiation of type α if the integrated or boundary term gives 0 at each stage. More precisely if f is in $C^{|\alpha|}(U)$, then the weak derivative of order α exists and is the pointwise derivative. To prove this, it is enough to handle a first-order partial derivative $D_j h$ for a function h in $C^1(U)$, showing that $\int_U hD_j\varphi\, dx = -\int_U (D_j h)\varphi\, dx$ for $\varphi \in C^\infty_{\text{com}}(U)$, i.e., that $\int_U D_j(h\varphi)\, dx = 0$. Because φ is compactly supported in U, $\psi = h\varphi$ makes sense as a compactly supported C^1 function on $\mathbb{R}^N$, and we are to prove that $\int_{\mathbb{R}^N} D_j\psi\, dx = 0$. The Fundamental Theorem of Calculus gives $\int_{-a}^a D_j\psi\, dx_j = [\psi]_{x_j=-a}^{x_j=a}$ for $a > 0$, and the compact support implies that this is 0 for a sufficiently large. Thus $\int_{\mathbb{R}} D_j\psi\, dx_j = 0$, and Fubini's Theorem gives $\int_{\mathbb{R}^N} D_j\psi\, dx = 0$.

The function g in the definition of weak derivative is unique up to sets of measure 0 if it exists. In fact, if g_1 and g_2 are both weak derivatives of order α, then $\int_U (g_1 - g_2)\varphi\, dx = 0$ for all φ in $C^\infty_{\text{com}}(U)$. Fix an open set V having compact closure contained in U. If f is in $C_{\text{com}}(V)$, then Corollary 3.6a produces a sequence of functions φ_n in $C^\infty_{\text{com}}(V)$ tending uniformly to f. Since $g_1 - g_2$ is integrable on V, the equalities $\int_V (g_1 - g_2)\varphi_n\, dx = 0$ for all n imply $\int_V (g_1 - g_2) f\, dx = 0$. By the uniqueness in the Riesz Representation Theorem, $g_1 = g_2$ a.e. on V. Since V is arbitrary, $g_1 = g_2$ a.e. on U.

EXAMPLE. In the open set $U = (-1, 1) \subseteq \mathbb{R}^1$, the function $e^{i/|x|}$ is locally integrable and is differentiable except at $x = 0$, but it does not have a weak derivative. In fact, if it had g as a weak derivative, we could use φ's vanishing in neighborhoods of the origin to see that $g(x)$ has to be $-ix^{-2}(\operatorname{sgn} x)e^{i/|x|}$ almost everywhere. But this function is not locally integrable on U.

If f has α^{th} weak derivative $D^\alpha f$ and $D^\alpha f$ has β^{th} weak derivative $D^\beta(D^\alpha f)$, then f has $(\beta+\alpha)^{\text{th}}$ weak derivative $D^{\beta+\alpha}f$ and $D^{\beta+\alpha}f = D^\beta(D^\alpha f)$. In fact, if φ is in $C^\infty_{\text{com}}(U)$, then this conclusion follows from the computation

$$\begin{aligned}\textstyle\int_U f D^{\beta+\alpha}\varphi\,dx &= \textstyle\int_U f D^\alpha(D^\beta\varphi)\,dx = (-1)^{|\alpha|}\int_U D^\alpha f D^\beta\varphi\,dx \\ &= \textstyle(-1)^{|\alpha|+|\beta|}\int_U D^\beta(D^\alpha f)\varphi\,dx.\end{aligned}$$

If f has weak j^{th} partial derivative $D_j f$ and if ψ is in $C^\infty(U)$, then $f\psi$ has a weak j^{th} partial derivative, and it is given by $(D_j f)\psi + f(D_j\psi)$. In fact, this conclusion holds because $\int_U f\psi(D_j\varphi)\,dx = \int_U f D_j(\psi\varphi)\,dx - \int_U f(D_j\psi)\varphi\,dx = -\int_U(D_j f)\psi\varphi\,dx - \int_U f(D_j\psi)\varphi\,dx = -\int_U(f(D_j\psi) + (D_j f)\psi)\varphi\,dx$.

If f has β^{th} weak derivative $D^\beta f$ for every β with $\beta \le \alpha$ and if ψ is in $C^\infty(U)$, then $f\psi$ has an α^{th} weak derivative. It is given by the **Leibniz rule**:

$$D^\alpha(f\psi) = \sum_{\beta\le\alpha}\frac{\alpha!}{\beta!(\alpha-\beta)!}(D^\beta f)(D^{\alpha-\beta}\psi).$$

This formula follows by iterating the formula for $D_j(f\psi)$ in the previous paragraph.

Now we can give the definition of Sobolev spaces. Let $k \ge 0$ be an integer, and let $1 \le p < \infty$. Define

$$L^p_k(U) = \big\{f \in L^p(U)\,\big|\,\text{all } D^\alpha f \text{ exist weakly for } |\alpha| \le k \text{ and are in } L^p(U)\big\}.$$

Then $L^p_k(U)$ is a vector space, and we make it into a normed linear space by defining

$$\|f\|_{L^p_k} = \Big(\sum_{|\alpha|\le k}\int_U |D^\alpha f|^p\,dx\Big)^{1/p}.$$

The normed linear spaces $L^p_k(U)$ are the **Sobolev spaces** for U. All the remaining results in this section concern these spaces.[5]

[5]The subject of partial differential equations makes use of a number of families that generalize these spaces in various ways. Of particular importance is a family H^s such that $H^s = L^2_k$ when s is an integer $k \ge 0$ but s is a continuous real parameter with $-\infty < s < \infty$. The spaces $H^s(\mathbb{R}^N)$ are introduced in Problems 8–12 at the end of the chapter. For an open set U, the two spaces $H^s_{\text{com}}(U)$ and $H^s_{\text{loc}}(U)$ are introduced in Chapter VIII. All of these spaces are called **Sobolev spaces**.

Proposition 3.7. If $k \geq 0$ is an integer and if $1 \leq p < \infty$, then the normed linear space $L_k^p(U)$ is complete.

PROOF. If $\{f_m\}$ is a Cauchy sequence in $L_k^p(U)$, then for each α with $|\alpha| \leq k$, the sequence $\{D^\alpha f_m\}$ is Cauchy in $L^p(U)$. Since $L^p(U)$ is complete, we can define $f^{(\alpha)}$ to be the $L^p(U)$ limit of $D^\alpha f_m$. For φ in $C_{\text{com}}^\infty(U)$, we then have

$$\textstyle\int_U f^{(\alpha)} \varphi \, dx = \int_U (\lim_m D^\alpha f_m) \varphi \, dx = \lim_m \int_U (D^\alpha f_m) \varphi \, dx,$$

the second equality holding since φ is in the dual space $L^{p'}(U)$. In turn, this expression is equal to

$$\textstyle(-1)^{|\alpha|} \lim_m \int_U (f_m)(D^\alpha \varphi) \, dx = (-1)^{|\alpha|} \int_U (f^{(0)})(D^\alpha \varphi) \, dx,$$

the second equality holding since $D^\alpha \varphi$ is in $L^{p'}(U)$. Therefore $f^{(\alpha)} = D^\alpha f^{(0)}$ and f_m tends to $f^{(0)}$ in $L_k^p(U)$. □

Proposition 3.8. If $k \geq 0$ is an integer and if $1 \leq p < \infty$, then a function f is in $L_k^p(U)$ if f is in $L^p(U)$ and there exists a sequence $\{f_m\}$ in $C^k(U)$ such that

(a) $\lim_m \|f - f_m\|_p = 0$,
(b) for each α with $|\alpha| \leq k$, the iterated pointwise partial derivative $D^\alpha f_m$ is in $L^p(U)$ and converges in $L^p(U)$ as m tends to infinity.

PROOF. By (b), $\|D^\alpha(f_l - f_m)\|_p^p$ for each fixed α tends to 0 as l and m tend to infinity. Summing on α and taking the p^{th} root, we see that $\|f_l - f_m\|_{L_k^p}$ tends to 0. In other words, $\{f_m\}$ is Cauchy in $L_k^p(U)$. By Proposition 3.7, $\{f_m\}$ converges to some g in $L_m^p(U)$. The limit function g has to have the property that $\|f_m - g\|_p$ tends to 0, and (a) shows that we must have $g = f$. Therefore f is in $L_k^p(U)$. □

The key theorem is the following converse to Proposition 3.8.

Theorem 3.9. If $k \geq 0$ is an integer and if $1 \leq p < \infty$, then $C^\infty(U) \cap L_k^p(U)$ is dense in $L_k^p(U)$.

On the other hand, despite Corollary 3.6b, it will be a consequence of Sobolev's Theorem that $C_{\text{com}}^\infty(U)$ is not dense in $L_k^p(U)$ if k is sufficiently large. The proof of the present theorem will be preceded by a lemma affirming that at least the members of $L_k^p(U)$ with compact support in U can be approximated by members of $C_{\text{com}}^\infty(U)$.

In addition, the proof of the theorem will make use of an "exhausting sequence" and a smooth partition of unity based on it. Since U is locally compact and σ-compact, we can find a sequence $\{K_n\}_{n=1}^\infty$ of compact subsets of U with union

U such that $K_n \subseteq K_{n+1}^o$ for all n. This sequence is called an **exhausting sequence** for U. We construct the partition of unity $\{\psi_n\}_{n\geq 1}$ as follows. For $n \geq 1$, we use Proposition 3.5f to choose a C^∞ function φ_n with values in [0, 1] such that

$$\varphi_1(x) = \begin{cases} 1 & \text{for } x \in K_3, \\ 0 & \text{for } x \in (K_4^o)^c, \end{cases}$$

and for $n \geq 2$,

$$\varphi_n(x) = \begin{cases} 1 & \text{for } x \in K_{n+2} - K_{n+1}^o, \\ 0 & \text{for } x \in (K_{n+3}^o)^c \cup K_n. \end{cases}$$

In the sum $\sum_{n=1}^\infty \varphi_n(x)$, each x has a neighborhood in which only finitely many terms are nonzero and some term is nonzero. Therefore $\varphi = \sum_{n=1}^\infty \varphi_n$ is a well-defined member of $C^\infty(U)$. If we put $\psi_n = \varphi_n \big/ \varphi$, then ψ_n is in $C^\infty(U)$, $\sum_{n=1}^\infty \psi_n = 1$ on U, $\psi_1(x)$ is > 0 on K_3 and is $= 0$ on $(K_4^o)^c$, and for $n \geq 2$,

$$\psi_n(x) \begin{cases} > 0 & \text{for } x \in K_{n+2} - K_{n+1}^o, \\ = 0 & \text{for } x \in (K_{n+3}^o)^c \cup K_n. \end{cases}$$

Lemma 3.10. Let φ be a member of $C_{\text{com}}^\infty(\mathbb{R}^N)$ vanishing for $|x| \geq 1$ and having total integral 1, put $\varphi_\varepsilon(x) = \varepsilon^{-N}\varphi(\varepsilon^{-1}x)$ for $\varepsilon > 0$, and let f be a function in $L_k^p(U)$ whose support is a compact subset of U. For ε sufficiently small, $\varphi_\varepsilon * f$ is in $C_{\text{com}}^\infty(U)$, and

$$\lim_{\varepsilon \downarrow 0} \|\varphi_\varepsilon * f - f\|_{L_k^p} = 0.$$

PROOF. As in the proof of Corollary 3.6, $\varphi_\varepsilon * f$ has compact support contained in U if $\varepsilon < \varepsilon_0$, where ε_0 is 1 if $U = \mathbb{R}^N$ and ε_0 is the distance of the support of f to the complement of U if $U \neq \mathbb{R}^N$. Moreover, the function $\varphi_\varepsilon * f$ is in $C^\infty(\mathbb{R}^N)$ with $D^\alpha(\varphi_\varepsilon * f) = (D^\alpha \varphi_\varepsilon) * f$ for each α. Thus $\varphi_\varepsilon * f$ is in $C_{\text{com}}^\infty(U)$ if $\varepsilon < \varepsilon_0$. By the first remark after the definition of weak derivative, $\varphi_\varepsilon * f$ has weak derivatives of all orders for $\varepsilon < \varepsilon_0$, and they are given by the ordinary derivatives $D^\alpha(\varphi_\varepsilon * f)$. For $\varepsilon < \varepsilon_0$,

$$\begin{aligned} D^\alpha(\varphi_\varepsilon * f)(x) &= \textstyle\int_U f(y)(D^\alpha \varphi_\varepsilon)(x-y)\,dy \\ &= (-1)^{|\alpha|} \textstyle\int_U f(y) D^\alpha(y \mapsto \varphi_\varepsilon(x-y))\,dy. \end{aligned}$$

Since f by assumption has weak derivatives through order k and since $y \mapsto \varphi_\varepsilon(x-y)$ has compact support in U, the right side is equal to

$$\textstyle\int_U D^\alpha f(y)\varphi_\varepsilon(x-y)\,dy = (\varphi_\varepsilon * D^\alpha f)(x)$$

for $|\alpha| \leq k$. Therefore, for $\varepsilon < \varepsilon_0$ and $|\alpha| \leq k$, we have

$$\|D^\alpha(\varphi_\varepsilon * f - f)\|_p = \|\varphi_\varepsilon * (D^\alpha f) - D^\alpha f\|_p.$$

For these same α's, Proposition 3.5a shows that the right side tends to 0 as ε tends to 0. Therefore $\varphi_\varepsilon * f - f$ tends to 0 in $L_k^p(U)$. $\square$

PROOF OF THEOREM 3.9. Let f be in $L^p_k(U)$. The idea is to break f into a countable sum of functions of compact support, apply the lemma to each piece, and add the results. The difficulty lies in arranging that each of the pieces of f have controlled weak derivatives through order k. Thus instead of using indicator functions to break up f, we shall use an exhausting sequence $\{K_n\}_{n\geq 1}$ and an associated partition of unity $\{\psi_n\}_{n\geq 1}$ of the kind described after the statement of the theorem. The discussion above concerning the Leibniz rule shows that each $\psi_n f$ has weak derivatives of all orders $\leq k$, and the construction shows that $\psi_n f$ has support in K_5^o for $n = 1$ and in $K_{n+4}^o - K_{n-1}$ for $n \geq 2$.

Let $\epsilon > 0$ be given, let φ be a member of $C^\infty_{\text{com}}(\mathbb{R}^N)$ vanishing for $|x| \geq 1$ and having total integral 1, and put $\varphi_\varepsilon(x) = \varepsilon^{-N}\varphi(\varepsilon^{-1}x)$ for $\varepsilon > 0$. Applying Lemma 3.10 to $\psi_n f$, choose $\varepsilon_n > 0$ small enough so that the function $u_n = \varphi_{\varepsilon_n} * (\psi_n f)$ has support in K_5^o for $n = 1$ and in $K_{n+4}^o - K_{n-1}$ for $n \geq 2$ and so that

$$\|u_n - \psi_n f\|_{L^p_k} < 2^{-n}\epsilon.$$

Put $u = \sum_{n=1}^\infty u_n$. Each x in U has a neighborhood on which only finitely many of the functions u_n are not identically 0, and therefore u is in $C^\infty(U)$. Also,

$$u = \sum_{n=1}^\infty (u_n - \psi_n f) + f \qquad \text{since } \sum_{n=1}^\infty \psi_n = 1.$$

Since for each compact subset of U, only finitely many $u_n - \psi_n f$ are not identically 0 on that set, the weak derivatives of order $\leq k$ satisfy $D^\alpha u = \sum_{n=1}^\infty D^\alpha(u_n - \psi_n f) + D^\alpha f$. Hence

$$D^\alpha(u - f) = \sum_{n=1}^\infty D^\alpha(u_n - \psi_n f).$$

Minkowski's inequality for integrals therefore gives

$$\|D^\alpha(u - f)\|_p \leq \sum_{n=1}^\infty \|D^\alpha(u_n - \psi_n f)\|_p \leq \sum_{n=1}^\infty \|u_n - \psi_n f\|_{L^p_k} \leq \sum_{n=1}^\infty \frac{\epsilon}{2^n} = \epsilon.$$

Finally we raise both sides to the p^{th} power, sum for α with $|\alpha| \leq k$, and extract the p^{th} root. If $m(k)$ denotes the number of such α's, we obtain

$$\|u - f\|_{L^p_k} \leq m(k)^{1/p}\,\epsilon,$$

and the proof is complete. □

Now we come to Sobolev's Theorem. For the remainder of the section, the open set U will be assumed bounded, and we shall impose a regularity condition on its boundary $\partial U = U^{\text{cl}} - U$. When we isolate one of the coordinates of points in $\mathbb{R}^N$, say the j^{th}, let us write y' for the other $N-1$ coordinates, so that $y = (y_j, y')$. We say that U **satisfies the cone condition** if there exist positive constants c and h such that for each x in U, there are a sign $\pm$ and an index j with $1 \leq j \leq N$ for which the closed truncated cone

$$\Gamma_x = x + \big\{y = (y_j, y') \bigm| \pm y_j \geq c|y'| \text{ and } |y| \leq h\big\}$$

lies in U for one choice of the sign $\pm$. See Figure 3.1. Problem 4 at the end of the chapter observes that if the bounded open set U has a C^1 boundary in a certain sense, then U satisfies the cone condition.

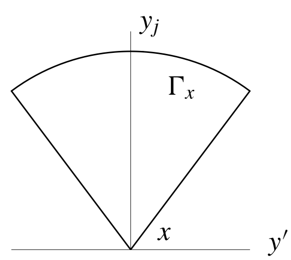

FIGURE 3.1. Cone condition for a bounded open set.

Theorem 3.11 (Sobolev's Theorem). Let U be a nonempty bounded open set in $\mathbb{R}^N$, and suppose that U satisfies the cone condition with constants c and h. If $1 \leq p < \infty$ and $k > N/p$, then there exists a constant $C = C(N, c, h, p, k)$ such that

$$\sup_{x \in U} |u(x)| \leq C \|u\|_{L^p_k}$$

for all u in $C^\infty(U) \cap L^p_k(U)$.

REMARK. Under the stated conditions on k and p, the theorem says that the inclusion of $C^\infty(U) \cap L^p_k(U)$ into the Banach space $C(U)$ of *bounded* continuous functions on U is a bounded linear operator relative to the norm of $L^p_k(U)$. Since $C^\infty(U) \cap L^p_k(U)$ is dense in $L^p_k(U)$ by Theorem 3.9 and since $C(U)$ is complete, the inclusion extends to a continuous map of $L^p_k(U)$ into $C(U)$. In other words, every member of $L^p_k(U)$ can be regarded as a bounded continuous function on U.

PROOF. Fix g in $C^\infty_{\text{com}}(\mathbb{R}^1)$ with $g(t)$ equal to 1 for $|t| \leq \frac{1}{2}$ and equal to 0 for $|t| \geq \frac{3}{4}$. Fix x in U and its associated sign $\pm$ and index j. We introduce spherical coordinates about x with the indices reordered so that j comes first, writing $x+y$ for a point near x with

$$\begin{aligned}
y_j &= \pm r\cos\varphi,\\
y_1 &= r\sin\varphi\cos\theta_1,\\
&\ \vdots \quad \text{(with } y_j \text{ omitted)}\\
y_{N-1} &= r\sin\varphi\sin\theta_1\cdots\sin\theta_{N-3}\cos\theta_{N-2},\\
y_N &= r\sin\varphi\sin\theta_1\cdots\sin\theta_{N-3}\sin\theta_{N-2},
\end{aligned}$$

when

$$\begin{aligned}
&0 \le \varphi \le \pi,\\
&0 \le \theta_i \le \pi \text{ for } i < N-2,\\
&0 \le \theta_{N-2} \le 2\pi.
\end{aligned}$$

All the points $x+y$ with $0 \le \varphi \le \Phi(c)$, where $\Phi(c)$ is some positive number and $0 \le r \le h$, lie in the cone Γ_x at x. For such φ's and for $0 \le t \le 1$, we define

$$F(t) = g\big(\tfrac{t}{h}\big)u\big(x + (\pm t\cos\varphi, t\sin\varphi\cos\theta_1, \dots)\big)$$

and expand F in a Taylor series through order $k-1$ with remainder about the point $t = h$. Because of the behavior of g, F and all its derivatives vanish at $t = h$. Therefore $F(t)$ is given by the remainder term:

$$F(t) = \tfrac{1}{(k-1)!}\int_h^t (t-s)^{k-1}F^{(k)}(s)\,ds.$$

Putting $t = 0$, we obtain

$$\begin{aligned}
u(x) &= \tfrac{1}{(k-1)!}\int_h^0 (-r)^{k-1}\,\tfrac{\partial^k}{\partial r^k}\big[g\big(\tfrac{r}{h}\big)u\big(x+(\cdots)\big)\big]\,dr\\
&= \tfrac{(-1)^k}{(k-1)!}\int_0^h r^{k-N}\,\tfrac{\partial^k}{\partial r^k}\big[g\big(\tfrac{r}{h}\big)u\big(x+(\cdots)\big)\big]\,r^{N-1}\,dr.
\end{aligned}$$

We regard the integral on the right side as taking place over the radial part of the spherical coordinates that describe the set of y's in Γ_x, and we want to extend the integration over all of Γ_x. To do so, we have to integrate over all values of $\theta_1, \dots, \theta_{N-2}$ and for $0 \le \varphi \le \Phi(c)$. We multiply by the spherical part of the Jacobian determinant for spherical coordinates and integrate both sides. The integrand on the left side is constant, being independent of y, and gives a *positive* multiple of $u(x)$. Dividing by that multiple, we get

$$u(x) = c_1\int_{\Gamma_x - x} |y|^{k-N}\,\tfrac{\partial^k}{\partial r^k}\big[g\big(\tfrac{|y|}{h}\big)u(x+y)\big]\,dy.$$

Suppose temporarily that $p > 1$. With p' still denoting the index dual to p, application of Hölder's inequality gives

$$|u(x)| \le c_1 \Big(\int_{\Gamma_x - x} |y|^{(k-N)p'}\, dy \Big)^{1/p'} \Big(\int_{\Gamma_x - x} \Big| \tfrac{\partial^k}{\partial r^k} \Big[g\big(\tfrac{|y|}{h}\big) u(x+y) \Big] \Big|^p\, dy \Big)^{1/p}.$$

The first integral on the right side is the critical one. The radius extends from 0 to h, and the integral is finite if and only if $(k-N)p' > -N > 0$, i.e., $k > N - N/p' = N/p$. This is the condition in the theorem.

The differentiation $\frac{\partial^k}{\partial r^k}$ in the second factor on the right can be expanded in terms of derivatives in Cartesian coordinates, and then the integration can be extended over all of U. The result is that the second factor is dominated by a multiple of $\|u\|_{L_k^p}$. This completes the proof when $p > 1$.

Now suppose that $p = 1$. Then the above result from applying Hölder's inequality is replaced by the inequality

$$|u(x)| \le c_1 \big\| |y|^{k-N} \big\|_{\infty, \Gamma_x - x} \int_{\Gamma_x - x} \Big| \tfrac{\partial^k}{\partial r^k} \Big[g\big(\tfrac{|y|}{h}\big) u(x+y) \Big] \Big|\, dy.$$

The first factor is finite if $k \ge N$, and the second factor is handled as before. This completes the proof if $p = 1$. □

Corollary 3.12. Suppose that U is a nonempty bounded open subset of $\mathbb{R}^N$ satisfying the cone condition, and suppose that $1 < p < \infty$ and that m and k are integers ≥ 0 such that $k > m + N/p$. If f is in $L_k^p(U)$, then f can be redefined on a set of measure 0 so as to be in $C^m(U)$.

PROOF. Choose by Theorem 3.9 a sequence $\{f_i\}$ in $C^\infty(U) \cap L_k^p(U)$ such that $\lim f_i = f$ in $L_k^p(U)$. For $|\alpha| \le m$, we apply Theorem 3.11 to see that

$$\sup_U |D^\alpha f_i - D^\alpha f_j|$$

tends to 0 as i and j tend to infinity. Thus all the $D^\alpha f_i$ converge uniformly. It follows that the uniform-limit function $\widetilde{f} = \lim f_i$ is in $C^m(U)$. Since $f_i \to f$ in $L^p(U)$ and $f_i \to \widetilde{f}$ uniformly, we conclude that $\widetilde{f} = f$ almost everywhere. Thus $\widetilde{f}$ tells how to redefine f on a set of measure 0 so as to be in $C^m(U)$. □

3. Harmonic Functions

Let U be an open set in $\mathbb{R}^N$. The discussion will not be very interesting for $N = 1$, and we exclude that case. A function u in $C^2(U)$ is **harmonic** in U if $\Delta u = 0$ identically in U. Harmonic functions were introduced already in Chapter I and investigated in connection with certain boundary-value problems. In the present section we examine properties of harmonic functions more generally. Harmonic functions in a half space, through their boundary values and the Poisson integral formula, become a tool in analysis for working with functions on the Euclidean boundary, and the behavior of harmonic functions on general open sets becomes a prototype for the behavior of solutions of further "elliptic" second-order partial differential equations.

Harmonic functions will be characterized shortly in terms of a certain mean-value property. To get at this characterization and its ramifications, we need the N-dimensional "Divergence Theorem" of Gauss for two special cases—a ball and a half space. The result for a ball will be formulated as in Lemma 3.13 below; we give a proof since this theorem was not treated in *Basic*. The argument for a half space is quite simple, and we will incorporate what we need into the proof of Proposition 3.15 below. For the case of a ball, recall[6] that the change-of-variables formula $x = r\omega$, with $r \geq 0$ and $|\omega| = 1$, for transforming integrals in Cartesian coordinates for $\mathbb{R}^N$ into spherical coordinates involves substituting $dx = r^{N-1}\,dr\,d\omega$, where $d\omega$ is a certain rotation-invariant measure on the unit sphere S^{N-1} that can be expressed in terms of $N-1$ angular variables. The open ball of radius x_0 and radius r is denoted by $B(r; x_0)$, and its boundary is $\partial B(r; x_0)$.

Lemma 3.13. If F is a C^1 function in an open set on $\mathbb{R}^N$ containing the closed ball $B(r; 0)^{\text{cl}}$ and if $1 \leq j \leq N$, then

$$\int_{x \in B(r;0)} \frac{\partial F}{\partial x_j}(x_0 + x)\,dx = \int_{r\omega \in \partial B(r;0)} x_j F(x_0 + r\omega) r^{N-2}\,d\omega.$$

REMARKS. The lemma is a special case of the **Divergence Theorem**, whose usual formula of is $\int_U \operatorname{div} \mathbf{F}\,dx = \int_{\partial U} (\mathbf{F} \cdot \mathbf{n})\,dS$, where U is a suitable bounded open set, $\partial U = U^{\text{cl}} - U$ is its boundary, $\mathbf{n}$ is the outward-pointing unit normal, $\mathbf{F}$ is a vector-valued C^1 function, and dS is surface area. In Lemma 3.13, U is specialized to the ball $B(r; 0)$, dS is the $(N-1)$-dimensional area measure $r^{N-1}\,d\omega$ on the surface $\partial B(r; 0)$ of the ball, $\mathbf{F}$ is taken to be the product of F by the j^{th} standard basis vector e_j, and $e_j \cdot \mathbf{n}$ is $r^{-1}x_j$.

PROOF. Without loss of generality, we may take $j = 1$ and $x_0 = 0$. Write $x = (x_1, x')$, where $x' = (x_2, \ldots, x_N)$, and write $\omega = (\omega_1, \omega')$ similarly. The left side in the displayed formula is equal to

$$\begin{aligned}\int_{|x'| \leq r} &\int_{x_1 = -\sqrt{r^2 - |x'|^2}}^{\sqrt{r^2 - |x'|^2}} \tfrac{\partial F}{\partial x_1}(x_1, x')\,dx_1\,dx' \\ &= \int_{|x'| \leq r} \Big[F(\sqrt{r^2 - |x'|^2}, x') - F(-\sqrt{r^2 - |x'|^2}, x')\Big]\,dx'.\end{aligned}$$

[6] From Section VI.5 of *Basic*.

Thus the lemma will follow if it is proved that

$$\int_{|x'|\le r} F(\sqrt{r^2-|x'|^2}, x')\,dx' = \int_{|\omega|=1,\ \omega_1\ge 0} x_1 F(r\omega) r^{N-2}\,d\omega \qquad (*)$$

and

$$-\int_{|x'|\le r} F(-\sqrt{r^2-|x'|^2}, x')\,dx' = \int_{|\omega|=1,\ \omega_1\le 0} x_1 F(r\omega) r^{N-2}\,d\omega. \qquad (**)$$

Let us use ordinary spherical coordinates for ω, with

$$\begin{pmatrix} r\omega_1 \\ \vdots \\ r\omega_N \end{pmatrix} = \begin{pmatrix} r\cos\theta_1 \\ r\sin\theta_1\cos\theta_2 \\ \vdots \\ r\sin\theta_1\cdots\sin\theta_{N-2}\cos\theta_{N-1} \\ r\sin\theta_1\cdots\sin\theta_{N-2}\sin\theta_{N-1} \end{pmatrix}$$

and

$$d\omega = \sin^{N-2}\theta_1 \sin^{N-3}\theta_2 \cdots \sin\theta_{N-2}\,d\theta_1\cdots d\theta_{N-1}.$$

The right side of $(*)$ is equal to

$$\begin{aligned} &\int_{|\omega|=1,\ \omega_1\ge 0} F(r\omega)\omega_1 r^{N-2}\,d\omega \\ &= \int_{\substack{0\le\theta_1\le\pi/2, \\ 0\le\theta_j\le\pi \text{ for } 1<j<N-1, \\ 0\le\theta_{N-1}\le 2\pi}} F(r\omega) r^{N-1}\cos\theta_1 \sin^{N-2}\theta_1 \sin^{N-3}\theta_2\cdots\sin\theta_{N-2}\,d\theta_1\cdots d\theta_{N-1}, \end{aligned}$$

and we show that it equals the left side of $(*)$ by carrying out for the left side of $(*)$ the change of variables $x' \leftrightarrow (\theta_1,\dots,\theta_{N-1})$ given with r constant by

$$x' = \begin{pmatrix} x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} r\sin\theta_1\cos\theta_2 \\ \vdots \\ r\sin\theta_1\cdots\sin\theta_{N-2}\cos\theta_{N-1} \\ r\sin\theta_1\cdots\sin\theta_{N-2}\sin\theta_{N-1} \end{pmatrix}.$$

The Jacobian matrix is the same as for the change to spherical coordinates $(r,\theta_2,\dots,\theta_{N-1})$ except that the first column has a factor $r\cos\theta_1$ instead of 1 and the other columns have an extra factor of $\sin\theta_1$. Consequently

$$dx' = r^{N-1}\big(|\cos\theta_1|\sin^{N-2}\theta_1\big)\big(\sin^{N-3}\theta_2\cdots\sin\theta_{N-2}\big)\,d\theta_1\cdots d\theta_{N-1}.$$

Therefore the measures match in the two transformed sides, the sets of integration for $(\theta_1,\dots,\theta_{N-1})$ are the same, and the integrands are the same because $\cos\theta_1 = |\cos\theta_1|$. This proves $(*)$. For $(**)$ we make the same computation but the interval of integration for θ_1 is $\pi/2\le\theta_1\le\pi$. To get a match, the minus sign is necessary because $\cos\theta_1 = -|\cos\theta_1|$. □

Proposition 3.14 (Green's formula[7] for a ball). Let B be an open ball in $\mathbb{R}^N$, let ∂B be its surface, and let $d\sigma$ be the surface-area measure of ∂B. If u and v are C^2 functions in an open set containing B^{cl}, then

$$\int_B (u\Delta v - v\Delta u)\,dx = \int_{\partial B} \Big(u\,\frac{\partial v}{\partial \mathbf{n}} - v\,\frac{\partial u}{\partial \mathbf{n}}\Big)\,d\sigma,$$

where $\mathbf{n} : \partial S \to \mathbb{R}^N$ is the outward-pointing unit normal vector.

PROOF. Apply Lemma 3.13 to $F = u\,\frac{\partial v}{\partial x_j}$ and then to $F = v\,\frac{\partial u}{\partial x_j}$, and subtract the results. Then sum on j. □

Let Ω_{N-1} be the surface area $\int_{S^{N-1}} d\omega$ of the unit sphere in $\mathbb{R}^N$. A continuous function u on an open subset U of $\mathbb{R}^N$ is said to have the **mean-value property** in U if the value of u at each point x in U equals the average value of u over each sphere centered at x and lying in U, i.e., if

$$u(x) = \frac{1}{\Omega_{N-1}} \int_{\omega\in S^{N-1}} u(x + t\omega)\,d\omega$$

for every x in U and for every positive t less than the distance from x to U^c.

The mean-value property over spheres implies a corresponding average-value property over balls. In fact, the volume $|B(t_0; 0)|$ of the ball $B(t_0; 0)$ is given by $\int_0^{t_0}\int_{S^{N-1}} t^{N-1}\,d\omega\,dt = N^{-1}t_0^N \int_{S^{N-1}} d\omega = N^{-1}t_0^N\Omega_{N-1}$. When the mean-value property over spheres is satisfied and t_0 is less than the distance from x to U^c, we can apply the operation $Nt_0^{-N}\int_0^{t_0}(-)\,dt$ to both sides of the mean-value formula and obtain

$$u(x) = \frac{Nt_0^{-N}}{\Omega_{N-1}} \int_0^{t_0}\int_{\omega\in S^{N-1}} u(x+t\omega)t^{N-1}\,d\omega\,dt = \frac{1}{|B(t_0; 0)|}\int_{B(t_0;0)} u(x+y)\,dy.$$

Proposition 3.15 (Green's formula for a half space). Let H be the subset of $\mathbb{R}^N = \{(x', x_N) \mid x' \in \mathbb{R}^{N-1} \text{ and } x_N \in \mathbb{R}\}$ with $x_N > 0$. Suppose that u and v are C^2 functions on an open subset of $\mathbb{R}^N$ containing the closure $\overline{H}$ and that at least one of u and v is compactly supported. Then

$$\int_{x\in H} (u\Delta v - v\Delta u)\,dx = \int_{x'\in\mathbb{R}^{N-1}} \Big(v\,\frac{\partial u}{\partial x_N} - u\,\frac{\partial v}{\partial x_N}\Big)\,dx'.$$

PROOF. Suppose F is a C^1 function compactly supported on an open subset of $\mathbb{R}^N$ containing $\overline{H}$. If $1 \le j \le N-1$, then $\int_H \frac{\partial F}{\partial x_j}\,dx = 0$ since the integral with

[7]This formula is related to but distinct from the formula with the same name at the beginning of Section I.3.

respect to dx_j is the difference between two values of F and since these are 0 by the compactness of the support. For $j = N$, however, one of the boundary terms may fail to be 0, and the result is that $\int_H \frac{\partial F}{\partial x_N}\,dx = -\int_{\mathbb{R}^{N-1}} F(x')\,dx'$.

Apply the j^{th} of these formulas first to $F = u\frac{\partial v}{\partial x_j}$ and then to $F = v\frac{\partial u}{\partial x_j}$, sum the results on j, and subtract the two sums. The result is the formula of the proposition. $\square$

Theorem 3.16. Let U be an open set in $\mathbb{R}^N$, and let u be a continuous scalar-valued function on U. If u is harmonic on U, then u has the mean-value property on U. Conversely if u has the mean-value property on U, then u is in $C^\infty(U)$ and is harmonic on U.

PROOF. Suppose that u is harmonic on U. We prove that u has the mean-value property. It is enough to treat $x = 0$. Green's formula, as in Proposition 3.14, directly extends from balls to the difference of two balls.[8] Thus we have

$$\int_E (u\Delta v - v\Delta u)\,dx = \int_{\partial E} \Big(u\,\frac{\partial v}{\partial \mathbf{n}} - v\,\frac{\partial u}{\partial \mathbf{n}}\Big)\,d\sigma \qquad (*)$$

whenever E is a closed ball B_t of radius t contained in U or is the difference $B_t - (B_\epsilon)^o$ of two concentric balls with $\epsilon < t$. Taking $E = B_t$ and $v = 1$ in $(*)$, we obtain

$$\int_{\partial B_t} \frac{\partial u}{\partial \mathbf{n}}\,d\sigma = 0. \qquad (**)$$

Routine computation shows that the function given by

$$v(x) = \begin{cases} |x|^{-(N-2)} & \text{for } N > 2, \\ \log|x| & \text{for } N = 2, \end{cases}$$

is harmonic for $x \neq 0$ and has $\frac{\partial v}{\partial r}$ equal to a nonzero multiple of $|x|^{-(N-1)}$, r being the spherical coordinate radius $|x|$. If we apply $(*)$ to this v and our harmonic u when $E = B_t - (B_\epsilon)^o$, we obtain

$$\int_{\partial(B_t - (B_\epsilon)^o)} \Big(u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}}\Big)\,d\sigma = 0.$$

Since v depends only on $|x|$, $(**)$ shows that the second term of the integrand yields 0. Thus this formula becomes

$$\int_{\partial(B_t - (B_\epsilon)^o)} u\frac{\partial v}{\partial \mathbf{n}}\,d\sigma = 0.$$

[8]For the extended result, suppose that the balls have radii $r_1 < r_2$. Then u and v are defined from radius $r_1 - \varepsilon$ to $r_2 + \varepsilon$ for some $\varepsilon > 0$. We can adjust u and v by multiplying by a suitable smooth function that is identically 1 for radius $\geq r_1 - \frac{1}{3}\varepsilon$ and identically 0 for radius $\leq r_1 - \frac{2}{3}\varepsilon$, and then u and v will extend as smooth functions for radius $< r_2 + \varepsilon$. Consequently Proposition 3.14 will apply on each ball to the adjusted functions, and subtraction of the results gives the desired version of Green's formula.

The normal vector for the inner sphere points toward the center. Hence we can rewrite our equality as

$$\int_{|x|=\epsilon} u\,\tfrac{\partial v}{\partial r}\,d\sigma = \int_{|x|=t} u\,\tfrac{\partial v}{\partial r}\,d\sigma.$$

Since $\frac{\partial v}{\partial r} = c|x|^{-(N-1)}$ with $c \neq 0$, we obtain

$$\epsilon^{-(N-1)} \int_{|x|=\epsilon} u\,d\sigma = t^{-(N-1)} \int_{|x|=t} u\,d\sigma.$$

On the left side, $d\sigma = \epsilon^{N-1}\,d\omega$, while on the right side, $d\sigma = t^{N-1}\,d\omega$. Therefore

$$\int_{|\omega|=1} u(\epsilon\omega)\,d\omega = \int_{|\omega|=1} u(t\omega)\,d\omega$$

whenever $0 < \epsilon < t$ and B_t is contained in U. Dividing by Ω_{N-1}, letting ϵ decrease to 0, and using the continuity of u, we see that $u(0) = \int_{\omega\in S^{N-1}} u(t\omega)\,d\omega$. Thus u has the mean-value property.

For the converse direction suppose initially that u is in $C^2(U)$. Define

$$m_t(u)(x) = \Omega_{N-1}^{-1} \int_{|\omega|=1} u(x+t\omega)\,d\omega$$

whenever x is in U and t is a positive number less than the distance of x to U^c. With x fixed, the function $m_t(u)(x)$ has two continuous derivatives. We shall show that

$$\frac{d^2}{dt^2}\,m_t(u)(x)\Big|_{t=0} = N^{-1}\Delta u(x), \tag{†}$$

the derivatives being understood to be one-sided derivatives as t decreases to 0. If u is assumed to have the mean-value property, $m_t(u)(x)$ is constant in t, and we can conclude from (†) that $\Delta u(x) = 0$. The computation of $\frac{d^2}{dt^2}\,m_t(u)(x)$ is

$$\begin{aligned} m_t(u)(x) &= \Omega_{N-1}^{-1} \int_{|\omega|=1} u(x_1+t\omega_1,\dots,x_N+t\omega_N)\,d\omega,\\ \tfrac{d}{dt}\,m_t(u)(x) &= \Omega_{N-1}^{-1} \int_{|\omega|=1} \textstyle\sum_{j=1}^N \omega_j D_j u(x+t\omega)\,d\omega,\\ \tfrac{d^2}{dt^2}\,m_t(u)(x) &= \Omega_{N-1}^{-1} \int_{|\omega|=1} \textstyle\sum_{j,k=1}^N \omega_j\omega_k D_j D_k u(x+t\omega)\,d\omega. \end{aligned}$$

Letting t decrease to 0, we obtain

$$\tfrac{d^2}{dt^2}\,m_t(u)(x)\Big|_{t=0} = \Omega_{N-1}^{-1} \textstyle\sum_{j,k=1}^N D_j D_k u(x) \int_{|\omega|=1} \omega_j\omega_k\,d\omega.$$

If $j \neq k$, then $\int_{|\omega|=1} \omega_j\omega_k\,d\omega = 0$ since the integrand is an odd function of the j^{th} variable taken over a set symmetric about 0. The integral $\int_{|\omega|=1} \omega_j^2\,d\omega$ is independent of j and has the property that N times it is equal to $\int_{|\omega|=1} |\omega|^2\,d\omega = \int_{|\omega|=1} d\omega = \Omega_{N-1}$. Thus $\int_{|\omega|=1} \omega_j^2\,d\omega = N^{-1}\Omega_{N-1}$, and

$$\tfrac{d^2}{dt^2}\, m_t(u)(x)\Big|_{t=0} = N^{-1}\textstyle\sum_{j=1}^N D_j^2 u(x) = N^{-1}\Delta u(x).$$

This proves (†) and completes the argument that a C^2 function in U with the mean-value property is harmonic.

Finally suppose that u has the mean-value property and is assumed to be merely continuous. Proposition 3.5e allows us to choose a function $\varphi \geq 0$ in $C^\infty_{\text{com}}(\mathbb{R}^N)$ with $\varphi(x) = \varphi_0(|x|)$, $\int_{\mathbb{R}^N} \varphi(x)\,dx = 1$, and $\varphi(x) = 0$ for $|x| \geq 1$. Put $\varphi_\varepsilon(x) = \varepsilon^{-N}\varphi(\varepsilon^{-1}x)$, and define $u_\varepsilon(x) = \int_{\mathbb{R}^N} u(x-y)\varphi_\varepsilon(y)\,dy$ in the open set $U_\varepsilon = \{x \in U \mid D(x, U^c) > \varepsilon\}$. Proposition 3.5c shows that u_ε is in $C^\infty(U_\varepsilon)$, and the mean-value property of u, in combination with the radial nature of φ_ε as expressed by the equality $\varphi_\varepsilon(t\omega) = \varphi_\varepsilon(te_1)$, forces $u_\varepsilon(x) = u(x)$ for all x in U_ε:

$$\begin{aligned} u_\varepsilon(x) &= \textstyle\int_{t=0}^{\varepsilon}\int_{|\omega|=1} u(x-t\omega)\varphi_\varepsilon(t\omega)t^{N-1}\,d\omega\,dt \\ &= \textstyle\int_{t=0}^{\varepsilon} \Omega_{N-1}u(x)\varphi_\varepsilon(te_1)t^{N-1}\,dt \\ &= u(x)\textstyle\int_{\mathbb{R}^N}\varphi_\varepsilon(y)\,dy = u(x). \end{aligned}$$

Since ε is arbitrary, u is in $C^\infty(U)$. The function u has now been shown to be in $C^2(U)$, and it is assumed to have the mean-value property. Therefore the previous case shows that it is harmonic. □

Corollary 3.17. If u is harmonic on an open subset U of $\mathbb{R}^N$, then u is in $C^\infty(U)$.

PROOF. This follows by using both directions of Theorem 3.16. □

A sequence of functions $\{u_n\}$ on a locally compact Hausdorff space X is said to converge **uniformly on compact subsets** of X if $\lim u_n = u$ pointwise on X and if for each compact subset K of X, the convergence is uniform on K. For example the sequence $\{x^n\}$ converges to the 0 function on $(0, 1)$ uniformly on compact subsets.

Corollary 3.18. If $\{u_n\}$ is a sequence of harmonic functions on an open subset U of $\mathbb{R}^N$ and if $\{u_n\}$ converges uniformly on compact subsets to u, then u is harmonic on U.

PROOF. About any point of U is a compact neighborhood lying in U, and the convergence is uniform on that neighborhood. Therefore u is continuous. Each integration needed for the mean-value property occurs on a compact subset of U, and the uniform convergence allows us to interchange limit and integral. Therefore the mean-value property for each u_n, valid because of one direction of Theorem 3.16, implies the mean-value property for u. Hence u is harmonic by the converse direction of Theorem 3.16. $\square$

Suppose that U is open in $\mathbb{R}^N$ and that u is harmonic on U. If B is an open ball in U, then $\int_U u\Delta\psi\,dx = 0$ for all $\psi \in C^\infty_{\text{com}}(B)$ by Green's formula (Proposition 3.14), since ψ and $\frac{\partial\psi}{\partial\mathbf{n}}$ are both identically 0 on the boundary of B. We shall use a smooth partition of unity to show that $\int_U u\Delta\psi\,dx$ is therefore 0 for all $\psi \in C^\infty_{\text{com}}(U)$. Corollary 3.19 below provides a converse; we shall use the converse in a crucial way in Corollary 3.23 below.

The argument to construct the partition of unity goes as follows. To each point of $K = \text{support}(\psi)$, we can associate an open ball centered at that point whose closure is contained in U. As the point varies, these open balls cover K, and we extract a finite subcover $\{U_1, \dots, U_k\}$. Lemma 3.15b of *Basic* constructs an open cover $\{W_1, \dots, W_k\}$ of K such that W_i^{cl} is a compact subset of U_i for each i. Now we argue as in the proof of Proposition 3.14 of *Basic*. A second application of Lemma 3.15b of *Basic* gives an open cover $\{V_1, \dots, V_k\}$ of K such that V_i^{cl} is compact and $V_i^{\text{cl}} \subseteq W_i$ for each i. Proposition 3.5f constructs a smooth function $g_i \geq 0$ that is 1 on V_i^{cl} and is 0 off W_i. Then $g = \sum_{i=1}^k g_i$ is smooth and ≥ 0 on $\mathbb{R}^N$ and is > 0 everywhere on K. A second application of Proposition 3.5f produces a smooth function $h \geq 0$ on $\mathbb{R}^N$ that is 1 on the set where g is 0 and is 0 on K. Then $g+h$ is everywhere positive on $\mathbb{R}^N$, and the functions $\varphi_i = g_i/(g+h)$ form the smooth partition of unity that we shall use.

To apply the partition of unity, we write $\psi = \sum_i \varphi_i\psi$. Then each term $\varphi_i\psi$ is smooth and compactly supported in an open ball whose closure is contained in U. Consequently we have $\int_U u\Delta(\varphi_i\psi)\,dx = 0$ for each i. Summing on i, we obtain $\int_U u\Delta\psi\,dx = 0$, which was what was being asserted.

Corollary 3.19. Suppose that U is open in $\mathbb{R}^N$, that u is continuous on U, and that $\int_U u\Delta\psi\,dx = 0$ for all $\psi \in C^\infty_{\text{com}}(U)$. Then u is harmonic on U.

PROOF. Let B be an open ball of radius r with closure contained in U, fix $\varepsilon > 0$ so as to be $< r$, and let B_ε be the open ball of radius $r-\varepsilon$ with the same center as B. Construct φ_ε as in the proof of Theorem 3.16, and let $u_\varepsilon = u * \varphi_\varepsilon$. Suppose that ψ is in $C^\infty_{\text{com}}(B_\varepsilon)$. For t and x in $\mathbb{R}^N$ with $|t| \leq \varepsilon$, define $\psi_t(x) = \psi(t+x)$. Since ψ is supported in B_ε, ψ_t is supported in B, and therefore

$$\int_B u(x-t)\Delta\psi(x)\,dx = \int_B u(x)\Delta\psi(x+t)\,dx = \int_B u\Delta\psi_t\,dx = 0,$$

the last equality holding by the hypothesis. Multiplying by $\varphi_\varepsilon(t)$, integrating for $|t| \leq \varepsilon$, and interchanging integrals, we obtain

$$0 = \int_B \int_{\mathbb{R}^N} u(x-t)\varphi_\varepsilon(t)\Delta\psi(x)\,dt\,dx = \int_B u_\varepsilon(x)\Delta\psi(x)\,dx.$$

Since ψ vanishes identically near the boundary of B, this identity and Green's formula (Proposition 3.14) together yield $\int_B \psi(x)\Delta u_\varepsilon(x)\,dx = 0$ for all ψ in $C^\infty_{\text{com}}(B_\varepsilon)$. Application of Corollary 3.6a allows us to extend this conclusion to all ψ in $C_{\text{com}}(B_\varepsilon)$, and then the uniqueness in the Riesz Representation Theorem shows that we must have $\Delta u_\varepsilon(x) = 0$ for all x in B_ε. As ε decreases to 0, u_ε tends to u uniformly on compact sets. By Corollary 3.18, u is harmonic in B. Since the ball B is arbitrary in U, u is harmonic in U. □

Corollary 3.20. Let U be a connected open set in $\mathbb{R}^N$. If u is harmonic in U and $|u|$ attains a maximum somewhere in U, then u is constant in U.

PROOF. Suppose that $|u|$ attains a maximum at x_0. Multiplying u by a suitable constant $e^{i\theta}$, we may assume that $u(x_0) = M > 0$. The subset E of U where $u(x)$ equals M is closed and nonempty. It is enough to prove that E is open. Let x_1 be in E, and choose an open ball B centered at x_1, say of some radius $r > 0$, that lies in U. We show that B lies in E. For $0 < t < r$, Theorem 3.16 says that u has the mean-value property

$$\Omega_{N-1}^{-1} \int_{S^{N-1}} u(x_1 + t\omega)\,d\omega = u(x_1) = M.$$

Arguing by contradiction, suppose that $u(x_1 + t_0\omega_0) \neq u(x_1)$ for some $t_0\omega_0$ with $0 < t_0 < r$. Then $\operatorname{Re} u(x_1 + t_0\omega_0) < M - \epsilon$ for some $\epsilon > 0$, and continuity produces a nonempty open set S in the sphere S^{N-1} such that $\operatorname{Re} u(x_1 + t_0\omega) < M - \epsilon$ for ω in S. If σ is the name of the measure on S^{N-1}, then we have

$$\begin{aligned} M\Omega_{N-1} &= \operatorname{Re}\Big(\int_{S^{N-1}} u(x_1 + t\omega)\,d\omega\Big) \\ &= \int_S \operatorname{Re} u(x_1 + t\omega)\,d\omega + \int_{S^{N-1}-S} \operatorname{Re} u(x_1 + t\omega)\,d\omega \\ &\le \int_S (M - \epsilon)\,d\omega + \int_{S^{N-1}-S} M\,d\omega \\ &= (M - \epsilon)\sigma(S) + M\sigma(S^{N-1} - S) \\ &= M\Omega_{N-1} - \epsilon\sigma(S), \end{aligned}$$

and we have arrived at a contradiction since $\sigma(S) > 0$. □

Corollary 3.21. Let U be a bounded open subset of $\mathbb{R}^N$, and let ∂U be its boundary. If u is harmonic in U and is u is continuous on U^{cl}, then $\sup_{x\in U} |u(x)| = \max_{x\in\partial U} |u(x)|$.

PROOF. Since u is continuous and U^{cl} is compact, $|u|$ assumes its maximum M somewhere on U^{cl}. If $|u(x_0)| = M$ for some x_0 in U, then Corollary 3.20 shows that u is constant on the component of U to which x_0 belongs. The closure of that component cannot equal that component since $\mathbb{R}^N$ is connected. Thus the closure of that component contains a point of ∂U, and $|u|$ must equal M at that point of ∂U. Consequently $\sup_{x\in U} |u(x)| \le \max_{x\in\partial U} |u(x)|$. Since every point of ∂U is the limit of a sequence of points in U, the reverse inequality is valid as well, and the corollary follows. □

Corollary 3.22 (Liouville). Any bounded harmonic function on $\mathbb{R}^N$ is constant.

REMARKS. The best-known result of Liouville of this kind is one from complex analysis—that a bounded function analytic on all of $\mathbb{C}$ is constant. This complex-analysis result is actually a consequence of Corollary 3.22 because the real and imaginary parts of a bounded analytic function on $\mathbb{C}$ are bounded harmonic functions on $\mathbb{R}^2$.

PROOF. Suppose that u is harmonic on $\mathbb{R}^N$ with $|u(x)| \leq M$. Let x_1 and x_2 be distinct points of $\mathbb{R}^N$, and let $R > 0$. Since u has the mean-value property over spheres by Theorem 3.16, u equals its average value over balls. Hence $u(x_1) = |B(R;0)|^{-1} \int_{B(R;x_1)} u(x)\,dx$ and $u(x_2) = |B(R;0)|^{-1} \int_{B(R;x_2)} u(x)\,dx$. Subtraction gives

$$\begin{aligned} u(x_1)-u(x_2) &= |B(R;0)|^{-1}\big(\textstyle\int_{B(R;x_1)} u(x)\,dx - \int_{B(R;x_2)} u(x)\,dx\big) \\ &= |B(R;0)|^{-1}\big(\textstyle\int_{B(R;x_1)-B(R;x_2)} u(x)\,dx - \int_{B(R;x_2)-B(R;x_1)} u(x)\,dx\big). \end{aligned}$$

Therefore

$$|u(x_1) - u(x_2)| \leq |B(R;0)|^{-1} \int_{B(R;x_1)\Delta B(R;x_2)} |u(x)|\,dx,$$

where $B(R;x_1)\Delta B(R;x_2)$ is the symmetric difference $(B(R;x_1) - B(R;x_2)) \cup (B(R;x_2) - B(R;x_1))$. Hence

$$|u(x_1)-u(x_2)| \leq \frac{M|B(R;x_1)\Delta B(R;x_2)|}{|B(R;0)|} = \frac{MR^N|B(1;x_1/R)\Delta B(1;x_2/R)|}{R^N|B(1;0)|}.$$

The right side is $|B(1;x_1/R)\Delta B(1;x_2/R)|$, apart from a constant factor, and the sets $B(1;x_1/R)\Delta B(1;x_2/R)$ decrease and have empty intersection as R tends to infinity. By complete additivity of Lebesgue measure, the measure of the symmetric difference tends to 0. We conclude that $u(x_1) = u(x_2)$. Therefore u is constant. $\square$

In the final two corollaries let $\mathbb{R}^{N+1}_+$ be the open half space of points (x,t) in $\mathbb{R}^{N+1}$ such that x is in $\mathbb{R}^N$ and $t > 0$.

Corollary 3.23 (Schwarz Reflection Principle). Suppose that $u(x,t)$ is harmonic in $\mathbb{R}^{N+1}_+$, that u is continuous on $(\mathbb{R}^{N+1}_+)^{\text{cl}}$, and that $u(x,0) = 0$ for all x. Then the definition $u(x,-t) = -u(x,t)$ for $t > 0$ extends u to a harmonic function on all of $\mathbb{R}^{N+1}$.

PROOF. Define

$$w(x,t) = \begin{cases} u(x,t) & \text{for } t \geq 0, \\ -u(x,-t) & \text{for } t \leq 0. \end{cases}$$

The function w is continuous. We shall show that $\int_{\mathbb{R}^N} w\Delta\psi\,dx = 0$ for all $\psi \in C^\infty_{\text{com}}(\mathbb{R}^{N+1})$, and then Corollary 3.19 shows that w is harmonic. Write ψ as the sum of functions even and odd in the variable t. Since w is odd in t, the contribution to $\int_{\mathbb{R}^N} w\Delta\psi\,dx$ from the even part of ψ is 0. We may thus assume that ψ is odd in t.

For $\varepsilon > 0$, let $R_\varepsilon = \{(x,t) \mid t > \varepsilon\}$. It is enough to show that $\int_{R_\varepsilon} u\Delta\psi\,dx\,dt$ has limit 0 as ε decreases to 0 since $\int_{\mathbb{R}^{N+1}} w\Delta\psi\,dx\,dt$ is twice this limit. We apply Green's formula for a half space (Proposition 3.15) with $v = \psi$ on the set $R_\varepsilon \subseteq \mathbb{R}^{N+1}$ except for one detail: to get the hypothesis of compact support to be satisfied, we temporarily multiply ψ by a smooth function that is identically 1 for $t \geq \varepsilon$ and is identically 0 for $t \leq \frac{1}{2}\varepsilon$. Since u is harmonic in R_ε, the result is that

$$-\int_{R_\varepsilon} u\Delta\psi\,dx\,dt = \int_{R_\varepsilon} (\psi\Delta u - u\Delta\psi)\,dx\,dt = \int_{\{(x,t)\,\mid\,t=\varepsilon\}} \big(u\tfrac{\partial\psi}{\partial t} - \psi\tfrac{\partial u}{\partial t}\big)\,dx.$$

On the right side, $\lim_{\varepsilon\downarrow 0}\int_{\{(x,t)\,\mid\,t=\varepsilon\}} u\frac{\partial\psi}{\partial t}\,dx = 0$ since $u(\,\cdot\,,\varepsilon)$ tends uniformly to 0 on the relevant compact set of x's in $\mathbb{R}^N$.

Thus it is enough to prove that $\lim_{\varepsilon\downarrow 0}\int_{\{(x,t)\,\mid\,t=\varepsilon\}} \psi\frac{\partial u}{\partial t}\,dx = 0$. Since $\psi(x,t)$ is of class C^2, is odd in x, and is compactly supported, we have $|\psi(x,t)| \leq Ct$ uniformly in x for small positive t. Thus it is enough to prove that

$$\lim_{t\downarrow 0}\Big|\,t\,\frac{\partial u}{\partial t}(x,t)\Big| = 0 \qquad (*)$$

uniformly on compact subsets of $\mathbb{R}^N$.

To prove $(*)$, let φ be a function as in Proposition 3.5e, and let $\varphi_\varepsilon(x,t) = \varepsilon^{-(N+1)}\varphi(\varepsilon^{-1}(x,t))$. Fix x_0 in $\mathbb{R}^N$, and define $X_0 = (x_0,t_0)$ and $X = (x_0,t)$. If $|X - X_0| < \frac{1}{3}t_0$, then the mean-value property of u in $\mathbb{R}^{N+1}_+$ gives $u(X) = (u * \varphi_{\frac{1}{3}t_0})(X)$. Hence we have

$$\begin{aligned}\tfrac{\partial u}{\partial t}(X) &= \tfrac{\partial}{\partial t}\int_{\mathbb{R}^{N+1}} \varphi_{\frac{1}{3}t_0}(X-Y)u(Y)\,dY \\ &= \int_{\mathbb{R}^{N+1}} \tfrac{\partial}{\partial t}\big[(\tfrac{1}{3}t_0)^{-(N+1)}\varphi\big((\tfrac{1}{3}t_0)^{-1}(X-Y)\big)\big]u(Y)\,dY.\end{aligned}$$

In the computation of the partial derivative on the right side, the variable t appears as the last coordinate of X. Therefore this expression is equal to

$$(\tfrac{1}{3}t_0)^{-1}\int_{\mathbb{R}^{N+1}} (\tfrac{1}{3}t_0)^{-(N+1)}\,\tfrac{\partial\varphi}{\partial t}\big((\tfrac{1}{3}t_0)^{-1}(X-Y)\big)u(Y)\,dY.$$

Changing variables in the integration by a dilation in Y shows that this expression is equal also to

$$(\tfrac{1}{3}t_0)^{-1}\int_{\mathbb{R}^{N+1}} \tfrac{\partial\varphi}{\partial t}\big((\tfrac{1}{3}t_0)^{-1}X - Y\big)u(\tfrac{1}{3}t_0Y)\,dY.$$

If we write $Y = (y, s)$ and take absolute values, we obtain

$$\big|\tfrac{\partial u}{\partial t}(x_0, t)\big| \le 3t_0^{-1}\big\|\tfrac{\partial\varphi}{\partial t}\big\|_1 \sup_{\substack{|s-t_0|<2t_0/3,\\ Y \text{ near } X_0}} |u(Y)|.$$

The required behavior of $t\frac{\partial u}{\partial t}$ follows from this estimate. $\square$

Corollary 3.24. Suppose that $u(x, t)$ is harmonic in $\mathbb{R}^{N+1}_+$, that u is continuous on $(\mathbb{R}^{N+1}_+)^{\text{cl}}$, and that $u(x, 0) = 0$ for all x. If u is bounded, then u is identically 0.

REMARK. Without the assumption of boundedness, the function $u(x, t) = t$ is a counterexample.

PROOF. Corollary 3.23 shows that u extends to a bounded harmonic function on all of $\mathbb{R}^{N+1}$, and Corollary 3.22 shows that the extended function is constant, hence identically 0. $\square$

4. $\mathcal{H}^p$ Theory

As was said at the beginning of Section 3, harmonic functions in a half space, through their boundary values and the Poisson integral formula, become a tool in analysis for working with functions on the Euclidean boundary. The Poisson integral formula, which was introduced in Chapters VIII and IX of *Basic*, generates harmonic functions from boundary values.

The details are as follows. Let $\mathbb{R}^{N+1}_+$ be the open half space of pairs (x, t) in $\mathbb{R}^{N+1}$ with $x \in \mathbb{R}^N$ and with $t > 0$ in $\mathbb{R}^1$. We view the boundary $\{(x, 0) \mid x \in \mathbb{R}^N\}$ as $\mathbb{R}^N$. The function

$$P(x, t) = P_t(x) = \frac{c_N\, t}{(t^2 + |x|^2)^{\frac{1}{2}(N+1)}},$$

for $t > 0$, with $c_N = \pi^{-\frac{1}{2}(N+1)}\Gamma\big(\frac{N+1}{2}\big)$, is called the **Poisson kernel** for $\mathbb{R}^{N+1}_+$. The **Poisson integral formula** for $\mathbb{R}^{N+1}_+$ is $u(x, t) = (P_t * f)(x)$, where f is any given function in $L^p(\mathbb{R}^N)$ and $1 \le p \le \infty$, and the function u is called the **Poisson integral** of f.

If f is in L^p, then u is harmonic on $\mathbb{R}^{N+1}_+$, $u(\,\cdot\,,t)$ is in L^p for each $t>0$, and $\|u(\,\cdot\,,t)\|_p \le \|f\|_p$. For $1 \le p < \infty$, $\lim_{t\downarrow 0} u(\,\cdot\,,t) = f$ in the norm topology of L^p, while for $p=\infty$, $\lim_{t\downarrow 0} u(\,\cdot\,,t) = f$ in the weak-star topology of L^∞ against L^1. In both cases, $\lim_{t\downarrow 0}\|u(\,\cdot\,,t)\|_p = \|f\|_p$, and $\lim_{t\downarrow 0} u(x,t) = f(x)$ a.e.; this latter result is known as **Fatou's Theorem**. When $p=\infty$, the a.e. convergence occurs at any point where f is continuous, and the pointwise convergence is uniform on any subset of $\mathbb{R}^N$ where f is uniformly continuous.

The L^p theory for $p=1$ extends from integrable functions to the Banach space $M(\mathbb{R}^N)$ of finite complex Borel measures. Specifically if ν is a finite complex Borel measure on $\mathbb{R}^N$, then the Poisson integral of ν is defined to be the function $u(x,t) = (P_t * \mu)(x) = \int_{\mathbb{R}^N} P_t(x-y)\,d\nu(y)$. Then u is harmonic on $\mathbb{R}^{N+1}_+$, $\|u(\,\cdot\,,t)\|_1 \le \|\nu\|$ for each $t>0$, $\lim_{t\downarrow 0} u(\,\cdot\,,t) = \nu$ in the weak-star topology of $M(\mathbb{R}^N)$ against $C_{\mathrm{com}}(\mathbb{R}^N)$, and $\lim_{t\downarrow 0}\|u(\,\cdot\,,t)\|_1 = \|\mu\|$.

The new topic for this section is a converse to the above considerations. For $1 \le p \le \infty$, we define $\mathcal{H}^p(\mathbb{R}^{N+1}_+)$ to be the vector space of functions $u(x,t)$ on $\mathbb{R}^{N+1}_+$ such that

(i) $u(x,t)$ is harmonic on $\mathbb{R}^{N+1}_+$,
(ii) $\sup_{t>0}\|u(\,\cdot\,,t)\|_p < \infty$.

With $\|u\|_{\mathcal{H}^p}$ defined as $\sup_{t>0}\|u(\,\cdot\,,t)\|_p$, the vector space $\mathcal{H}^p(\mathbb{R}^{N+1}_+)$ is a normed linear space. If f is in $L^p(\mathbb{R}^N)$, then the facts about the Poisson integral formula show that the Poisson integral of f is in $\mathcal{H}^p(\mathbb{R}^{N+1}_+)$ and its $\mathcal{H}^p(\mathbb{R}^{N+1}_+)$ norm matches the $L^p(\mathbb{R}^N)$ norm of f. For $p=1$, we readily produce further examples. Specifically if ν is any member of $M(\mathbb{R}^N)$, then the Poisson integral of ν is in $\mathcal{H}^1(\mathbb{R}^{N+1}_+)$, with the $\mathcal{H}^1(\mathbb{R}^{N+1}_+)$ norm matching the $M(\mathbb{R}^N)$ norm. The theorem of this section will say that there are no other examples.

The members of $\mathcal{H}^\infty(\mathbb{R}^{N+1}_+)$ are exactly the bounded harmonic functions in the half space $\mathbb{R}^{N+1}_+$, and the tool for obtaining an L^∞ function on $\mathbb{R}^N$ from this harmonic function is the preliminary form of Alaoglu's Theorem proved in *Basic*:[9] any norm-bounded sequence in the dual of a separable normed linear space has a weak-star convergent subsequence.[10] We shall use Corollary 3.24 to see that the harmonic function has to be the Poisson integral of this L^∞ function.

Theorem 3.25. If $1 < p \le \infty$, then any harmonic function in $\mathcal{H}^p(\mathbb{R}^{N+1}_+)$ is the Poisson integral of a function in $L^p(\mathbb{R}^N)$. For $p=1$, any harmonic function in $\mathcal{H}^1(\mathbb{R}^{N+1}_+)$ is the Poisson integral of a finite complex measure in $M(\mathbb{R}^N)$.

PROOF. We begin by proving that $u(x,t)$ is bounded for $t \ge t_0$. For this step we may assume that $p<\infty$. Theorem 3.16 shows that u has the mean-value

[9] Theorem 5.58 of *Basic*.

[10] The full-fledged version of Alaoglu's Theorem will be stated and proved in Chapter IV.

property. We know as a consequence that if B denotes the ball with center (x,t) and radius $\frac{1}{2}t_0$, then the value of u at (x,t) equals the average value over B:

$$u(x,t)=\tfrac{1}{|B|}\int_B u(y,s)\,dy\,ds.$$

Since the measure $|B|^{-1}\,dy\,ds$ on B has total mass 1, Hölder's inequality gives

$$\begin{aligned}|u(x,t)|^p&\le \tfrac{1}{|B|}\int_B |u(y,s)|^p\,dy\,ds\\&\le \tfrac{1}{|B|}\int_{|s-t|\le\frac12 t_0}\int_{y\in\mathbb{R}^N}|u(y,s)|^p\,dy\,ds\\&\le [(\tfrac12 t_0)^{N+1}\Omega_N]^{-1}(N+1)t_0\|u\|^p_{\mathcal{H}^p},\end{aligned}$$

and the boundedness is proved.

For each positive integer k, define $f_k(x)=u(x,1/k)$ and $w(x,t)=(P_t*f_k)(x)$. Then the function $w_k(x,t)-u(x,t+1/k)$ is

(i) harmonic in (x,t) for $t>0$ since w_k and any translate of u are harmonic,

(ii) bounded as a function of (x,t) for $t\ge 0$ since $u(x,t+1/k)$ is bounded for $t\ge 0$, according to the previous paragraph, and since w_k is the Poisson integral of the bounded function f_k,

(iii) continuous in (x,t) for $t\ge 0$ since $u(x,t+1/k)$ and $w_k(x,t)$ both have this property, the latter because f_k is continuous and bounded.

By Corollary 3.24, $w_k(x,t)-u(x,t+1/k)=0$. That is,

$$u(x,t+1/k)=\int_{\mathbb{R}^N}P_t(x-y)f_k(y)\,dy.$$

Now suppose $p>1$, so that L^p is the dual space to $L^{p'}$ if $p^{-1}+p'^{-1}=1$. Since u is in $\mathcal{H}^p$, $\|f_k\|_p\le M$ for the constant $M=\|u\|_{\mathcal{H}_p}$. By the preliminary form of Alaoglu's Theorem, there exists a subsequence $\{f_{k_j}\}$ of $\{f_k\}$ that is weak-star convergent to some function f in L^p. Since for each fixed t, P_t is in $L^1\cap L^\infty$ and hence is in $L^{p'}$, each (x,t) has the property that

$$u(x,t+1/k_j)=\int_{\mathbb{R}^N}P_t(x-y)f_{k_j}(y)\,dy\to\int_{\mathbb{R}^N}P_t(x-y)f(y)\,dy.$$

But $u(x,t+1/k_j)\to u(x,t)$ by continuity of u. We conclude that $u(x,t)=\int_{\mathbb{R}^N}P_t(x-y)f(y)\,dy$.

This proves the theorem for $p>1$. If $p=1$, the above argument falls short of constructing a function f in L^1 since L^1 is not the dual of L^∞. Instead, we treat f_k as a complex measure $f_k(x)\,dx$. The norm of $f_k(x)\,dx$ in $M(\mathbb{R}^N)$ equals $\|f_k\|_1$, and thus the norms of the complex measures $f_k(x)\,dx$ are bounded. The space $M(\mathbb{R}^N)$ is the dual of $C_{\text{com}}(\mathbb{R}^N)$ and hence also of its uniform closure, which is the Banach space $C_0(\mathbb{R}^N)$ of continuous functions on $\mathbb{R}^N$ vanishing at infinity. Let $\{f_{k_j}(x)\,dx\}$ be a weak-star convergent subsequence of $\{f_k(x)\,dx\}$, with limit ν in $M(\mathbb{R}^N)$. Since each function $y\mapsto P_t(x-y)$ is in $C_0(\mathbb{R}^N)$, we have $\lim_k\int_{\mathbb{R}^N}P_t(x-y)f_{k_j}(y)\,dy=\int_{\mathbb{R}^N}P_t(x-y)\,d\nu(y)$. This completes the proof. $\square$

For $N = 1$, every analytic function in the upper half plane $\mathbb{R}^2_+$ is automatically harmonic, and one can ask for a characterization of the subspace of analytic members of $\mathcal{H}^p(\mathbb{R}^2_+)$. Aspects of the corresponding theory are discussed in Problems 13–20 at the end of the chapter.

5. Calderón–Zygmund Theorem

The Calderón–Zygmund Theorem asserts the boundedness of certain kinds of important operators on $L^p(\mathbb{R}^N)$ for $1 < p < \infty$. It is an N-dimensional generalization of the theorem giving the boundedness of the Hilbert transform, which was proved in Chapters VIII and IX of *Basic*. We state and prove the Calderón–Zygmund Theorem in this section, and we give some applications to partial differential equations in the next section.

Theorem 3.26 (Calderón–Zygmund Theorem). Let $K(x)$ be a C^1 function on $\mathbb{R}^N - \{0\}$ homogeneous[11] of degree 0 with mean value 0 over the unit sphere, i.e., with

$$\int_{S^{N-1}} K(\omega)\, d\omega = 0.$$

For each $\varepsilon > 0$, define

$$T_\varepsilon f(x) = \int_{|t|\geq\varepsilon} \frac{K(t)}{|t|^N} f(x-t)\, dt$$

whenever $1 < p < \infty$ and f is in $L^p(\mathbb{R}^N)$. Then

(a) $\|T_\varepsilon f\|_p \leq A_p \|f\|_p$ for a constant A_p independent of ε and f,
(b) $\lim\limits_{\varepsilon\downarrow 0} T_\varepsilon f = Tf$ exists as an L^p limit,
(c) $\|Tf\|_p \leq A_p \|f\|_p$ for a constant A_p independent of f.

REMARKS. If $1 \leq p < \infty$ and if p' is the dual index to p, then the function equal to $K(t)/|t|^N$ for $|t| \geq \varepsilon$ and equal to 0 for $|t| < \varepsilon$ is in $L^{p'}$. Therefore, for each such p, $T_\varepsilon f$ is the convolution of an $L^{p'}$ function and an L^p function and is a well-defined bounded uniformly continuous function. In proving the theorem, we shall use less about $K(x)$ than the assumed C^1 condition on $\mathbb{R}^N - \{0\}$ but more than continuity. The precise condition that we shall use is that $|K(x) - K(y)| \leq \psi(|x-y|)$ on S^{N-1} for a nondecreasing function $\psi(\delta)$ of one variable that satisfies $\int_0^1 \frac{\psi(\delta)}{\delta}\, d\delta < \infty$.

[11] A function F of several variables is **homogeneous of degree** m if $F(rx) = r^m F(x)$ for all $r > 0$ and all $x \neq 0$.

The main steps in the proof are to show that the operator T_1 equal to T_ε for $\varepsilon = 1$ is bounded on L^2 and is of weak-type $(1, 1)$ in the sense that $\big|\{x \mid |(T_1 f)(x) > \xi\}\big| \leq C\|f\|_1/\xi$. The remainder of the argument is qualitatively similar to the argument with the Hilbert transform, not really involving any new ideas. We handle matters in the following order: First we prove as Lemma 3.27 two facts needed in the L^2 analysis, second we give the proof of the boundedness of T_1 on L^2, third we establish in Lemmas 3.28 and 3.29 a weak-type (1, 1) result for a wide class of operators, and fourth we show as a special case that T_1 is of weak-type $(1, 1)$. Finally we tend to the remaining details of the proof.

Lemma 3.27. There is a constant C such that for all $R \geq 1$, all ε with $0 < \varepsilon \leq 1$, and all nonzero real a and b,

(a) $\displaystyle \Big|\int_\varepsilon^R \frac{\sin ar\,dr}{r}\Big| \leq C,$

(b) $\displaystyle \Big|\int_\varepsilon^R \frac{(\cos ar - \cos br)\,dr}{r}\Big| \leq C\big(1 + \big|\log(|a/b|)\big|\big).$

PROOF. In (a) and (b), the signs of a and b make no difference, and we may therefore assume that $a > 0$ and $b > 0$.

In (a), the change of variables $s = ar$ converts the integral into $\int_{a\varepsilon}^{aR} \frac{\sin s\,ds}{s}$. Since $s^{-1}\sin s$ is integrable near 0, it is enough to consider $\int_0^S \frac{\sin s\,ds}{s}$. Integration by parts shows that this integral equals $\big[\frac{1-\cos s}{s}\big]_0^S - \int_0^S \frac{(\cos s - 1)\,ds}{s^2}$. The integrated term tends to a finite limit as S tends to infinity, and the integral is absolutely convergent. Hence (a) follows.

In (b), possibly by interchanging a and b, we may assume that $c = b/a$ is ≤ 1. The change of variables $s = ar$ converts the integral into $\int_{a\varepsilon}^{aR} \frac{(\cos s - \cos cs)\,ds}{s}$. Since $|1 - \cos s| \leq \frac{1}{2}s^2$ for all s, we have $|1 - \cos cs| \leq \frac{1}{2}c^2s^2 \leq \frac{1}{2}s^2$. So the integrand is $\leq s$ in absolute value everywhere and in particular is integrable for s near 0. It is therefore enough to show that $\big|\int_1^S \frac{(\cos s - \cos cs)\,ds}{s}\big| \leq C(1 + \log(c^{-1}))$. Integration by parts gives $\int_1^S \frac{\cos s\,ds}{s} = \big[\frac{\sin s}{s}\big]_1^S + \int_1^S \frac{\sin s\,ds}{s^2}$. The integrated term tends to a finite limit, and the integral is absolutely convergent. Hence the term $\int_1^S \frac{\cos s\,ds}{s}$ is bounded, and it is enough to handle $\int_1^S \frac{\cos cs\,ds}{s}$. Putting $t = cs$ changes this integral to $\int_c^{cS} \frac{\cos t\,dt}{t}$. If $cS \geq 1$, the integral from 1 to cS contributes a bounded amount, as is seen by integrating by parts, and the integral from c to 1 contributes in absolute value at most $\int_c^1 \frac{dt}{t} = \log c^{-1}$. If $cS \leq 1$, the integral from c to cS contributes in absolute value at most $\int_c^1 \frac{dt}{t} + \int_{cS}^1 \frac{dt}{t} = \log c^{-1} + \log(cS)^{-1} \leq 2\log c^{-1}$. $\square$

PROOF FOR THEOREM 3.26 THAT T_1 IS BOUNDED ON L^2. Define $k(x)$ to be

$K(x)/|x|^N$ for $|x| \geq 1$ and to be 0 for $|x| < 1$. Then k is an L^2 function, and $T_1 f = k * f$. We show that T_1 is bounded on L^2 by showing that the Fourier transform $\mathcal{F}k$ of k is an L^∞ function.

If I_n denotes the indicator function of $\{|x| \leq n\}$, then the sequence $\{kI_n\}$ converges to k in L^2. By the Plancherel formula, $\{\mathcal{F}(kI_n)\}$ converges to $\mathcal{F}k$ in L^2. Thus a subsequence converges almost everywhere. To simplify the notation, let n run through the indices of the subsequence. We have just shown that

$$(\mathcal{F}k)(x) = \lim_n \textstyle\int_{|x|\leq n} k(x) e^{-2\pi i x\cdot y}\, dx,$$

the limit existing almost everywhere. Write $x = r\omega$ and $y = r'\omega'$, where $r = |x|$ and $r' = |y|$. Then $x \cdot y = rr' \cos\gamma$, where $\gamma = \omega \cdot \omega'$, and $(\mathcal{F}k)(x)$ is the limit on n of

$$\begin{aligned}
&\textstyle\int_{S^{N-1}} \int_1^n \frac{K(\omega)}{r^N} e^{-2\pi i r r' \cos\gamma} r^{N-1}\, dr\, d\omega \\
&\qquad = \textstyle\int_{S^{N-1}} \Big[\int_1^n \frac{e^{-2\pi i r r' \cos\gamma}\, dr}{r} \Big] K(\omega)\, d\omega \\
&\qquad = \textstyle\int_{S^{N-1}} \Big[\int_1^n \frac{(e^{-2\pi i r r' \cos\gamma} - \cos 2\pi r r')\, dr}{r} \Big] K(\omega)\, d\omega \qquad \text{since } K \text{ has mean value } 0 \\
&\qquad = \textstyle\int_{S^{N-1}} \Big[\int_1^n \frac{\big(\cos(2\pi r r' \cos\gamma) - \cos 2\pi r r'\big)\, dr}{r} \Big] K(\omega)\, d\omega \\
&\qquad\quad - i \textstyle\int_{S^{N-1}} \Big[\int_1^n \frac{\sin(2\pi r r' \cos\gamma)\, dr}{r} \Big] K(\omega)\, d\omega.
\end{aligned}$$

Let us call the terms on the right side Term I and $-i$ Term II. The inner integral for Term II is bounded independently of r, r', γ, n by Lemma 3.27a. Since K is bounded, Term II is bounded.

The inner integral for Term I is bounded by $C\big(1 + \log(|\cos\gamma|^{-1})\big)$, according to Lemma 3.27b. Since K is bounded, the contribution from C by itself yields a bounded contribution to Term I and is harmless. We are left with a term that in absolute value is

$$\leq C \textstyle\int_{S^{N-1}} \log(|\cos\gamma|^{-1}) |K(\omega)|\, d\omega = C \int_{S^{N-1}} \log(|\cos(\omega\cdot\omega')|^{-1}) |K(\omega)|\, d\omega.$$

Since K is bounded, it is enough to estimate $\int_{S^{N-1}} \log(|\cos(\omega\cdot\omega')|^{-1})\, d\omega$. This integral is independent of ω'. We introduce spherical coordinates

$$\begin{aligned}
\omega_1 &= \cos\theta_1, \\
\omega_2 &= \sin\theta_1 \cos\theta_2, \\
&\ \vdots
\end{aligned}$$

and take $\omega' = (1, 0, \ldots, 0)$. The integral becomes

$$\int\limits_{\substack{0 \le \theta_j \le \pi \text{ for } j < N-1, \\ 0 \le \theta_{N-1} \le 2\pi}} \log(|\cos\theta_1|^{-1}) \ \sin^{N-2}\theta_1 \cdots \sin\theta_{N-2}\, d\theta_{N-1} \cdots d\theta_1,$$

which is a constant times $\int_0^\pi \log(|\cos\theta|^{-1}) \sin^{N-2}\theta\, d\theta$. This integral in turn is $\le \int_0^\pi \log(|\cos\theta|^{-1})\, d\theta$, whose finiteness reduces to the local integrability of $\log(|x|^{-1})$ on the line. Thus Term I is bounded, and the boundedness of $\mathcal{F}k$ follows. $\square$

Lemma 3.28 (Calderón–Zygmund decomposition). Let f be in $L^1(\mathbb{R}^N)$, and let ξ be a positive real number. Then there exists a finite or infinite disjoint sequence $\{E_n\}_{n\ge 1}$ of Borel subsets of $\mathbb{R}^N$ such that

(a) for each E_n, there exists a ball $B_n = B(r_n; x_n)$ such that the balls B_n and $B_n^* = B(5r_n; x_n)$ have $B_n \subseteq E_n \subseteq B_n^*$,

(b) $\sum_n |E_n| \le 5^N \|f\|_1 \big/ \xi$,

(c) $|f(x)| \le \xi$ almost everywhere off $\bigcup_n E_n$,

(d) $\dfrac{1}{|E_n|} \displaystyle\int_{E_n} |f(y)|\, dy \le 5^N \xi$ for each n.

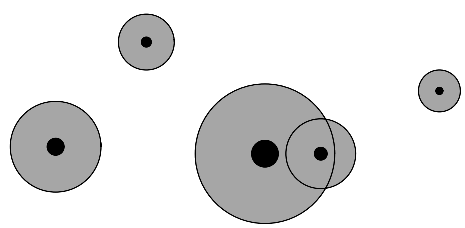

FIGURE 3.2. Calderón–Zygmund decomposition of $\mathbb{R}^N$ relative to a function at a certain height. The set where the maximal function of f exceeds ξ lies in the union of the gray balls. The gray balls have radii 5 times those of the black balls, and the black balls are disjoint. The function $|f|$ is $\le \xi$ almost everywhere off the union of the gray balls, and the sum of the volumes of the gray balls is controlled.

REMARKS. In the 1-dimensional case, this result was embedded in the proof of Theorem 8.25 of *Basic*. The sets E_n were open intervals. Extending that argument too literally to the N-dimensional case is unnecessarily complicated for current purposes. Instead, we settle for an n^{th} set that contains a ball of some radius about a point and is contained in a ball of 5 times that radius. Thus the n^{th}

set E_n consists of a black ball and part of the corresponding gray ball in Figure 3.2. The fact that E_n has not been precisely located makes the proof of weak-type (1, 1) in the present section more difficult than the proof of Theorem 8.25 of *Basic*.

PROOF. Let f^* be the Hardy–Littlewood maximal function

$$f^*(x) = \sup_{0<r<\infty} |B(r;x)|^{-1} \int_{B(r;x)} |f(y)|\,dy,$$

and let $E = \{x \mid f^*(x) > \xi\}$. If x is in E, then $|B(r;x)|^{-1} \int_{B(r;x)} |f(y)|\,dy > \xi$ for some $r > 0$. On the other hand, $\lim_{r\to\infty} |B(r;x)|^{-1} \int_{B(r;x)} |f(y)|\,dy = 0$ since f is integrable. Thus, for each x in E, there exists an $r = r_x$ depending on x such that

$$|B(r_x;x)|^{-1} \int_{B(r_x;x)} |f(y)|\,dy > \xi$$

and

$$|B(5r_x;x)|^{-1} \int_{B(5r_x;x)} |f(y)|\,dy \le \xi.$$

Since $\|f\|_1 \ge \int_{B(r_x;x)} |f(y)|\,dy > \xi|B(r_x;x)| = r_x^N \xi |B(1;0)|$, the radii r_x are bounded. Applying the Wiener Covering Lemma[12] to the cover $\{B(r_x;x) \mid x \in E\}$ of E, we obtain a finite or infinite sequence of points $x_1, x_2, \dots$ such that the balls $B(r_{x_n};x_n)$ are disjoint and

$$E \subseteq \bigcup_n B(5r_{x_n};x_n). \tag{$*$}$$

Write r_n for r_{x_n}. Put $E_1 = B(5r_1;x_1) - \bigcup_{j\ne 1} B(r_j;x_j)$, and define inductively

$$E_n = B(5r_n;x_n) - \bigcup_{j=1}^{n-1} E_j - \bigcup_{j\ne n} B(r_j;x_j).$$

By inspection

(i) the sets E_n are disjoint,
(ii) $B(r_n;x_n) \subseteq E_n \subseteq B(5r_n;x_n)$ for each n,
(iii) $\bigcup_n E_n = \bigcup_n B(5r_n;x_n)$.

Property (ii) immediately yields (a). The second inclusion of (ii) gives $\xi|E_n| \le \xi|B(5r_n;x_n)| = 5^N\xi|B(r_n;x_n)| \le 5^N \int_{B(r_n;x_n)} |f(y)|\,dy$. Summing on n and taking into account the disjointness of the sets $B(r_n;x_n)$, we obtain $\xi\sum_n |E_n| \le 5^N \int_{\bigcup_n B(r_n;x_n)} |f(y)|\,dy \le 5^N\|f\|_1$. This proves (b). The two inclusions of (ii) together yield $\int_{E_n} |f(y)|\,dy \le \int_{B(5r_n;x_n)} |f(y)|\,dy \le \xi|B(5r_n;x_n)| = 5^N\xi|B(r_n;x_n)| \le 5^N\xi|E_n|$, and this proves (d). Finally ($*$) and (iii) together show that $E \subseteq \bigcup_n E_n$. Therefore $f^*(x) \le \xi$ everywhere off $\bigcup_n E_n$. Since

$$\lim_{r\downarrow 0} |B(r;x)|^{-1} \int_{B(r;x)} |f(y)|\,dy = f(x)$$

almost everywhere on $\mathbb{R}^N$, we see that $|f(x)| \le \xi$ almost everywhere off $\bigcup_n E_n$. This proves (c). □

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[12] Lemma 6.41 of *Basic*.

Lemma 3.29. Let k be in $L^2(\mathbb{R}^N)$, and define $Tf = k * f$ for f in $L^1 + L^2$. If

(a) $\|Tf\|_2 \le A\|f\|_2$ and
(b) there exist constants B and $\alpha > 0$ such that

$$\int_{|x|\ge\alpha|y|} |k(x-y) - k(x)|\,dx \le B$$

independently of y,

then the operator T is of weak-type (1, 1) with a constant depending only on A, B, α, and N.

PROOF. We are to estimate the measure of the set of x where $|(Tf)(x)| > \xi$. Fix f and ξ, and apply Lemma 3.28 to obtain disjoint Borel sets E_n and balls $B_n = B(r_n; x_n)$ and $B_n^* = B(5r_n; x_n)$ with $B_n \subseteq E_n \subseteq B_n^*$ and with the other properties listed in the lemma. Now that the sets E_n have been determined, we decompose f into the sum $f = g + b$ of a "good" function and a "bad" function by

$$g(x) = \begin{cases} \frac{1}{|E_n|}\int_{E_n} f(y)\,dy & \text{for } x \in E_n, \\ f(x) & \text{for } x \notin \bigcup_n E_n, \end{cases}$$

$$b(x) = \begin{cases} f(x) - \frac{1}{|E_n|}\int_{E_n} f(y)\,dy & \text{for } x \in E_n, \\ 0 & \text{for } x \notin \bigcup_n E_n. \end{cases}$$

Since $\big\{x \bigm| |Tf(x)| > \xi\big\} \subseteq \big\{x \bigm| |Tg(x)| > \xi/2\big\} \cup \big\{x \bigm| |Tb(x)| > \xi/2\big\}$, it is enough to prove

(i) $\big|\big\{x \bigm| |Tg(x)| > \xi/2\big\}\big| \le C\|f\|_1\big/\xi$ and
(ii) $\big|\big\{x \bigm| |Tb(x)| > \xi/2\big\}\big| \le C\|f\|_1\big/\xi$

for some constant C independent of ξ and f.

The definition of g shows that $\int_{E_n} |g(x)|\,dx \le \int_{E_n} |f(x)|\,dx$ for all n and that $|g(x)| = |f(x)|$ for $x \notin \bigcup_n E_n$; therefore $\int_{\mathbb{R}^N} |g(x)|\,dx \le \int_{\mathbb{R}^N} |f(x)|\,dx$. Also, properties (b) and (c) of the E_n's show that $|g(x)| \le 5^N\xi$ a.e. These two inequalities, together with the bound $\|Tg\|_2 \le A\|g\|_2$, give

$$\begin{aligned}\int_{\mathbb{R}^N} |Tg(x)|^2\,dx &\le A^2 \int_{\mathbb{R}^N} |g(x)|^2\,dx \\ &\le 5^N\xi A^2 \int_{\mathbb{R}^N} |g(x)|\,dx \le 5^N\xi A^2 \int_{\mathbb{R}^N} |f(x)|\,dx.\end{aligned}$$

Combining this result with Chebyshev's inequality

$$\big|\big\{x \bigm| |F(x)| > \beta\big\}\big| \le \beta^{-2}\int_{\mathbb{R}^N} |F(x)|^2\,dx$$

for the function $F = Tg$ and the number $\beta = \xi/2$, we obtain

$$\big|\{x \mid |Tg(x)| > \xi/2\}\big| \le \frac{4}{\xi^2} 5^N \xi A^2 \int_{\mathbb{R}^N} |f(x)|\,dx = \frac{4 \cdot 5^N A^2 \|f\|_1}{\xi}.$$

This proves (i).

For the function b, let b_n be the product of b with the indicator function of E_n. Then we have $b = \sum_n b_n$ with the sum convergent in L^1. Inspection of the definition shows that $\|b_n\|_1 \le 2\int_{E_n} |f(y)|\,dy$, and therefore $\|b\|_1 \le 2\|f\|_1$. Since T is convolution by the L^2 function k and since $b = \sum_n b_n$ in L^1, $Tb = \sum_n Tb_n$ with the sum convergent in L^2. A subsequence of partial sums therefore converges almost everywhere. Inserting absolute values consistently with the subsequence and then inserting absolute values around each term, we see that

$$|Tb(x)| \le \textstyle\sum_n |Tb_n(x)| \qquad \text{a.e.}$$

Let α be the constant in hypothesis (b). The measure of $\bigcup_n B(5\alpha r_n; x_n)$ is

$$\begin{aligned}\big|\textstyle\bigcup_n B(5\alpha r_n; x_n)\big| &\le \textstyle\sum_n |B(5\alpha r_n; x_n)| = \sum_n 5^N \alpha^N |B(r_n; x_n)| \\ &\le 5^N \alpha^N \textstyle\sum_n |E_n| \le 5^{2N}\alpha^N \|f\|_1 \big/ \xi.\end{aligned}$$

Let $X = \mathbb{R}^N - \bigcup_n B(5\alpha r_n; x_n)$. If we show that $\int_X |Tb(x)|\,dx \le C'\|f\|_1$, then we will have

$$\big|\{x \mid |Tb(x)| > \xi/2\}\big| \le (5^{2N}\alpha^N + 2C')\|f\|_1 \big/ \xi, \qquad (*)$$

and (ii) will be proved. Put $\tau_n(X) = \{x - x_n \mid x \in X\}$. Since $\int_{E_n} b(y)\,dy = 0$ for each n,

$$\begin{aligned}\textstyle\int_X |Tb(x)|\,dx &\le \textstyle\sum_n \int_X |Tb_n(x)|\,dx \\ &= \textstyle\sum_n \int_X \big|\int_{E_n} k(x-y)b(y)\,dy\big|\,dx \\ &= \textstyle\sum_n \int_X \big|\int_{E_n} [k(x-y) - k(x-x_n)]b(y)\,dy\big|\,dx \\ &\le \textstyle\sum_n \int_X \int_{E_n} |k(x-y) - k(x-x_n)||b(y)|\,dy\,dx \\ &\overset{x-x_n\to x}{=} \textstyle\sum_n \int_{E_n} \big[\int_{\tau_n(X)} |k(x+x_n-y) - k(x)|\,dx\big]|b(y)|\,dy \\ &\le \textstyle\sum_n \int_{E_n} \big[\int_{B(5\alpha r_n;0)^c} |k(x+x_n-y) - k(x)|\,dx\big]|b(y)|\,dy.\end{aligned}$$

In the n^{th} term on the right side, y is in $E_n \subseteq B_n^*$, and hence $|x_n - y| \le 5r_n$; meanwhile, $|x| \ge 5\alpha r_n$. Therefore $|x| \ge 5\alpha r_n \ge \alpha|x_n - y|$. The right side in the display is not decreased by increasing the region of integration in the x variable, and hence the right side is

$$\begin{aligned}&\le \textstyle\sum_n \int_{E_n} \big[\int_{|x|\ge\alpha|x_n-y|} |k(x+x_n-y) - k(x)|\,dx\big]|b(y)|\,dy \\ &\le \textstyle\sum_n \int_{E_n} B|b(y)|\,dy = B\|b\|_1 \le 2B\|f\|_1.\end{aligned}$$

Therefore $(*)$ is proved with $C' = 2B$, and the proof of (ii) is complete. $\square$

PROOF FOR THEOREM 3.26 THAT T_1 IS OF WEAK-TYPE $(1, 1)$. With $k(x)$ taken to be $K(x)/|x|^N$ for $|x| \geq 1$ and to be 0 for $|x| < 1$, Lemma 3.29 shows that it is enough to prove that

$$\int_{|x|\geq 2|y|} |k(x-y) - k(x)|\,dx \leq B \tag{$*$}$$

with B independent of y. The function k is bounded, and thus the contribution to the integral in $(*)$ from the bounded set of x's where $|x| < 1$ is bounded independently of y. The set of x's where $|x - y| < 1$ is a ball whose measure is bounded as a function of y, and thus this set too contributes a bounded term to the integral in $(*)$. It is therefore enough to prove that

$$\int\limits_{\substack{|x|\geq 2|y|,\\ |x-y|\geq 1,\ |x|\geq 1}} \Big|\frac{K(x-y)}{|x-y|^N} - \frac{K(x)}{|x|^N}\Big|\,dx$$

is bounded as a function of y. If M is an upper bound for $|K|$, then this expression is

$$\begin{aligned} &\leq \textstyle\int |K(x-y)|\Big|\frac{1}{|x-y|^N} - \frac{1}{|x|^N}\Big|\,dx + \int \frac{|K(x-y)-K(x)|}{|x|^N}\,dx \\ &\leq M \int\limits_{\substack{|x|\geq 2|y|,\\ |x|\geq 1}} \Big|\frac{1}{|x-y|^N} - \frac{1}{|x|^N}\Big|\,dx + \int\limits_{\substack{|x|\geq 2|y|,\\ |x|\geq 1}} \frac{|K(x-y)-K(x)|}{|x|^N}\,dx. \end{aligned} \tag{$**$}$$

We use the two estimates

$$|x-y| \leq |x| + |y| \leq |x| + \tfrac{1}{2}|x| = \tfrac{3}{2}|x|$$

and

$$|x-y| \geq |x| - |y| = (\tfrac{1}{2}|x| - |y|) + \tfrac{1}{2}|x| \geq \tfrac{1}{2}|x|.$$

The integrand in the first term of $(**)$ is equal to

$$\begin{aligned} &\Big|\tfrac{1}{|x-y|^N} - \tfrac{1}{|x|^N}\Big| = \Big|\tfrac{|x|^N - |x-y|^N}{|x|^N|x-y|^N}\Big| \leq 2^N \Big|\tfrac{|x|^N - |x-y|^N}{|x|^{2N}}\Big| \\ &\quad\leq 2^N\, \tfrac{\big||x|-|x-y|\big|(|x|^{N-1}+|x|^{N-2}|x-y|+\cdots+|x-y|^{N-1})}{|x|^{2N}} \\ &\quad\leq 2^N\, \tfrac{|y|(|x|^{N-1}+|x|^{N-2}|x-y|+\cdots+|x-y|^{N-1})}{|x|^{2N}} \leq 2^N(\tfrac{3}{2})^N\, \tfrac{|y|(|x|^{N-1}+|x|^{N-1}+\cdots+|x|^{N-1})}{|x|^{2N}} \\ &\quad= N3^N \tfrac{|y|}{|x|^{N+1}}. \end{aligned}$$

Thus the integral in the first term of $(**)$ is

$$\begin{aligned} &\leq N3^N \textstyle\int_{|x|\geq \max\{1,2|y|\}} \frac{|y|}{|x|^{N+1}}\,dx = N3^N\Omega_{N-1} \int_{\max\{1,2|y|\}}^\infty \frac{|y|}{r^{N+1}}\, r^{N-1}\,dr \\ &= N3^N\Omega_{N-1} \tfrac{|y|}{\max\{1,2|y|\}} \leq \tfrac{1}{2}N3^N\Omega_{N-1}, \end{aligned}$$

and this is bounded independently of y.

For the second term of $(**)$, we start from the estimate

$$\left|\tfrac{z}{|z|}-\tfrac{w}{|w|}\right| \le \tfrac{|z-w|}{\min\{|z|,|w|\}}. \tag{$\dagger$}$$

To verify $(\dagger)$, we may assume that $|z| \ge |w|$. Then $\frac{|z|}{|w|}+1 \ge \frac{2z\cdot w}{|z||w|}$ because the left side is ≥ 2 and the right side is ≤ 2. Multiplying by $\frac{|z|}{|w|}-1$, we obtain $\frac{|z|^2}{|w|^2}-1 \ge \frac{2z\cdot w}{|w|^2}-\frac{2z\cdot w}{|z||w|}$. Hence $1-\frac{2z\cdot w}{|z||w|}+1 \le \frac{|z|^2}{|w|^2}-\frac{2z\cdot w}{|w|^2}+1$, which is the square of $(\dagger)$.

Using $(\dagger)$ and the definition and monotonicity of the function ψ that is defined in the remarks with the theorem and that captures the smoothness of K, we have

$$|K(x-y)-K(x)| = \left|K\big(\tfrac{x-y}{|x-y|}\big)-K\big(\tfrac{x}{|x|}\big)\right| \le \psi\Big(\left|\tfrac{x-y}{|x-y|}-\tfrac{x}{|x|}\right|\Big) \le \psi\Big(\tfrac{|y|}{\min\{|x-y|,|x|\}}\Big).$$

Since $|x-y| \ge \frac{1}{2}|x|$, $\min\{|x-y|,|x|\} \ge \frac{1}{2}|x|$. Thus $\psi\big(\frac{|y|}{\min\{|x-y|,|x|\}}\big) \le \psi\big(\frac{2|y|}{|x|}\big)$, and the computation

$$\begin{aligned}
\int_{\substack{|x|\ge 2|y|,\\ |x|\ge 1}} \tfrac{|K(x-y)-K(x)|}{|x|^N}\,dx &\le \int_{\substack{|x|\ge 2|y|,\\ |x|\ge 1}} \tfrac{\psi(2|y|/|x|)}{|x|^N}\,dx = \int_{\substack{|z|\ge 1,\\ |z|\ge 1/2|y|}} \tfrac{\psi(1/|z|)}{|z|^N}\,dz \\
&= \Omega_{N-1}\int_{\max\{1,1/2|y|\}}^{\infty} \psi(1/r)r^{-1}\,dr \\
&= \Omega_{N-1}\int_0^{\min\{1,2|y|\}} \psi(\delta)\delta^{-1}\,d\delta \\
&\le \Omega_{N-1}\int_0^1 \psi(\delta)\delta^{-1}\,d\delta
\end{aligned}$$

shows that the second term of $(**)$ is bounded independently of y. □

PROOF OF REMAINDER OF THEOREM 3.26. We can now argue in the same way that the Hilbert transform was handled in Chapter IX of *Basic*. Since T_1 has been shown to be bounded on L^2 and to be of weak-type $(1,1)$, the Marcinkiewicz Interpolation Theorem given in Theorem 9.20 of *Basic* shows that $\|T_1 f\|_p \le A_p\|f\|_p$ for $1 < p \le 2$ with A_p independent of f. Lemma 9.22 of *Basic* extends this conclusion to $1 < p < \infty$. The argument that proves Theorem 9.23a in *Basic* applies here and shows that $\|T_\varepsilon f\|_p \le A_p\|f\|_p$ for $1 < p < \infty$ with A_p independent of f and ε. This proves Theorem 3.26a.

The same argument as in Lemma 9.24 of *Basic* shows that if f is a C^1 function of compact support on $\mathbb{R}^N$, then

$$\lim_{\varepsilon\downarrow 0}\int_{|y|\ge\varepsilon} \tfrac{K(y)f(x-y)\,dy}{|y|^N}$$

exists uniformly and in L^p for every $p > 1$. This proves (b) of Theorem 3.26 for the dense set of C^1 functions f of compact support.

To prove the norm convergence when we are given a general f in L^p with $1 < p < \infty$, we choose a sequence f_n in the dense set with $f_n \to f$ in L^p. Then

$$\begin{aligned}\|T_\varepsilon f - T_{\varepsilon'} f\|_p &\le \|T_\varepsilon(f - f_n)\|_p + \|T_\varepsilon f_n - T_{\varepsilon'} f_n\|_p + \|T_{\varepsilon'}(f_n - f)\|_p \\ &\le A_p\|f_n - f\|_p + \|T_\varepsilon f_n - T_{\varepsilon'} f_n\|_p + A_p\|f_n - f\|_p.\end{aligned}$$

Choose n to make the first and third terms small on the right, and then choose ε and ε' sufficiently close to 0 so that the second term on the right is small. The result is that $T_{\varepsilon_n} f$ is Cauchy in L^p along any sequence $\{\varepsilon_n\}$ tending to 0. This proves Theorem 3.26b.

For any f in L^p with $1 < p < \infty$, we have just seen that $T_\varepsilon f \to Tf$ in L^p. Then (a) gives $\|Tf\|_p = \lim_{\varepsilon\downarrow 0} \|T_\varepsilon f\|_p \le \limsup_{\varepsilon\downarrow 0} A_p\|f\|_p = A_p\|f\|_p$. This proves Theorem 3.26c. □

6. Applications of the Calderón–Zygmund Theorem

EXAMPLE 1. Riesz transforms. These are a more immediate N-dimensional analog of the Hilbert transform than is the operator in the Calderón–Zygmund Theorem. In $\mathbb{R}^1$, the Poisson kernel and conjugate Poisson kernel are given by

$$P(x, y) = P_y(x) = \frac{1}{\pi}\frac{y}{x^2 + y^2} \qquad \text{and} \qquad Q(x, y) = Q_y(x) = \frac{1}{\pi}\frac{x}{x^2 + y^2}.$$

The conjugate Poisson kernel Q may be obtained starting from the Poisson kernel P by applying the Cauchy–Riemann equations in the form

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} \qquad \text{and} \qquad \frac{\partial Q}{\partial x} = -\frac{\partial P}{\partial y}$$

and by requiring that Q vanish at infinity. The differential equations lead to the solution

$$Q(x, y) = \int_\infty^{(x,y)} \frac{\partial P}{\partial x}\, dy.$$

The Hilbert transform kernel may be obtained by letting y decrease to 0 in $Q(x, y)$. The resulting formal convolution formula

$$Hf(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x - t)}{t}\, dt$$

is to be interpreted in such a way as to represent passage from the boundary values of $P_y * f$ to the boundary values of $Q_y * f$. We know that a valid way of arriving at this interpretation is to take the integral for $|t| \ge \varepsilon$ and let ε decrease to 0.

In N dimensions the Poisson kernel for $\mathbb{R}^{N+1}_+$ is

$$P(x,t) = P_t(x) = \frac{c_N t}{(|x|^2+t^2)^{\frac{1}{2}(N+1)}}, \qquad x \in \mathbb{R}^N,\ t > 0,$$

with $c_N = \pi^{-\frac{1}{2}(N+1)}\Gamma\big(\frac{N+1}{2}\big)$. If we write x_{N+1} in place of t, the natural extension of the Cauchy–Riemann equations is the system for the $(N+1)$-component function $u = (u_1, \dots, u_{N+1})$ given by

$$\operatorname{div} u = 0 \qquad \text{and} \qquad \operatorname{curl} u = 0,$$

i.e.,

$$\sum_{i=1}^{N+1} \frac{\partial u_i}{\partial x_i} = 0 \qquad \text{and} \qquad \frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i} \text{ when } i \neq j.$$

A solution is $(Q_1, \dots, Q_N, P)$, where

$$Q_j(x,t) = \frac{c_N x_j}{(|x|^2+t^2)^{\frac{1}{2}(N+1)}}, \qquad x \in \mathbb{R}^N,\ t > 0.$$

Imitating the procedure summarized above for the Hilbert transform, we let t decrease to 0 here and arrive at the kernel

$$\frac{c_N x_j}{|x|^{N+1}}.$$

Accordingly, we define the j^{th} **Riesz transform** for $1 \le j \le N$ by

$$R_j f(x) = c_N \lim_{\varepsilon \downarrow 0} \int_{|y| \ge \varepsilon} \frac{y_j}{|y|^{N+1}} f(x-y)\, dy.$$

The Calderón–Zygmund Theorem (Theorem 3.26) shows that R_j is a bounded operator on $L^p(\mathbb{R}^N)$ for $1 < p < \infty$. The multiplier on the Fourier transform side can be obtained routinely from the formula for the Fourier transform of $P_t(x)$, namely $\widehat{P_t}(y) = e^{-2\pi t|y|}$, by using the differential equations and letting t decrease to 0. The result is

$$\widehat{R_j f}(y) = -\frac{i x_j}{|x|}\, \widehat{f}(y).$$

A sample application of the Riesz transforms is to an inequality asserting that the Laplacian controls all mixed second derivatives for smooth functions of compact support:

$$\Big\| \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \varphi \Big\|_p \le A_p \|\Delta\varphi\|_p \qquad \text{for } 1 < p < \infty \text{ and } \varphi \in C^\infty_{\text{com}}(\mathbb{R}^N).$$

The argument works as well for all Schwartz functions φ: the partial derivatives satisfy the identity $\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\varphi = -R_j R_k \Delta\varphi$ because the equality

$$-4\pi^2 y_j y_k\, \widehat{\varphi}(y) = -\Big(-\frac{i y_j}{|y|}\Big)\Big(-\frac{i y_k}{|y|}\Big)(-4\pi^2|y|^2)\, \widehat{\varphi}(y)$$

shows that the Fourier transforms are equal.

EXAMPLE 2. Beltrami equation. This will be an application in which the L^p theory of the Calderón–Zygmund Theorem is essential for some $p \neq 2$. We deal with functions on $\mathbb{R}^2$. Define

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\Big) \qquad \text{and} \qquad \frac{\partial}{\partial \bar z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\Big).$$

We shall use the abbreviations $f_z = \frac{\partial f}{\partial z}$ and $f_{\bar z} = \frac{\partial f}{\partial \bar z}$. The Cauchy–Riemann equations, testing whether a complex-valued function on $\mathbb{R}^2$ is analytic, become the single equation $f_{\bar z} = 0$.

We shall use weak derivatives on $\mathbb{R}^2$ in the sense of Section 2. Let μ be in $L^\infty(\mathbb{R}^2)$ with $\|\mu\|_\infty = k < 1$. In the sense of weak derivatives, the **Beltrami equation** is

$$f_{\bar z} = \mu f_z.$$

This equation is fundamental in dealing with Riemann surfaces, since solutions to it provide "quasiconformal mappings" with certain properties. For simplicity we assume that μ has compact support. We seek a solution f such that $f(0) = 0$ and $f_z - 1$ is in some L^p class.

The equation is solved by first putting it in another form. Let

$$Ph(\zeta) = -\frac{1}{\pi}\int_{\mathbb{R}^2}\Big(\frac{1}{z-\zeta} - \frac{1}{z}\Big)h(z)\,dx\,dy.$$

The factor in parentheses is in $L^q(\mathbb{R}^2)$ for $1 \le q < 2$, and Hölder's inequality shows that Ph is therefore well defined for h in $L^p(\mathbb{R}^2)$ if $p > 2$. In fact, one can show that $|Ph(\zeta_1) - Ph(\zeta_2)| \le C\|h\|_p|\zeta_1 - \zeta_2|^{1-\frac{2}{p}}$, and therefore Ph is continuous for such h. Observe that $Ph(0) = 0$ for all h. Also, one can show that

$$(Ph)_{\bar z} = h \qquad \text{in the sense of weak derivatives.} \tag{$*$}$$

However, the definition of P falls apart for $p = 2$. Now define

$$Th(\zeta) = \lim_{\varepsilon \downarrow 0} -\frac{1}{\pi}\int_{|z-\zeta|\ge\varepsilon}\frac{h(z)}{(z-\zeta)^2}\,dx\,dy.$$

The operator T is bounded on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$ by the Calderón–Zygmund Theorem, and we shall be interested in h as above, thus interested in $p > 2$. One can show that

$$(Ph)_z = Th \qquad \text{in the sense of weak derivatives if } h \in L^p \text{ with } p > 2. \tag{$**$}$$

Now we can transform the Beltrami equation. Suppose that f is a weak solution of the Beltrami equation with $f(0) = 0$ and $f_z - 1$ in L^p for some p with $p > 2$.

Since μ is in L^∞, $\mu f_z - \mu$ is in L^p, and since μ has compact support, μf_z is in L^p. Then $f_{\bar z} = \mu f_z$ is in L^p, and $P(f_{\bar z})$ is defined. The function $f - P(f_{\bar z})$ is analytic because $(*)$ shows that $\frac{\partial}{\partial \bar z}(f - P(f_{\bar z})) = f_{\bar z} - f_{\bar z} = 0$. One can easily show that this analytic function has to be z, i.e., that

$$f = P(f_{\bar z}) + z.$$

Differentiating with respect to z and using $(**)$, we obtain $f_z = T(f_{\bar z}) + 1 = T(\mu f_z) + 1$. The equation

$$f_z = T(\mu f_z) + 1 \qquad (\dagger)$$

is the transformed equation.

Assuming that f is a solution of the Beltrami equation and therefore of $(\dagger)$, we shall manipulate $(\dagger)$ a little and arrive at a formula for f. Multiply $(\dagger)$ by μ and apply T to get $T(\mu f_z) = T\mu T\mu f_z + T\mu$. Adding 1 and substituting from $(\dagger)$ gives

$$f_z = T\mu T\mu f_z + T\mu + 1.$$

Iteration of this procedure yields

$$f_z = (T\mu)^n f_z + [1 + T\mu + \cdots + (T\mu)^{n-1}].$$

We want to arrange that the first term on the right side tends to 0 in the limit on n. The operations of P and T have together made sense only on L^p for $p > 2$. The linear operator $g \mapsto \mu g$ on L^p has norm $\|\mu\|_\infty = k < 1$, and T has norm A_p, say. It can be shown that T is unitary on L^2, so that $A_2 = 1$. The Marcinkiewicz Interpolation Theorem does not reveal good limiting behavior for the bounds of operators at the endpoints of an interval of p's where it is applied, but the Riesz Convexity Theorem[13] does. Consequently we can conclude that $\limsup_{p\downarrow 2} A_p = 1$. Therefore the operator $g \mapsto T\mu g$, with norm $\leq kA_p$ on L^p for $p > 2$, has norm < 1 if p is sufficiently close to 2 (but is greater than 2). Fix such a p. Then we have

$$\|(T\mu)^n f_z\|_p \leq \|T\mu\|^{n-1}\|T\mu f_z\|_p \longrightarrow 0,$$

and

$$f_z = \lim_n [1 + T\mu + \cdots + (T\mu)^{n-1}].$$

The function $f_z - 1 = \lim_n [T\mu + \cdots + (T\mu)^{n-1}]$ is certainly in L^p. As a solution of the Beltrami equation, f has $f_{\bar z} = \mu f_z = \mu + \mu \lim_n [T\mu + \cdots + (T\mu)^{n-1}]$.

[13]The Riesz Convexity Theorem uses complex analysis. It was stated in Chapter IX of *Basic*, but the proof was omitted.

We saw above that any solution f of the Beltrami equation with $f(0)$ and with $f_z - 1$ in L^p has to satisfy $f = P(f_{\bar z}) + z$. Thus our formula for f is

$$f = P\big(\mu + \mu \lim_n [T\mu + \cdots + (T\mu)^{n-1}]\big) + z.$$

Finally we can turn things around and check that this process actually gives a solution. Define $g = \mu + \mu \lim_n [T\mu + \cdots + (T\mu)^{n-1}]$ in L^p, and put $f = Pg + z$. Application of $(*)$ and $(**)$ gives $f_{\bar z} = g$ and $f_z = Tg + 1$. Substitution of the formula for g into these yields

$$\begin{aligned} f_{\bar z} &= \mu + \mu \lim_n [T\mu + \cdots + (T\mu)^{n-1}] = \mu(1 + \lim_n [T\mu + \cdots + (T\mu)^{n-1}]) \\ &= \mu(1 + T(\lim_n \mu + \mu T\mu + \cdots + \mu(T\mu)^{n-2}])) = \mu(1 + Tg) = \mu f_z, \end{aligned}$$

as required. The equality $f_z = Tg + 1$ shows that $f_z - 1$ is in L^p, and the fact that $Ph(0) = 0$ for all h shows that $f(0) = (Pg + z)(0) = 0$.

7. Multiple Fourier Series

Fourier series in several variables are a handy tool for local problems with linear differential equations. One isolates a problem in a bounded subset of $\mathbb{R}^N$ and then reproduces it periodically in each variable, using a large period. Multiple Fourier series for potentially rough functions is a complicated subject, but we have no need for it. What is required is information about Fourier series of smooth functions. The relevant theory is presented in this section, using 2π for the period in each variable, and a relatively simple application is given in the next section. A more decisive application appears in Chapter VII, where we establish local solvability of linear partial differential equations with constant coefficients.

If f is a locally integrable function on $\mathbb{R}^N$ that is periodic of period 2π in each variable, its **multiple Fourier series** is given by

$$f(x) \sim \sum_k c_k e^{ik\cdot x},$$

the sum being over all integer N-tuples and the coefficients c_k being given by

$$c_k = (2\pi)^{-N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(x) e^{-ik\cdot x}\, dx.$$

Let us write $\mathbb{Z}^N$ for the set of all integer N-tuples and $[-\pi, \pi]^N$ for the region of integration. Such series have the following properties.

Proposition 3.30. If f is a locally integrable function on $\mathbb{R}^N$ that is periodic of period 2π in each variable, then

(a) $|c_k| \le \|f\|_1$ relative to $L^1([-\pi,\pi]^N, (2\pi)^{-N}\,dx)$,
(b) $|c_k| \le C_M|k|^{-M}$ for every positive integer M if f is smooth,
(c) $\sum_{k\in\mathbb{Z}^N} c_k e^{ik\cdot x}$ is smooth and periodic if $|c_k| \le C_M|k|^{-M}$ for every positive integer M,
(d) $\{e^{ik\cdot x}\}_{k\in\mathbb{Z}^N}$ is an orthonormal basis of $L^2([-\pi,\pi]^N, (2\pi)^{-N}\,dx)$,
(e) $f(x) = \sum_{k\in\mathbb{Z}^N} c_k e^{ik\cdot x}$ if f is smooth.

PROOF. Conclusion (a) is evident by inspection of the definition. For (b), integration by parts shows that any C^1 periodic function f has the property that

$$(ik_j)\int_{[-\pi,\pi]^N} f(x)e^{-ik\cdot x}\,dx = \int_{[-\pi,\pi]^N} D_j f(x)e^{-ik\cdot x}\,dx.$$

Apart from the factor of $(2\pi)^{-N}$, the right side is a Fourier coefficient, and its size is controlled by (a). Iterating this formula, we see, in the case that f is smooth, that the Fourier coefficients c_k of f have the property that $\{P(k)c_k\}_{k\in\mathbb{Z}^N}$ is bounded for every polynomial P. Then (b) follows.

Conclusion (c) is immediate from the standard theorem about interchanging sums and derivatives. The result (d) is known in the 1-dimensional case, and the N-dimensional case then follows from Proposition 12.9 of *Basic*. In (e), the series converges to f in L^2 as a consequence of (d), and hence a subsequence converges almost everywhere to f. On the other hand, the series converges uniformly to something smooth by (c). The smooth limit must be almost everywhere equal to f, and it must equal f since f is smooth. $\square$

8. Application to Traces of Integral Operators

We return to the topic of traces of linear operators on Hilbert spaces, which was introduced in Section II.5. That section defined trace-class operators as a subset of the compact operators, and the trace of such an operator L is then given by $\sum_i (Lu_i, u_i)$, where $\{u_i\}$ is an orthonormal basis. The defining condition for trace class was hard to check, but Proposition 2.9 gave a sufficient condition: if $L : V \to V$ is bounded and if $\sum_{i,j} |(Lu_i, v_j)| < \infty$ for some orthonormal bases $\{u_i\}$ and $\{v_j\}$, then L is of trace class.

In this section we use multiple Fourier series to show how traces can be computed for simple integral operators in a Euclidean setting. The setting for realistic applications is to be a compact smooth manifold. Such manifolds are introduced in Chapter VIII, and the present result is to be regarded as the main step toward a theorem about traces of integral operators on smooth manifolds.[14]

[14]Traces of integral operators play a role in the representation theory of noncompact locally com-

Proposition 3.31. Let $K(\cdot\,,\,\cdot)$ be a complex-valued smooth function on $\mathbb{R}^N \times \mathbb{R}^N$ that is periodic of period 2π in each of the $2N$ variables, and suppose that the subset of $[-\pi,\pi]^N \times [-\pi,\pi]^N$ where K is nonzero is contained in $[-\frac{\pi}{8},\frac{\pi}{8}]^N \times [-\frac{\pi}{8},\frac{\pi}{8}]^N$. Define a bounded linear operator L on the Hilbert space $L^2([-\pi,\pi]^N, (2\pi)^{-N}\,dx)$ by

$$Lf(x) = \frac{1}{(2\pi)^N}\int_{[-\pi,\pi]^N} K(x,y)f(y)\,dy.$$

Then L is of trace class, and its trace is given by

$$\operatorname{Tr} L = \frac{1}{(2\pi)^N}\int_{[-\pi,\pi]^N} K(x,x)\,dx.$$

PROOF. For each k in $\mathbb{Z}^N$, the effect of L on the function $x \mapsto e^{ik\cdot x}$ is

$$L(e^{ik\cdot(\cdot)})(x) = \frac{1}{(2\pi)^N}\int_{[-\pi,\pi]^N} K(x,y)e^{ik\cdot y}\,dy.$$

Taking the inner product in $L^2([-\pi,\pi]^N, (2\pi)^{-N}\,dx)$ with $x \mapsto e^{il\cdot x}$ gives

$$(L(e^{ik\cdot(\cdot)}), e^{il\cdot(\cdot)}) = \frac{1}{(2\pi)^{2N}}\iint_{[-\pi,\pi]^{2N}} K(x,y)e^{ik\cdot y}e^{-il\cdot x}\,dy\,dx. \qquad (*)$$

The right side is a multiple-Fourier-series coefficient of the function K, and it is estimated by Proposition 3.30b. Proposition 3.30c shows that the corresponding trigonometric series converges absolutely. The functions $e^{ik\cdot x}$ are an orthonormal basis of $L^2([-\pi,\pi]^N, (2\pi)^{-N}\,dx)$ as a consequence of Proposition 3.30d, and therefore the sufficient condition of Proposition 2.9 is met for L to be of trace class.

To compute the trace, we start from $(*)$ with $k = l$. We change variables, letting $u = y - x$ and $v = y + x$, and the right side of $(*)$ becomes

$$\frac{1}{(2\pi)^{2N}}\iint_{[-\pi,\pi]^{2N}} 2^{-N}K\big(\tfrac{1}{2}(v-u), \tfrac{1}{2}(v+u)\big)e^{ik\cdot u}\,du\,dv$$

because of the small support of K. We sum on k in $\mathbb{Z}^N$, moving the sum under the integration with respect to v and recognizing the sum inside as the sum of the multiple-Fourier-series coefficients in the u variable, i.e., the sum

pact groups and in index theory. Both these topics are beyond the scope of this book. Consequently Chapter VIII does not carry out the easy argument to extend the Euclidean result to compact smooth manifolds.

of the series at the origin. Since the functions $e^{ik\cdot u}$ are an orthonormal basis of $L^2([-\pi,\pi]^N, (2\pi)^{-N}\,dx)$, the sum of the uniformly convergent multiple Fourier series has to be the function itself. Thus we find that

$$\operatorname{Tr} L = \frac{1}{(4\pi)^N}\int_{[-\pi,\pi]^N} K\big(\tfrac{1}{2}v, \tfrac{1}{2}v\big)\,dv.$$

Replacing $\frac{1}{2}v$ by v and again taking into account the small support of K, we obtain the formula asserted. $\square$

9. Problems

1. Check that $(1+4\pi^2|y|^2)^{-1}g$ is in the Schwartz space $\mathcal{S}$ if g is in $\mathcal{S}$, so that $(1-\Delta)u = f$ is solvable in $\mathcal{S}$ if f is in $\mathcal{S}$.
2. Show that the Schwartz space $\mathcal{S}$ is closed under pointwise product and convolution, and show that these operations are continuous from $\mathcal{S}\times\mathcal{S}$ into $\mathcal{S}$.
3. If Ω is the open disk in $\mathbb{R}^2$ with $x^2+y^2<\frac{1}{2}$, prove the following:
   (a) The function $(x,y)\mapsto \log\big((x^2+y^2)^{-1}\big)$ is in $L_1^p(\Omega)$ for $1\le p<2$ but is not in $L_1^2(\Omega)$.
   (b) The unbounded function $(x,y)\mapsto \log\log\big((x^2+y^2)^{-1}\big)$ is in $L_1^2(\Omega)$.
4. Let Ω be a nonempty bounded open set in $\mathbb{R}^n$, and suppose that there exists a real-valued C^1 function h on $\mathbb{R}^n$ such that h is positive on Ω, h is negative on $(\Omega^{\mathrm{cl}})^c$, and the first partial derivatives of h do not simultaneously vanish at any point of the boundary $\Omega^{\mathrm{cl}}-\Omega$. Prove that Ω satisfies the cone condition of Section 2.

Problems 5–7 compute explicitly the Fourier transforms of the members of a family of tempered distributions.

5. Show that the function $|x|^{-(N-\alpha)}$ on $\mathbb{R}^N$ is a tempered distribution if $0<\alpha<N$. For what values of α is it the sum of an L^1 function and an L^2 function?
6. Verify the identity $\int_0^\infty t^{\beta-1}e^{-\pi|x|^2t}dt = \int_0^\infty t^{-\beta-1}e^{-\pi|x|^2/t}\,dt = \Gamma(\beta)(\pi|x|^2)^{-\beta}$.
7. Let φ be in $\mathcal{S}(\mathbb{R}^N)$. Taking the formula $\mathcal{F}(e^{-\pi t|x|^2}) = t^{-N/2}e^{-\pi|x|^2/t}$ as known and applying the multiplication formula, obtain the identity

   $$\int_{\mathbb{R}^N} e^{-\pi t|x|^2}\widehat{\varphi}(x)\,dx = t^{-N/2}\int_{\mathbb{R}^N} e^{-\pi|x|^2/t}\varphi(x)\,dx.$$

   Multiply both sides by $t^{\frac{1}{2}(N-\alpha)-1}$ and integrate in t. Dropping dx from the notation for tempered distributions that are given by functions, conclude from the resulting formula that

$$\mathcal{F}(|x|^{-\alpha}) = \frac{\pi^{-\frac{1}{2}N+\alpha}\Gamma(\frac{1}{2}(N-\alpha))}{\Gamma(\frac{1}{2}\alpha)}\,|x|^{-(N-\alpha)}$$

as tempered distributions if $0 < \alpha < N$.

Problems 8–12 introduce a family $H^s = H^s(\mathbb{R}^N)$ of Hilbert spaces for s real that are known as spaces of **Bessel potentials**. Because of Problem 8 below, these spaces are sometimes called "Sobolev spaces." The space H^s consists of all tempered distributions $T \in \mathcal{S}'(\mathbb{R}^N)$ whose Fourier transforms $\mathcal{F}(T)$ are locally square integrable functions such that $\int_{\mathbb{R}^N} |\mathcal{F}(T)|^2(1+|\xi|^2)^s\,d\xi$ is finite, the norm $\|T\|_{H^s}$ being the square root of this expression. The spaces H^s get larger as s decreases.

8. Let $s \geq 0$ be an integer, and let T be a tempered distribution.
   (a) Prove that if T is in H^s, then all distributions $D^\alpha T$ with $|\alpha| \leq s$ are L^2 functions. In this situation, if T is the L^2 function f, conclude that f is in $L^2_s(\mathbb{R}^N)$.
   (b) Prove conversely that if $D^\alpha T$ is given by an L^2 function whenever $|\alpha| \leq s$, then T is in H^s.
   (c) As a consequence of (a) and (b), H^s can be identified with $L^2_s(\mathbb{R}^N)$ if $s \geq 0$ is an integer. Prove that the respective norms are bounded above and below by constant multiples of each other.
9. (a) Prove for each s that the operator $A_s(T) = \mathcal{F}^{-1}\big((1+|\xi|^2)^{s/2}\mathcal{F}(T)\big)$ is a linear isometry of H^s onto $H^0 \cong L^2$, and conclude that the inner-product space H^s is a Hilbert space.
   (b) Prove that A_s^{-1} carries the subspace $\mathcal{S}(\mathbb{R}^N)$ of Schwartz functions, i.e., tempered distributions of the form T_φ with $\varphi \in \mathcal{S}(\mathbb{R}^N)$, onto itself.
   (c) Prove that $\mathcal{S}(\mathbb{R}^N)$ is dense in H^s for all s.
10. Suppose that T is in H^{-s} and φ is in $\mathcal{S}(\mathbb{R}^N) \subseteq H^s$. Prove that $|\langle T, \varphi\rangle| \leq \|T\|_{H^{-s}}\|\varphi\|_{H^s}$.
11. Conversely suppose that s is real and that T is a tempered distribution such that $|\langle T, \varphi\rangle| \leq C\|\varphi\|_{H^s}$ for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$. Show that $\mathcal{F}(T)$ defines a bounded linear functional on the Hilbert space $L^2(\mathbb{R}^N, (1+|\xi|^2)^s\,d\xi)$, and deduce that T is in H^{-s} with $\|T\|_{-s} \leq C$.
12. Let $s > N/2$.
   (a) Prove that if the tempered distribution T given by the function $\varphi \in \mathcal{S}(\mathbb{R}^N)$ is regarded as a member T_φ of H^s, then $\|\varphi\|_{\text{sup}} \leq \|\mathcal{F}(\varphi)\|_1 \leq C\|T_\varphi\|_{H^s}$, where C is the constant $\big(\int_{\mathbb{R}^N}(1+|\xi|^2)^{-s}\,d\xi\big)^{1/2}$ independent of φ.
   (b) **(Sobolev's Theorem)** Deduce from (a) that any member T of H^s with $s > N/2$ is given by a bounded continuous function.
   (c) Extend the above argument to show for each integer $m \geq 0$ that any member T of H^s with $s > N/2 + m$ is given by a function of class C^m.

Problems 13–20 concern the **Hardy spaces** $H^p(\mathbb{R}^2_+)$ for the upper half plane $\mathbb{R}^2_+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. These problems use complex analysis in one variable, and some familiarity with the Poisson and conjugate Poisson kernels as in Chapters VIII and IX of *Basic* will be helpful. The space $H^p(\mathbb{R}^2_+)$ is defined to be the vector subspace of analytic functions in the space $\mathcal{H}^p(\mathbb{R}^2_+)$. Let f^* be the Hardy–Littlewood maximal function of f on $\mathbb{R}^1$. Take as known the result from *Basic* that the Poisson integral $P_y * f$ satisfies $|P_y * f(x)| \le Cf^*(x)$ with C independent of f and y.

13. Suppose that p satisfies $1 < p < \infty$, and let $H : L^p(\mathbb{R}^1) \to L^p(\mathbb{R}^1)$ be the Hilbert transform.
    (a) Prove that if $u_0(x)$ is in $L^p(\mathbb{R}^1)$, then the Poisson integral of the function $u_0(x) + i(Hu_0)(x)$ is in $H^p(\mathbb{R}^1)$.
    (b) Conversely suppose that $f(x + iy)$ is in $H^p(\mathbb{R}^1_+)$. Applying Theorem 3.25, let $f(x + iy)$ be the Poisson integral of the member $f_0(x)$ of $L^p(\mathbb{R}^1_+)$. If $\operatorname{Re} f_0 = u_0$, prove that $\operatorname{Im} f_0 = Hu_0$.
14. Prove that the functions f in $L^2(\mathbb{R}^1)$ whose Poisson integrals are in the subspace $H^2(\mathbb{R}^2_+)$ of $\mathcal{H}^2(\mathbb{R}^2_+)$ are exactly the functions for which $\mathcal{F}f(x) = 0$ a.e. for $x < 0$.
15. Let $F = (f_1, \dots, f_n)$ be an n-tuple of analytic functions on an open subset of $\mathbb{C}$, and let $(\,\cdot\,,\,\cdot\,)$ be the usual inner product on $\mathbb{C}^n$. For a function on an open set in $\mathbb{C}$, define $f_z = \frac{1}{2}(f_x - if_y)$ and $f_{\bar z} = \frac{1}{2}(f_x + if_y)$, so that the condition for analyticity is $f_{\bar z} = 0$ and so that $\Delta f = 4 f_{z\bar z}$. Suppose that F is nowhere 0 on an open set. Prove for all $q > 0$ that
$$\begin{aligned}\Delta(|F|^q) &= q^2|F|^{q-4}|(F, F')|^2 + 2q|F|^{q-4}\big(- |(F, F')|^2 + |F|^2|F'|^2\big)\\ &\ge q^2|F|^{q-4}|(F, F')|^2 \ge 0.\end{aligned}$$
16. Suppose that u is a smooth real-valued function on an open set in $\mathbb{R}^N$ containing the ball $B(r; x_0)^{\text{cl}}$ such that $\Delta u \ge 0$ on $B(r; x_0)$ and $u \le 0$ on $\partial B(r; x_0)$. By considering $u + c(|x - x_0|^2 - r^2)$ for a suitable c, prove that $u \le 0$ on $B(r; x_0)^{\text{cl}}$.
17. Let f be in $H^1(\mathbb{R}^2_+)$, and define $F_\varepsilon : \{\operatorname{Im} z \ge 0\} \to \mathbb{C}^2$ for $\varepsilon > 0$ by $F_\varepsilon(z) = (f(z + i\varepsilon), \varepsilon(z + i)^{-2})$. Define $g_\varepsilon(x) = |F_\varepsilon(x)|^{1/2}$ for $x \in \mathbb{R}$.
    (a) Prove that $\|g_\varepsilon\|_2^2 \le \|f\|_{H^1} + \varepsilon\|(x + i)^{-2}\|_1$.
    (b) Let $g_\varepsilon(z)$ be the Poisson integral of $g_\varepsilon(x)$. Show that $|F_\varepsilon(z)|^{1/2}$ and $g_\varepsilon(z)$ both tend to 0 as $|x|$ or y tends to infinity in $\mathbb{R}^2_+$.
    (c) By applying the previous two problems to $|F_\varepsilon(z)|^{1/2} - g_\varepsilon(z)$ on large disks in $\mathbb{R}^2_+$, prove that $|F_\varepsilon(z)|^{1/2} \le g_\varepsilon(z)$ on $\mathbb{R}^2_+$.
18. By Alaoglu's Theorem let $g(x)$ be a weak-star limit in $L^2(\mathbb{R}^1)$ of a sequence $g_{\varepsilon_n}(x)$ with $\varepsilon_n \downarrow 0$, and let $g(z)$ be the Poisson integral of $g(x)$.
    (a) Prove that $|f(z)|^{1/2} \le g(z) \le Cg^*(x)$, with $g^*(x)$ being the Hardy–Littlewood maximal function of $g(x)$.

(b) Conclude that $|f(x+iy)|$ is dominated by the fixed integrable function $g^*(x)^2$ as $y \downarrow 0$.

19. Let X be a locally compact separable metric space, let μ be a finite Borel measure on X, and suppose that $\{g_n\}$ is a sequence of Borel functions on X with $|g_n| \le 1$ such that the sequence $\{g_n(x)\,d\mu(x)\}$ of complex Borel measures converges weak-star against $C_{\text{com}}(X)$ to a complex Borel measure ν. Prove that ν is absolutely continuous with respect to μ.

20. **(F. and M. Riesz Theorem)** Deduce from the above facts that each member of $H^1(\mathbb{R}^2_+)$ is the Poisson integral of an L^1 function on $\mathbb{R}^1$.

Problems 21–24 show that the limit $Tf = \lim_{\varepsilon\downarrow 0} T_\varepsilon f$ defining a Calderón–Zygmund operator T exists almost everywhere for $f \in L^p$ and $1 < p < \infty$, as well as in L^p. Let notation be as in the statement of Theorem 3.26 and Lemma 3.29: $K(x)$ is a C^1 function on $\mathbb{R}^N - \{0\}$ homogeneous of degree 0 with mean value 0 over the unit sphere, $k(x)$ is $K(x)/|x|^N$ for $|x| \ge 1$ and is 0 for $|x| < 1$. For any function φ on $\mathbb{R}^N$, define $\varphi_\varepsilon(x) = \varepsilon^{-N}\varphi(\varepsilon^{-1}x)$. The operator $T_\varepsilon f$ is $k_\varepsilon * f$. Let f^* be the Hardy–Littlewood maximal function of f. Take as known from *Basic* that if $\Psi \ge 0$ is an integrable function on $\mathbb{R}^N$ of the form $\Psi(x) = \Psi_0(|x|)$ with Ψ_0 nonincreasing and finite at 0, then $\sup_{\varepsilon>0}(\Psi_\varepsilon * f)(x) \le C_\Psi f^*(x)$ for some finite constant C_Ψ. Let f be in L^p with $1 < p < \infty$.

21. Let φ be as in Proposition 3.5e. Define $\Phi = T(\varphi) - k$.
    (a) Taking into account the fact that φ is in $C^\infty_{\text{com}}(\mathbb{R}^N)$, prove that $T(\varphi)$ is in $C^\infty(\mathbb{R}^N)$, and conclude that Φ is locally bounded.
    (b) By taking into account the compact support of φ, prove that $|\Phi(x)|$ is bounded by a multiple of $|x|^{-N-1}$ for large $|x|$.
    (c) Deduce that $|\Phi(x)|$ is dominated for all x by an integrable function $\Psi(x)$ on $\mathbb{R}^N$ of the form $\Psi(x) = \Psi_0(|x|)$ with Ψ_0 nonincreasing and finite at 0.

22. Let φ and Φ be as in the previous problem.
    (a) Prove that $(T\varphi)_\varepsilon = T\varphi_\varepsilon$.
    (b) Prove the associativity formula $T\varphi_\varepsilon * f = \varphi_\varepsilon * (Tf)$.
    (c) Deduce that $\varphi_\varepsilon * (Tf) - k_\varepsilon * f = \Phi_\varepsilon * f$.

23. Conclude from the previous problem that there are constants C_1 and C_2 independent of f such that $\sup_{\varepsilon>0}|T_\varepsilon f(x)| \le C_1 f^*(x) + C_2(Tf)^*(x)$.

24. Why does it follow that $\lim_{\varepsilon\downarrow 0} T_\varepsilon f(x)$ exists almost everywhere?

Problems 25–34 introduce Sobolev spaces in the context of multiple Fourier series. In this set of problems, periodic functions are understood to be defined on $\mathbb{R}^N$ and to be periodic of period 2π in each variable. Write T for the circle $\mathbb{R}/2\pi\mathbb{Z}$, and let $C^\infty(T^N)$ be the complex vector space of all smooth periodic functions. Let $L^2(T^N)$ be the space of all periodic functions (modulo functions that are 0 almost everywhere) that are in $L^2([-\pi,\pi]^N)$. If $\alpha = (\alpha_1,\dots,\alpha_N)$ is a multi-index, a member f of $L^2(T^N)$

is said to have a **weak** α^{th} **derivative** in $L^2(T^N)$ if there exists a function $D^\alpha f$ in $L^2(T^N)$ with

$$\int_{[-\pi,\pi]^N} (D^\alpha f)\varphi\,dx = (-1)^{|\alpha|} \int_{[-\pi,\pi]} f D^\alpha \varphi\,dx$$

for all φ in $C^\infty(T^N)$. Define the **Sobolev space** $L^2_k(T^N)$ for each integer $k \geq 0$ to consist of all members of $L^2(T^N)$ having α^{th} derivative in $L^2(T^N)$ for all α with $|\alpha| \leq k$. The norm on $L^2_k(T^N)$ is given by

$$\|f\|^2_{L^2_k(T^N)} = \sum_{|\alpha|\leq k} (2\pi)^{-N} \int_{[-\pi,\pi]^N} |D^\alpha f|^2\,dx.$$

25. Prove that $L^2_k(T^N)$ is complete.
26. Prove that $C^\infty(T^N)$ is dense in $L^2_k(T^N)$ for all $k \geq 0$.
27. Prove for each multi-index α and each $k \geq 0$ that there exists a constant $C_{\alpha,k}$ such that
$$\|D^\alpha f\|_{L^2_k(T^N)} \leq C_{\alpha,k} \|f\|_{L^2_{k+|\alpha|}(T^N)}$$
for all f in $C^\infty(T^N)$.
28. Prove for each $k \geq 0$ that there is a constant A_k such that every member f of $L^2_k(T^N)$ has
$$\|f\|_{L^2_k(T^N)} \leq A_k \sum_{|\alpha|\leq k} \sup_{x\in[-\pi,\pi^N]} |D^\alpha f(x)|.$$
29. Prove for each integer $k \geq 0$ that there exist positive constants B_k and C_k such that $B_k \sum_{|\alpha|\leq k} l^{2\alpha} \leq (1+|l|^2)^k \leq C_k \sum_{|\alpha|\leq k} l^{2\alpha}$.
30. Prove that if f is periodic and locally integrable on $\mathbb{R}^N$ with multiple Fourier series $f(x) \sim \sum_{l\in\mathbb{Z}^N} c_l e^{il\cdot x}$, then f is in $L^2_k(T^N)$ if and only if
$$\sum_{l\in\mathbb{Z}^N} |c_l|^2 (1+|l|^2)^k < \infty.$$
31. With notation as in the previous problem, prove for each $k \geq 0$ that there exist positive constants B_k and C_k independent of f such that
$$B_k \|f\|^2_{L^2_k(T^N)} \leq \sum_{l\in\mathbb{Z}^N} |c_l|^2 (1+|l|^2)^k \leq C_k \|f\|^2_{L^2_k(T^N)}$$
for all f in $L^2_k(T^N)$.
32. **(Sobolev's Theorem)** Suppose that K is an integer with $K > N/2$. Prove that $\sum_{l\in\mathbb{Z}^N} (1+|l|^2)^{-K} < \infty$, and deduce that any f in $L^2_K(T^N)$ can be adjusted on a set of measure 0 so as to be continuous.

33. Prove for each multi-index α that there exist some integer $m(\alpha)$ and constant C_α such that

$$\sup_{x\in[-\pi,\pi]} |D^\alpha f(x)| \le C_\alpha \|f\|_{L^2_{m(\alpha)}(T^N)}$$

for all f in $C^\infty(T^N)$.

34. Prove that the separating family of seminorms $\|\cdot\|_{L^2_k(T^N)}$ on $C^\infty(T^N)$, indexed by k, is equivalent to the family of seminorms $\sup_{x\in[-\pi,\pi]^N} |D^\alpha(\,\cdot\,)(x)|$, indexed by α. Here "is equivalent to" is to mean that the identity map is uniformly continuous from the one metric space to the other.