HINTS FOR SOLUTIONS OF PROBLEMS

Chapter I

1. We start from

$$\int_0^l \sin p_n x \sin p_m x \, dx = -\tfrac{1}{2}\int_0^l \cos(p_n + p_m)x \, dx + \tfrac{1}{2}\int_0^l \cos(p_n - p_m)x \, dx.$$

The first term on the right is equal to

$$\begin{aligned} -\tfrac{1}{2}\tfrac{1}{p_n+p_m}\sin(p_n+p_m)l &= -\tfrac{1}{2}\tfrac{1}{p_n+p_m}(\sin p_n l \cos p_m l + \cos p_n l \sin p_m l) \\ &= -\tfrac{1}{2}\tfrac{1}{p_n+p_m}\big(-\tfrac{p_n}{h}\cos p_n l \cos p_m l - \tfrac{p_m}{h}\cos p_n l \cos p_m l\big) \\ &= \tfrac{1}{2h}\tfrac{1}{p_n+p_m}(p_n+p_m)\cos p_n l \cos p_m l = \tfrac{1}{2h}\cos p_n l \cos p_m l. \end{aligned}$$

Similarly the second term on the right is $-\frac{1}{2h}\cos p_n l \cos p_m l$. The two terms cancel, and the desired orthogonality follows.

2. In (a), the adjusted operator is $L(u) = ((1-t^2)u')'$, and Green's formula gives

$$\begin{aligned} (\lambda_n - \lambda_m)\int_{-1}^{1} P_n(t)P_m(t)\,dt &= (L(P_n), P_m) - (P_n, L(P_m)) \\ &= \big[(1-t^2)(P_n'(t)P_m(t) - P_n(t)P_m'(t))\big]_{-1}^{1}, \end{aligned}$$

where λ_n and λ_m are the values $\lambda_n = -n(n+1)$ and $\lambda_m = -m(m+1)$ such that $L(P_n) = \lambda_n P_n$ and $L(P_m) = \lambda_m P_m$. The right side is 0 because $1 - t^2$ vanishes at -1 and 1.

In (b), the adjusted operator is $L(u) = (tu')' + tu$, and $L(J_0(k\,\cdot))$ equals $-k^2 t$ if $J_0(k) = 0$. Green's formula gives

$$\begin{aligned} (-k_n^2 + k_m^2)&\textstyle\int_0^1 J_0(k_n t)J_0(k_m t)t\,dt \\ &= \big(L(J_0(k_n\,\cdot)), J_0(k_m\,\cdot)\big) - \big(J_0(k_n\,\cdot), L(J_0(k_m\,\cdot))\big) \\ &= \big[t\big(\tfrac{d}{dt}(J_0(k_n\,\cdot))(t)J_0(k_m t) - J_0(k_n t)\tfrac{d}{dt}(J_0(k_n\,\cdot))(t)\big)\big]_0^1. \end{aligned}$$

The expression in brackets on the right side is 0 at $t = 1$ because $J_0(k_n) = J_0(k_m) = 0$, and it is 0 at $t = 0$ because of the factor t.

3. With $L(u) = (p(t)u')' - q(t)u$, the formula for $u^*(t) = \int_a^t G_0(t,s)f(s)\,ds$ in the proof of Lemma 4.4 is

$$u^*(t) = p(c)^{-1}\big(-\varphi_1(t)\int_a^t \varphi_2(s)f(s)\,ds + \varphi_2(t)\int_a^t \varphi_1(s)f(s)\,ds\big).$$

As is observed in the proof of Lemma 4.4, the derivative of this involves terms in which the integrals are differentiated at their upper limits, and these terms drop out. Thus

$$u^{*\prime}(t) = p(c)^{-1}\big(-\varphi_1'(t)\int_a^t \varphi_2(s)f(s)\,ds + \varphi_2'(t)\int_a^t \varphi_1(s)f(s)\,ds\big).$$

For the second derivative, the terms do not drop out, and we obtain

$$\begin{aligned} u^{*\prime\prime}(t) &= p(c)^{-1}\big(-\varphi_1''(t)\int_a^t \varphi_2(s)f(s)\,ds + \varphi_2''(t)\int_a^t \varphi_1(s)f(s)\,ds\big) \\ &\quad + p(c)^{-1}\big(-\varphi_1'(t)\varphi_2(t)f(t) + \varphi_2'(t)\varphi_1(t)f(t)\big). \end{aligned}$$

When we combine these expressions to form $p(t)u^{*\prime\prime}(t) + p'(t)u^{*\prime}(t) - q(t)u^*(t)$, the coefficient of $\int_a^t \varphi_2(s)f(s)\,ds$ is $-p(c)^{-1}L(\varphi_1) = 0$, and similarly the coefficient of $\int_a^t \varphi_1(s)f(s)\,ds$ is $p(c)^{-1}L(\varphi_2) = 0$. Thus

$$\begin{aligned} L(u^*) &= p(c)^{-1}p(t)f(t)\big(-\varphi_1'(t)\varphi_2(t) + \varphi_2'(t)\varphi_1(t)\big) \\ &= p(c)^{-1}p(t)f(t)\det W(\varphi_1,\varphi_2)(t) = f(t), \end{aligned}$$

the value of $\det W(\varphi_1,\varphi_2)$ having been computed in the proof. This completes (a).

For (b), we can take $\varphi_1(t) = \cos t$ and $\varphi_2(t) = \sin t$. Since $p(t) = 1$, we obtain

$$G_0(t,s) = \begin{cases} \sin t\cos s - \cos t\sin s & \text{if } s \le t, \\ 0 & \text{if } s > t. \end{cases}$$

The conditions $u(0) = 0$ and $u(\pi/2) = 0$ mean that $a = 0$, $b = \pi/2$, $c_1 = d_1 = 1$, and $c_2 = d_2 = 0$ in (SL2). Thus the system of equations $(*)$ in the proof of Lemma 4.4 reads

$$\begin{pmatrix} \cos 0 & \sin 0 \\ \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} -u^*(0) \\ -u^*(\pi/2) \end{pmatrix},$$

and we obtain $k_1 = -u^*(0) = 0$ and $k_2 = -u^*(\pi/2) = -\int_0^{\pi/2} f(s)\cos s\,ds$. The proof of Lemma 4.4 says to take $K_1(s) = 0$ and $K_2(s) = -\cos s$. The formula for $G_1(t,s)$ is $G_1(t,s) = G_0(t,s) + K_1(s)\varphi_1(t) + K_2(s)\varphi_2(t)$, and therefore

$$G_1(t,s) = \left\{ \begin{matrix} \sin t\cos s - \cos t\sin s \\ 0 \end{matrix} \right\} - \sin t\cos s = \left\{ \begin{matrix} -\cos t\sin s \\ -\sin t\cos s \end{matrix} \right\}.$$

In particular, $G_1(t,s)$ is symmetric, as it is supposed to be!

4. We have $\int_{t_1}^{t_2} \big((py_1')'y_2 - (py_2')'y_1\big)\,dt = \int_{t_1}^{t_2} (g_2 - g_1)y_1y_2\,dt > 0$ as a result of the outlined steps. Since $\big((py_1')'y_2 - (py_2')'y_1\big) = \frac{d}{dt}\big(p(y_1'y_2 - y_1y_2')\big)$, we conclude that $\big[p(y_1'y_2 - y_1y_2'))\big]_{t_1}^{t_2} > 0$. This proves (a).

Since $y_1(t_1) = y_1(t_2) = 0$, the expression $p(t)y_1'(t)y_2(t) - p(t)y_1(t)y_2'(t)$ is $p(t_2)y_1'(t_2)y_2(t_2)$ at $t = t_2$. Here $p(t_2) > 0$ and $y_2(t_2) \ge 0$. Since $y_1(t_2) = 0$ and since $y_1(t) > 0$ for all t slightly less than t_2, we obtain $y_1'(t_2) \le 0$. Thus $p(t_2)y_1'(t_2)y_2(t_2) \le 0$. Similarly the same expression is $p(t_1)y_1'(t_1)y_2(t_1)$ at $t = t_1$. We have $p(t_1) > 0$ and $y_2(t_1) \ge 0$. Since $y_1(t_1) = 0$ and $y_1(t) > 0$ for t slightly greater than t_1, we obtain $y_1'(t_1) \ge 0$. Thus $p(t_1)y_1'(t_1)y_2(t_1) \ge 0$. This gives the desired contradiction and completes (b).

Part (c) is just the special case in which $g_1(t) = -q(t) + \lambda_1 r(t)$ and $g_2(t) = -q(t) + \lambda_2 r(t)$. The hypothesis on $g_2 - g_1$ is satisfied because $g_2(t) - g_1(t) = (\lambda_2 - \lambda_1)r(t) > 0$.

5. For (a), substitute for $\Psi(x,t)$ and get $-\psi''(x)\varphi(t) + V(x)\psi(x)\varphi(t) = i\psi(x)\varphi'(t)$. Divide by $\psi(x)\varphi(t)$ to obtain $-\frac{\psi''(x)}{\psi(x)} + V(x) = i\frac{\varphi'(t)}{\varphi(t)}$. The left side depends only on x, and the right side depends only on t. So the two sides must be some constant E. Then $-\frac{\psi''(x)}{\psi(x)} + V(x) = E$ yields $\psi'' + (E - V(x))\psi = 0$.

For (b), the equation for φ is $i\frac{\varphi'(t)}{\varphi(t)} = E$. Then $\varphi' = -iE\varphi$, and $\varphi(t) = ce^{-iEt}$.

6. We substitute $\psi(x) = e^{-x^2/2}H(x)$, $\psi'(x) = -xe^{-x^2/2}H(x) + e^{-x^2/2}H'(x)$, and $\psi''(x) = x^2e^{-x^2/2}H(x) - 2xe^{-x^2/2}H'(x) + e^{-x^2/2}H''(x) - e^{-x^2/2}H(x)$, and we are led to Hermite's equation.

7. Write $H(x) = \sum_{k=0}^{\infty} c_k x^k$. We find that c_0 and c_1 are arbitrary and that $(k+2)(k+1)c_{k+2} - (2n-2k)c_k = 0$ for $k \ge 0$. To get a polynomial of degree d, we must have $c_d \ne 0$ and $c_{d+2} = 0$. Since $c_{d+2} = c_d(2n-2d)/((d+2)(d+1))$, this happens if and only if $d = n$.

8. We have $L(H_n(x)e^{-x^2/2}) = -(2n+1)H_n(x)e^{-x^2/2}$. Define an inner product by integrating over $[-N, N]$. Then

$$\begin{aligned}
-2(n-m)&\int_{-N}^{N} H_n(x)H_m(x)e^{-x^2}\,dx \\
&= (L(H_n(x)e^{-x^2/2}), H_m(x)e^{-x^2/2}) - (H_n(x)e^{-x^2/2}, L(H_m(x)e^{-x^2/2})) \\
&= \big[(H_n(x)e^{-x^2/2})'(H_m(x)e^{-x^2/2}) - (H_n(x)e^{-x^2/2})(H_m(x)e^{-x^2/2})'\big]_{-N}^{N}.
\end{aligned}$$

As N tends to infinity, the right side tends to 0. Since $n \ne m$, we obtain the desired orthogonality.

Chapter II

1. A condition in (a) is that f take on some value on a set of positive measure. A condition in (b) is that f take on only countably many values, these tending to 0,

and that the set E where f is nonzero be the countable union of sets E_n of positive measure such that no E_n decomposes as the disjoint union of two sets of positive measure.

2. Let v_n be in image$(\lambda I - L)$ with $v_n \to v$, and choose u_n with $(\lambda I - L)u_n = v_n$. We are to show that v is in the image. We may assume that $v \neq 0$, so that $\|v_n\|$ is bounded below by a positive constant for large n. Since $\|v_n\| \leq \|\lambda I - L\|\|u_n\|$, $\|u_n\|$ is bounded below for large n. Passing to a subsequence, we may assume either that $\|u_n\|$ tends to infinity or that $\|u_n\|$ is bounded.

If $\|u_n\|$ is bounded, then we may assume by passing to a subsequence that $\{Lu_n\}$ is convergent, say with limit w. From $\lambda u_n = Lu_n + v_n$, we see that $\lambda u_n \to w + v$. Put $u = \lambda^{-1}(w + v)$. Then $(\lambda I - L)u = (w + v) - \lim Lu_n = w + v - w = v$, and v is in the image.

If $\|u_n\|$ tends to infinity, choose a subsequence such that $\{L(\|u_n\|^{-1}u_n)\}$ is convergent, say to w. Then we have $\|u_n\|^{-1}\lambda u_n - L(\|u_n\|^{-1}u_n) = \|u_n\|^{-1}v_n$. Passing to the limit and using that $v_n \to v$, we see that $\|u_n\|^{-1}\lambda u_n \to w$. Applying L, we obtain $\lambda w = L(w)$. Thus $(\lambda I - L)w = 0$. Since $\lambda I - L$ is one-one, $w = 0$. Then $\|u_n\|^{-1}\lambda u_n \to 0$, and we obtain a contradiction since $\|u_n\|^{-1}\lambda u_n$ has norm $|\lambda|$ for all n.

3. It was shown in Section 4 that the set of Hilbert–Schmidt operators is a normed linear space with norm $\|\cdot\|_{\mathrm{HS}}$. Since $\|L\| \leq \|L\|_{\mathrm{HS}}$, any Cauchy sequence $\{L_n\}$ in this space is Cauchy in the operator norm. The completeness of the space of bounded linear operators in the operator norm shows that $\{L_n\}$ converges to some L in the operator norm. In particular, $\lim_n (L_n u, v) = (Lu, v)$ for all u and v. By Fatou's Lemma,

$$\begin{aligned}\|L\|_{\mathrm{HS}} &= \textstyle\sum_j \|Lu_j\|^2 = \sum_j \liminf_n \|L_n u_j\|^2 \\ &\leq \liminf_n \textstyle\sum_j \|L_n u_j\|^2 = \liminf_n \|L_n\|_{\mathrm{HS}}.\end{aligned}$$

The right side is finite since Cauchy sequences are bounded, and hence L is a Hilbert–Schmidt operator. A second application of Fatou's Lemma gives

$$\begin{aligned}\|L_m - L\|_{\mathrm{HS}} &= \textstyle\sum_j \|(L_m - L)u_j\|^2 = \sum_j \liminf_n \|(L_m - L_n)u_j\|^2 \\ &\leq \liminf_n \textstyle\sum_j \|(L_m - L_n)u_j\|^2 = \liminf_n \|L_m - L_n\|_{\mathrm{HS}}.\end{aligned}$$

Since the given sequence is Cauchy, the lim sup on m of the right side is 0, and hence $\{L_m\}$ converges to L in the Hilbert–Schmidt norm.

4. If L and M are of trace class, then $\sum_i |((L + M)u_i, v_i)| \leq \sum_i (|(Lu_i, v_i)| + |(Mu_i, v_i)|) \leq \|L\|_{\mathrm{TC}} + \|M\|_{\mathrm{TC}}$. Taking the supremum over all orthonormal bases $\{u_i\}$ and $\{v_i\}$, we obtain the triangle inequality.

5. Once we know that $\mathrm{Tr}(AL) = \mathrm{Tr}(LA)$, then $\mathrm{Tr}(BLB^{-1}) = \mathrm{Tr}(B^{-1}(BL)) = \mathrm{Tr}(L)$. To prove that $\mathrm{Tr}(AL) = \mathrm{Tr}(LA)$, fix an orthonormal basis $\{u_i\}$. The formal

computation is

$$\begin{aligned}\operatorname{Tr}(AL) &= \textstyle\sum_j (ALu_j, u_j) = \sum_j (Lu_j, A^*u_j) = \sum_j \sum_i (Lu_j, u_i)\overline{(A^*u_j, u_i)} \\ &= \textstyle\sum_j \sum_i (Au_i, u_j)\overline{(L^*u_i, u_j)} = \sum_i \sum_j (Au_i, u_j)\overline{(L^*u_i, u_j)} \\ &= \textstyle\sum_i (Au_i, L^*u_i) = \sum_i (LAu_i, u_i) = \operatorname{Tr}(LA),\end{aligned}$$

and justification is needed for the interchange of order of summation within the second line. It is enough to have absolute convergence in *some* orthonormal basis, and this will be derived from the estimate

$$\begin{aligned}\textstyle\sum_{i,j} |(Au_i, u_j)(L^*u_i, u_j)| &\le \textstyle\sum_i \big(\sum_j |(Au_i, u_j)|^2\big)^{1/2}\big(\sum_j |(L^*u_i, u_j)|^2\big)^{1/2} \\ &= \textstyle\sum_i \|Au_i\|\|L^*u_i\| \le \|A\| \sum_i \|L^*u_i\|.\end{aligned}$$

The proof of Proposition 2.8, applied to L^* instead of L, produces operators U and T, orthonormal bases $\{w_i\}$ and $\{f_i\}$, and scalars $\lambda_i \ge 0$ such that $L^* = UT$, $\|U\| \le 1$, $Tw_i = \sqrt{\lambda_i}\,w_i$, and $\sum |(L^*w_i, f_i)| = \sum (Tw_i, w_i)$. Taking $u_i = w_i$, we have $\|L^*w_i\| = \|UTw_i\| \le \|Tw_i\| = \sqrt{\lambda_i} = (Tw_i, w_i)$. Hence for this orthonormal basis, $\sum \|L^*w_i\| \le \sum (Tw_i, w_i) = \sum |(L^*w_i, f_i)|$. The right side is finite since L^* is of trace class.

6. If v is a nonzero vector in the λ eigenspace of L_α and if $L_\beta L_\alpha = L_\alpha L_\beta$, then $L_\alpha L_\beta(v) = L_\beta L_\alpha(v) = \lambda L_\beta v$. Thus the λ eigenspace of L_α is invariant under L_β. We apply Theorem 2.3 to the compact operator L_β on each eigenspace of L_α, obtaining an orthonormal basis of simultaneous eigenvectors under L_α and L_β. Iterating this procedure by taking into account one new operator at a time, we obtain the desired basis.

7. In (a), the operators $L + L^*$ and $-i(L - L^*)$ are self adjoint, and they commute since L commutes with L^*. Compactness is preserved under passage to adjoints and under taking linear combinations, and (b) follows.

8. If U is unitary, then $U^* = U^{-1}$. Then $UU^{-1} = I = U^{-1}U$ shows that U is normal. Since U preserves norms, every eigenvalue λ has $|\lambda| = 1$. If U is also compact, then the eigenvalues tend to 0. Hence U is compact if and only if the Hilbert space is finite-dimensional.

9. The solutions of the homogeneous equation are spanned by $\cos \omega t$ and $\sin \omega t$. Then the result follows by applying variation of parameters.

10. Take $g(s) = \rho(s)u(s)$ in Problem 9.

11. In (a), let $t < t'$. Then

$$\begin{aligned}(Tf)(t') - (Tf)(t) &= \textstyle\int_s^{t'} K(t', s)f(s)\,ds - \int_a^t K(t, s)f(s)\,ds \\ &= \textstyle\int_t^{t'} K(t', s)f(s)ds + \int_a^t [K(t', s) - K(t, s)]f(s)\,ds.\end{aligned}$$

The first term on the right tends to 0 as $t' - t$ tends to 0 because the integrand is bounded, and the second term tends to 0 by the boundedness of f and the uniform continuity of $K(t', s) - K(t, s)$ on the set of (s, t, t') where $a \le s \le t \le t'$.

In (b), for $n = 1$, we have $|(Tf)(t)| = \big|\int_a^t K(t,s) f(s)\,ds\big| \le M\int_a^t |f(s)|\,ds \le CM$ as required. Assume the result for $n-1 \ge 1$, namely that $|(T^{n-1}f)(t)| \le \frac{1}{(n-2)!}CM^{n-1}(t-a)^{n-2}$. Then $|(T^n f)(t)| = \big|\int_a^t K(t,s)(T^{n-1}f)(s)\,ds\big| \le M\int_a^t |(T^{n-1}f)(s)|\,ds \le M\frac{1}{(n-2)!}CM^{n-1}\int_a^t (s-a)^{n-2}\,ds = \frac{1}{(n-1)!}CM^n(t-a)^{n-1}$. Thus the n^{th} term of the series is $\le \frac{1}{(n-1)!}CM^n(b-a)^{n-1}$.

In (c), the uniform convergence follows from the estimate in (b) and the Weierstrass M test.

12. The operator T is bounded as a linear operator from $C([a,b])$ into itself. Because of the uniform convergence, we can apply the operator term by term to the series defining u. The result is $Tu = Tf + T^2 f + T^3 f + \cdots = u - f$. Therefore $u - Tu = f$.

13. Subtracting, we are to investigate solutions of $u - Tu = 0$. Problem 11 showed for each continuous u that the series $u + Tu + T^2u + \cdots$ is uniformly convergent. If $u = Tu$, then all the terms in this series equal u, and the only way that the series can converge uniformly is if $u = 0$.

Chapter III

1. Let $D_j = \partial/\partial y_j$. Let $\widetilde{\mathcal{S}}$ be the vector space of all linear combinations of functions $(1+4\pi^2|y|^2)^{-n}h$ with n a positive integer and h in the Schwartz space $\mathcal{S}$. Then $D_j\big((1+4\pi^2|y|^2)^{-n}h\big) = -8n\pi^2 y_j(1+4\pi^2|y|^2)^{-(n+1)}h + (1+4\pi^2|y|^2)^{-n}D_j h$. The first term on the right side is in $\widetilde{\mathcal{S}}$ because $y_j h$ is in $\mathcal{S}$, and the second term on the right side is in $\widetilde{\mathcal{S}}$ because $D_j h$ is in $\mathcal{S}$. Thus $\widetilde{\mathcal{S}}$ is closed under all partial derivatives. Since the product of a polynomial and a Schwartz function is a Schwartz function, $\widetilde{\mathcal{S}}$ is closed under multiplication by polynomials. Since the members of $\widetilde{\mathcal{S}}$ are bounded, we must have $\widetilde{\mathcal{S}} \subseteq \mathcal{S}$. In particular, $(1+4\pi^2|y|^2)^{-1}g$ is in $\mathcal{S}$ if g is in $\mathcal{S}$.

2. Since the Fourier transform and its inverse are continuous, it is enough to handle pointwise product. Pointwise product is handled directly.

3. In (a), the ordinary partial derivatives are $D_x\big(\log((x^2+y^2)^{-1})\big) = \frac{-2x}{x^2+y^2}$ and $D_y\big(\log((x^2+y^2)^{-1})\big) = \frac{-2y}{x^2+y^2}$. These are also weak derivatives. In fact, use of polar coordinates shows that they are integrable near $(0,0)$, hence locally integrable on $\mathbb{R}^2$. If φ is in $C^\infty_{\text{com}}(\Omega)$, we are to show that $\int_\Omega \log((x^2+y^2)^{-1})D_x\varphi(x,y)\,dx\,dy = \int_\Omega \frac{2x\varphi(x,y)}{x^2+y^2}\,dx\,dy$ and similarly for y. For each $y \ne 0$, the integrals over x are equal, and the set where $y = 0$ is of measure 0 in Ω. The argument with the variables interchanged is similar. Thus $\log((x^2+y^2)^{-1})$ has weak derivatives of order 1. In

polar coordinates the p^{th} power of $\left|\frac{x\varphi(x,y)}{x^2+y^2}\right|$ is $\frac{r^p|\cos\theta|^p}{r^{2p}} = r^{-p}|\cos\theta|^p$, which is integrable near $r = 0$ relative to $r\,dr$ for $p < 2$ but not $p = 2$.

In (b), the argument for the existence of the weak derivative of $\log\log((x^2+y^2)^{-1})$ is similar to the argument for (a), the ordinary x derivative being

$$\frac{-2x}{(x^2+y^2)\log((x^2+y^2)^{-1})}.$$

In polar coordinates the square of this is $\dfrac{4\cos^2\theta}{r^2\log^2(r^{-2})}$, which is integrable relative to $r\,dr$.

4. The idea is to use the Implicit Function Theorem to obtain, for each point of the boundary, a neighborhood of the point for which some coordinate has the property that the cone of a particular size and orientation based at any point in that neighborhood lies in the region. These neighborhoods cover the boundary, and we extract a finite subcover. Then we obtain a single size of cone such that every point of the boundary has some coordinate where the cone lies in Ω. The cones based at the boundary points cover all points within some distance $\epsilon > 0$ of the boundary, and cones of half the height based at interior points within those cones and within distance $\epsilon/2$ of the boundary lie within the cones for the boundary points. The remaining points of the region can then be covered by a cone with any orientation such that its vertex is at distance $< \epsilon/2$ from all its other points.

5. For $0 < \alpha < N$, $|x|^{-(N-\alpha)}$ is the sum of an L^1 function and an L^∞ function and hence is a tempered distribution. It is the sum of an L^1 function and an L^2 function for $0 < \alpha < N/2$.

6. The second expression is converted into the first by changing t into $1/t$. The first expression is evaluated as the third by replacing $t|x|^2$ by s.

7. The formula obtained from the first displayed identity is

$$\int_{\mathbb{R}^N} (\pi|x|^2)^{-\frac{1}{2}(N-\alpha)}\Gamma(\tfrac{1}{2}(N-\alpha))\widehat{\varphi}(x)\,dx = \int_{\mathbb{R}^N} (\pi|x|^2)^{-\frac{1}{2}\alpha}\Gamma(\tfrac{1}{2}\alpha)\varphi(x)\,dx,$$

which sorts out as

$$\pi^{-\frac{1}{2}(N-\alpha)}\Gamma(\tfrac{1}{2}(N-\alpha))\int_{\mathbb{R}^N} |x|^{-(N-\alpha)}\widehat{\varphi}(x)\,dx = \pi^{-\frac{1}{2}\alpha}\Gamma(\tfrac{1}{2}\alpha)\int_{\mathbb{R}^N} |x|^{-\alpha}\varphi(x)\,dx.$$

8. In (a), we check directly that $\mathcal{F}(D^\alpha T) = (2\pi i)^{|\alpha|}\xi^\alpha\mathcal{F}(T)$. Since T is in H^s, $\int_{\mathbb{R}^N} |\mathcal{F}(T)(\xi)|^2(1+|\xi|^2)^s\,d\xi$ is finite. Now $|\xi_j| \le |\xi| \le (1+|\xi|^2)^{1/2}$ for every j, and hence $|\xi^\alpha| \le (1+|\xi|^2)^{s/2}$ for $|\alpha| = s$. Since $(1+|\xi|^2)^{1/2} \ge 1$, $(1+|\xi|^2)^{t/2}$ is an increasing function of t, and thus $|\xi^\alpha| \le (1+|\xi|^2)^{s/2}$ for $|\alpha| \le s$. Consequently $(2\pi i)^{|\alpha|}\xi^\alpha\mathcal{F}(T)$ is square integrable for $|\alpha| \le s$. Thus the Fourier transform of $D^\alpha T$ is a square integrable function for $|\alpha| \le s$. By the Plancherel formula, $D^\alpha T$ is a square integrable function for $|\alpha| \le s$.

Let T be the L^2 function f, and let $D^\alpha T$ be the L^2 function g_α for $|\alpha| \le s$. The statement that f has g_α as weak derivative of order α is the statement that $\int_{\mathbb{R}^N} f D^\alpha \varphi\, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^N} g_\alpha \varphi\, dx$ for $\varphi \in C^\infty_{\text{com}}(\mathbb{R}^N)$; this is proved for $\psi = \overline{\varphi}$ by the following computation, which uses the polarized version of the Plancherel formula twice:

$$\begin{aligned}(-1)^{|\alpha|}\textstyle\int_{\mathbb{R}^N} g_\alpha \overline{\psi}\, dx &= (-1)^{|\alpha|} \textstyle\int_{\mathbb{R}^N} (2\pi i)^{|\alpha|} \xi^\alpha \mathcal{F}(f)\overline{\mathcal{F}(\psi)}\, d\xi \\ &= \textstyle\int_{\mathbb{R}^N} \mathcal{F}(f)\overline{(2\pi i)^{|\alpha|}\xi^\alpha \mathcal{F}(\psi)}\, d\xi = \int_{\mathbb{R}^N} \mathcal{F}(f)\overline{\mathcal{F}(D^\alpha \psi)}\, d\xi = \int_{\mathbb{R}^N} f\overline{D^\alpha \psi}\, dx.\end{aligned}$$

Since f and its weak derivatives g_α through $|\alpha| \le s$ are all in L^2, f is in $L^2_s(\mathbb{R}^N)$.

In (b), if T is given by an L^2 function, then $\mathcal{F}(T) = \mathcal{F}(f)$ is an L^2 function. Hence $\mathcal{F}(T)$ is locally square integrable. We are assuming that $D^\alpha T$ is given by an L^2 function g_α for $|\alpha| \le s$. The formula $\mathcal{F}(g_\alpha) = \mathcal{F}(D^\alpha T) = (2\pi i)^{|\alpha|}\xi^\alpha \mathcal{F}(T)$ shows that $\xi^\alpha \mathcal{F}(f)$ is in L^2 for $|\alpha| \le s$. Now $|\xi|^2|\mathcal{F}(f)|^2 = \sum_j |\xi_j \mathcal{F}(f)|^2$ and similarly $|\xi|^{2k}|\mathcal{F}(f)|^2 = \sum_{j_1,\dots,j_k} |\xi_{j_1}\cdots\xi_{j_k}\mathcal{F}(f)|^2 = \sum_{|\alpha|=k}\binom{|\alpha|}{\alpha_1,\dots,\alpha_N}|\xi^\alpha \mathcal{F}(f)|^2$. Hence

$$(1+|\xi|^2)^s|\mathcal{F}(f)|^2 = \textstyle\sum_{k=0}^s \binom{s}{k}\sum_{|\alpha|=k}\binom{|\alpha|}{\alpha_1,\dots,\alpha_N}|\xi^\alpha\mathcal{F}(f)|^2 \le s!\sum_{|\alpha|\le s}|\xi^\alpha \mathcal{F}(f)|^2,$$

and f is in H^s.

For (c), in one direction the argument for (a) gives

$$\begin{aligned}\|f\|^2_{L^2_s} &= \textstyle\sum_{|\alpha|\le s}\|D^\alpha f\|^2_{L^2} = \sum_{|\alpha|\le s}\|(2\pi i)^{|\alpha|}\xi^\alpha \mathcal{F}(f)\|^2_{L^2} \\ &\le \big(\textstyle\sum_{|\alpha|\le s}(2\pi)^{2|\alpha|}\big)\|(1+|\xi|^2)^{s/2}\mathcal{F}(f)\|^2_{L^2} \le \big(\sum_{|\alpha|\le s}(2\pi)^{2|\alpha|}\big)\|f\|^2_{H^s}.\end{aligned}$$

In the other direction the displayed formula for (b), when integrated, gives

$$\|f\|^2_{H^s} \le s!\textstyle\sum_{|\alpha|\le s}|2\pi i|^{-|\alpha|}\|D^\alpha f\|^2_{L^2} \le s!\|f\|^2_{L^2_s}.$$

9. In (a), let T be in H^s. Then the computation

$$\|T\|^2_{H^s} = \|(1+|\xi|^2)^{s/2}\mathcal{F}(T)\|^2_{L^2} = \|\mathcal{F}^{-1}\big((1+|\xi|^2)^{s/2}\mathcal{F}(T)\big)\|^2_{L^2} = \|A_s(T)\|^2_{L^2}$$

shows that A_s preserves norms. To see that A_s is onto L^2, let f be in L^2. Then $\mathcal{F}(f)$ is in L^2 and hence acts as a tempered distribution. Then $(1+|\xi|^2)^{-s/2}\mathcal{F}(f)$ is a tempered distribution also. Since $\mathcal{F}$ carries $\mathcal{S}'(\mathbb{R}^N)$ onto itself, $T = \mathcal{F}^{-1}\big((1+|\xi|^2)^{-s/2}\mathcal{F}(f)\big)$ is a tempered distribution. This tempered distribution has the property that $A_s(T) = f$.

In (b), the relevant formula is that $(A_s)^{-1}(\varphi) = \mathcal{F}^{-1}\big((1+|\xi|^2)^{-s/2}\mathcal{F}(\varphi)\big)$. If φ is in $\mathcal{S}(\mathbb{R}^N)$, then so is $\mathcal{F}(\varphi)$. An easy induction shows that any iterated derivative of $(1+|\xi|^2)^{-s/2}$ is a sum of products of polynomials in ξ times powers (possibly negative) of $1+|\xi|^2$. Application of the Leibniz rule therefore shows that any iterated derivative of $(1+|\xi|^2)^{-s/2}\mathcal{F}(\varphi)$ is a sum of products of polynomials in ξ times derivatives of $\mathcal{F}(\varphi)$, all divided by powers of $1+|\xi|^2$. Consequently $(1+|\xi|^2)^{-s/2}\mathcal{F}(\varphi)$ is a Schwartz function, and so is its inverse Fourier transform.

For (c), we know that $C^\infty_{\text{com}}(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$, and hence $\mathcal{S}(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$ also. Applying the operator $(A_s)^{-1}$, which must carry $\mathcal{S}(\mathbb{R}^N)$ onto itself, we see that $\mathcal{S}(\mathbb{R}^N)$ is dense in H^s.

10. If T is in H^{-s} and φ is in $\mathcal{S}(\mathbb{R}^N)$, then the definition of Fourier transform on $\mathcal{S}(\mathbb{R}^N)$, together with the Schwarz inequality, implies that

$$\begin{aligned}|\langle T,\varphi\rangle| &= |\langle \mathcal{F}(T),\mathcal{F}^{-1}(\varphi)\rangle| = \left|\textstyle\int_{\mathbb{R}^N}\mathcal{F}(T)(\xi)\mathcal{F}^{-1}(\varphi)(\xi)\,d\xi\right|\\ &= \left|\textstyle\int_{\mathbb{R}^N}[(1+|\xi|^2)^{-s/2}\mathcal{F}(T)(\xi)]\,[(1+|\xi|^2)^{s/2}\mathcal{F}^{-1}(\varphi)(\xi)]\,d\xi\right|\\ &\le \|(1+|\xi|^2)^{-s/2}\mathcal{F}(T)\|_{L^2}\|(1+|\xi|^2)^{s/2}\mathcal{F}^{-1}(\varphi)\|_{L^2} = \|T\|_{H^{-s}}\|\varphi\|_{H^s}.\end{aligned}$$

11. For ψ in $\mathcal{S}(\mathbb{R}^N)$, we have $|\langle\mathcal{F}(T),\psi\rangle| = |\langle T,\mathcal{F}(\psi)\rangle| \le C\|\mathcal{F}(\psi)\|_{H^s} = C\big(\int_{\mathbb{R}^N}|\mathcal{F}(\mathcal{F}(\psi))(\xi)|^2(1+|\xi|^2)^s\,d\xi\big)^{1/2} = C\big(\int_{\mathbb{R}^N}|\psi(-\xi)|^2(1+|\xi|^2)^s\,d\xi\big)^{1/2} = C\|\psi\|_{L^2(\mathbb{R}^N,(1+|\xi|^2)^s\,d\xi)}$. Thus $\mathcal{F}(T)$ acts as a bounded linear functional on the dense vector subspace $\mathcal{S}(\mathbb{R}^N)$ of $L^2(\mathbb{R}^N,(1+|\xi|^2)^s\,d\xi)$. Extending this linear functional continuously to the whole space and applying the Riesz Representation Theorem for Hilbert spaces, we obtain a function f in $L^2(\mathbb{R}^N,(1+|\xi|^2)^s\,d\xi)$ such that

$$\langle\mathcal{F}(T),\psi\rangle = \textstyle\int_{\mathbb{R}^N}\psi(\xi)\overline{f(\xi)}(1+|\xi|^2)^s\,d\xi$$

for all ψ in $\mathcal{S}(\mathbb{R}^N)$. Put $\psi_0(\xi)=\overline{f(\xi)}(1+|\xi|^2)^s$. Then $\int_{\mathbb{R}^N}|\psi_0(\xi)|^2(1+|\xi|^2)^{-s}\,d\xi = \int_{\mathbb{R}^N}|f(\xi)|^2(1+|\xi|^2)^s\,d\xi < \infty$, and the above displayed formula shows that $\mathcal{F}(T)$ agrees with the function ψ_0 on $\mathcal{S}(\mathbb{R}^N)$. Thus T is in H^{-s}. To estimate $\|T\|_{H^{-s}}$, we twice use the fact that $\mathcal{S}(\mathbb{R}^N)$ is dense: $\|T\|_{H^{-s}} = \|\psi_0\|_{L^2(\mathbb{R}^N,(1+|\xi|^2)^{-s}\,d\xi)} = \|f\|_{L^2(\mathbb{R}^N,(1+|\xi|^2)^s\,d\xi)} = \sup_{\|\psi\|_{L^2(\mathbb{R}^N,(1+|\xi|^2)^s\,d\xi)}\le 1}|\langle\mathcal{F}(T),\psi\rangle| = \sup_{\|\varphi\|_{H^s}\le 1}|\langle T,\varphi\rangle|$. Thus $\|T\|_{H^{-s}}\le C$.

12. In (a), we apply the Schwarz inequality: $\|\varphi\|_{\sup}\le\|\mathcal{F}^{-1}(\varphi)\|_1 = \|\mathcal{F}(\varphi)\|_1 = \int_{\mathbb{R}^N}|[\mathcal{F}(\varphi)(\xi)(1+|\xi|^2)^{s/2}]\,[(1+|\xi|^2)^{-s/2}]|\,d\xi \le \|T_\varphi\|_{H^s}\big(\int_{\mathbb{R}^N}\big|1+|\xi|^2\big|^{-s}\,d\xi\big)^{1/2}$.

For (b), the last integral in (a) is finite for $s>N/2$. Thus we have $\|\varphi\|_{\sup}\le C\|T_\varphi\|_{H^s}$ for all φ in $\mathcal{S}(\mathbb{R}^N)$. If T is in H^s, we know from Problem 9c that we can find a sequence φ_k in $\mathcal{S}(\mathbb{R}^N)$ such that T_{φ_k} tends to T in H^s. For $p\le q$, we then have $\|\varphi_p-\varphi_q\|_{\sup}\le C\|T_{\varphi_p}-T_{\varphi_q}\|_{H^s}$. Letting q tend to infinity, we see that φ_p converges uniformly to some function f, necessarily continuous and bounded. Let T_f be the tempered distribution given by f. We show that $T=T_f$. If ψ is in $\mathcal{S}(\mathbb{R}^N)$, then $\mathcal{F}(\psi)$ is integrable, being a Schwartz function, and the uniform convergence of φ_p to f implies that $\langle T_f,\mathcal{F}(\psi)\rangle = \lim_p\langle T_{\varphi_p},\mathcal{F}(\psi)\rangle$. On the other hand, $|\langle T_{\varphi_p}-T,\mathcal{F}(\psi)\rangle|\le\|T_{\varphi_p}-T\|_{H^s}\|\mathcal{F}(\psi)\|_{H^{-s}}$, and thus $\langle T_{\varphi_p},\mathcal{F}(\psi)\rangle$ tends to $\langle T,\mathcal{F}(\psi)\rangle$. Therefore $\langle T_f,\mathcal{F}(\psi)\rangle = \langle T,\mathcal{F}(\psi)\rangle$, and $T=T_f$.

For (c), it follows from Problem 8 that for any fixed s and any multi-index α with $|\alpha| \le s$, $\|D^\alpha\varphi\|_{H^{s-|\alpha|}} \le C_\alpha \|\varphi\|_{H^s}$ for all φ in $\mathcal{S}(\mathbb{R}^N)$. Fix m with $s > N/2 + m$, and let T be in H^s. By Problem 9c we can choose a sequence φ_k in $\mathcal{S}(\mathbb{R}^N)$ such that T_{φ_k} tends to T in H^s. For $p \le q$ and for each α with $|\alpha| \le m$, (a) shows that $\|D^\alpha\varphi_p - D^\alpha\varphi_q\|_{\sup} \le C'_\alpha \|T_{D^\alpha\varphi_p} - T_{D^\alpha\varphi_q}\|_{H^{s-|\alpha|}} \le C'_\alpha \|T_{D^\alpha\varphi_p} - T_{D^\alpha\varphi_q}\|_{H^{s-m}}$. Letting q tend to infinity, we see that $D^\alpha\varphi_p$ converges uniformly to some function f_α, necessarily continuous and bounded. By the theorem on interchange of limit and derivative, $f_0 = \lim_p D^0\varphi_p$ is of class C^m with $f_\alpha = D^\alpha f_0$ for all α with $|\alpha| \le m$. Then we can argue as in (b) to see that $T = T_f$, and (c) is proved.

13. In (a), $P_y * (u_0 + iHu_0)(x) = P_y * u_0(x) + iQ_y * u_0(x) = \frac{i\bar z}{\pi|z|^2} * u_0(x) = ((-i\pi z)^{-1}) * u_0(x)$. The left side is in $\mathcal{H}^p$ since H is bounded on L^p, and the form of the right side shows that the result is analytic in the upper half plane. Hence the expression is in H^p.

In (b), we know that $f(x+iy) = P_y * u_0(x) + iQ_y * u_0(x) = P_y * u_0(x) + iP_yHu_0(x)$. Taking the L^p limit as $y \downarrow 0$, we obtain $f_0 = u_0 + iHu_0$. Hence iHu_0 is the imaginary part of f_0.

14. According to the previous problem, the functions in H^2 are those of the form $P_y * (u_0 + iHu_0)$ with u_0 in L^2. That is, they are the functions of the form $u_0 + iHu_0$ with u_0 in L^2. The operator H acts on the Fourier transform side by multiplication by $-i \operatorname{sgn} x$. Hence the Fourier transforms of the functions of interest are all expressions $\widehat{u}_0(x) + i(-i\operatorname{sgn} x)\widehat{u}_0(x)$ a.e. This function is $2\widehat{u}_0(x)$ for $x > 0$ and is 0 for $x < 0$. Conversely any function in L^2 is the Fourier transform of an L^2 function, and thus if g is given that vanishes a.e. for $x < 0$, we can find u_0 with $\widehat{u}_0 = \frac{1}{2}g$. Then $\widehat{u}_0 + i(-i\operatorname{sgn} x)\widehat{u}_0 = g$.

15. The first inequality is by the Schwarz inequality, and the second inequality is evident. For the equality we make the calculation

$$\begin{aligned}
\Delta(|F|^q) &= 4\tfrac{\partial}{\partial\bar z}\tfrac{\partial}{\partial z}(|F|^2)^{q/2} = 2q\tfrac{\partial}{\partial\bar z}[(|F|^2)^{\frac{q}{2}-1}\tfrac{\partial}{\partial z}(F,F)] \\
&= 2q\tfrac{\partial}{\partial\bar z}[(|F|^2)^{\frac{q}{2}-1}(F',F)] \\
&= q(q-2)(|F|^2)^{\frac{q}{2}-2}(F,F')(F',F) + 2q(|F|^2)^{\frac{q}{2}-1}(F',F') \\
&= q^2|F|^{q-4}|(F,F')|^2 - 2q|F|^{q-4}|(F,F')|^2 + 2q|F|^{q-2}|F'|^2 \\
&= q^2|F|^{q-4}|(F,F')|^2 + 2q|F|^{q-4}\big(-|(F,F')|^2 + |F|^2|F'|^2\big).
\end{aligned}$$

16. Arguing by contradiction, suppose that $u(x_1) > 0$ with $|x_1 - x_0| < r$. For any $c > 0$, the function $v_c(x) = u(x) + c(|x - x_0|^2 - r^2)$ has $\Delta v_c > 0$ on $B(r; x_0)$ and $v = u \le 0$ on $\partial B(r; x_0)$. We can choose the positive number c sufficiently small so that $v_c(x_1) > 0$. Fix that c, and choose x_2 in $B(r; x_0)^{\mathrm{cl}}$ where v_c is a maximum. Then x_2 is in $B(r; x_0)$, and all the first partial derivatives of v_c must be 0 there. Since $\Delta v_c(x_2) > 0$, we must have $D_j^2 v_c(x_2) > 0$ for some j, and then the presence of a maximum for $v - x$ at x_2 contradicts the second derivative test.

17. For (a), we calculate $\|g_\varepsilon\|_2^2 = \int_{\mathbb{R}} |g_\varepsilon(x)|^2\,dx = \int_{\mathbb{R}} |F_\varepsilon(x)|\,dx \leq \int_{\mathbb{R}} |f(x+i\varepsilon)|\,dx + \varepsilon \int_{\mathbb{R}} |x+i|^{-2} \leq \|f\|_{H^1} + \varepsilon\|(x+i)^{-2}\|_1$.

In (b), the functions $x \mapsto g_\varepsilon(x+iy)$ and $x \mapsto F_\varepsilon(x+iy)$ are Poisson integrals of the functions with y replaced by $y/2$, and then are iterated Poisson integrals in passing from $y/2$ to $3y/4$ and to y. In the first case the starting function is in L^2, and in the second case the starting function is in L^1. The function at $3y/4$ is then in L^2 since $L^1 * L^2 \subseteq L^2$, and the function at y is continuous vanishing at infinity since $L^2 * L^2 \subseteq C_0(\mathbb{R})$. This handles the dependence for large x. For large y, we refer to the proof of Theorem 3.25, where we obtained the estimate $|u(x,t)|^p \leq [(\tfrac{1}{2}t_0)^{N+1}\Omega_1]^{-1}(N+1)t_0\|u\|_{\mathcal{H}^p}^p$ if u is in $\mathcal{H}^p$ and $t \geq t_0$.

In (c), the functions $|F_\varepsilon(z)|^{1/2}$ and $g_\varepsilon(z)$ are equal for $z = x$. Hence the continuous function $u(z) = |F_\varepsilon(z)|^{1/2} - g_\varepsilon(z)$ on $\mathbb{R}^2_+$ vanishes at $y = 0$ and tends to 0 as $|x|+|y|$ tends to infinity. Given $\delta > 0$, choose an open ball B large enough in $\mathbb{R}^2_+$ so that $u(z) \leq \delta$ off this ball. Since the second component of $F_\varepsilon(z)$ is nowhere vanishing, $|F_\varepsilon(z)|^{1/2}$ is everywhere smooth for $y > 0$. Problem 15 shows that $\Delta(|F_\varepsilon(z)|^{1/2}) \geq 0$, and we know that $\Delta g_\varepsilon(z) = 0$ since g_ε is a Poisson integral. Hence $\Delta u(z) \geq 0$. Applying Problem 16 on the ball B, we see that $u(z) \leq \delta$ on B. Hence $u(z) \leq \delta$ on $\mathbb{R}^2_+$. Since δ is arbitrary, $u(z) \leq 0$ on $\mathbb{R}^2_+$. Therefore $|F_\varepsilon(z)|^{1/2} \leq g_\varepsilon(z)$ on $\mathbb{R}^2_+$.

18. In (a), the fact that P_y is in L^2 implies that $\lim_n \int_{\mathbb{R}} P_y(x-t)g_{\varepsilon_n}(t)\,dt = \int_{\mathbb{R}} P_y(x-t)g(t)\,dt$. Thus $g_{\varepsilon_n}(z) \to g(z)$ pointwise for $\operatorname{Im} z > 0$. Then we have $|f(z)|^{1/2} \leq \limsup_n |f(z+i\varepsilon_n)|^{1/2} \leq \limsup_n g_\varepsilon(z) = g(z)$. Since $g(z)$ is the Poisson integral of $g(x)$, the inequality $g(x+iy) \leq Cg^*(x)$ is known from the given facts at the beginning of this group of problems.

In (b), we have $|f(x+iy)| \leq C^2 g^*(x)^2$, and we know that $\|g^*\|_2 \leq A_2\|g\|_2$. From Problem 17a we have $\|g\|_2^2 \leq \limsup_n \|g_{\varepsilon_n}\|_2^2 \leq \limsup_n \big(\|f\|_{H^1} + \varepsilon\|(x+i)^{-2}\|_1\big) = \|f\|_{H^1}$.

19. Every f in $C_{\text{com}}(X)$ has $\big|\int_X f(x)\,d\nu(x)\big| = \lim_n \big|\int_X f(x)g_n(x)\,d\mu(x)\big| \leq \limsup_n \int_X |f(x)||g_n(x)|\,d\mu(x) \leq \int_X |f(x)|\,d\mu(x)$. If K is compact in X, we can find a sequence $\{f_k\}$ of functions ≥ 0 in $C_{\text{com}}(X)$ decreasing pointwise to the indicator function of K, and dominated convergence implies that $\big|\int_K d\nu(x)\big| \leq \int_K d\mu(x)$. In other words, $|\nu(K)| \leq \mu(K)$. Separating the real and imaginary parts of ν and then working with subsets of a maximal positive set for ν and a maximal negative set for ν, we reduce to the case that $\nu \geq 0$. Since ν is automatically regular, we obtain $\nu(E) \leq \mu(E)$ for all Borel sets E, and the absolute continuity follows.

20. Since f is in H^1, it is in $\mathcal{H}^1$ and hence is the Poisson integral of a finite complex Borel measure ν, and the complex measures $f(x+i/n)\,dx$ converge weak-star against $C_{\text{com}}(\mathbb{R})$ to ν. Meanwhile, we have $|f(x+i/n)| \leq C^2 g^*(x)^2$ for all n. In Problem 19 take $d\mu(x) = C^2 g^*(x)^2\,dx$. Then the complex measures $f(x+i/n)[C^2 g^*(x)^2]^{-1}\,d\mu(x)$ converge weak-star to ν. Problem 19 shows that ν is absolutely continuous with respect to $C^2 g^*(x)^2\,dx$. Hence ν is absolutely continuous with respect to Lebesgue measure.

21. For (a), $\mathcal{F}(T\varphi)$ is the product of an L^∞ function and a Schwartz function. The rapid decrease of the Fourier transform translates into the existence of derivatives of all orders for the function itself. Hence Φ is locally bounded.

For (b), any x with $|x| \geq 1$ has

$$\Phi(x) = \lim_{y\downarrow 0} \int_{|y|\geq\varepsilon} \big(\tfrac{K(x-y)}{|x-y|^N} - \tfrac{K(x)}{|x|^N}\big)\varphi(y)\,dy.$$

Hence $|\Phi(x)|$ is

$$\leq \limsup_{y\downarrow 0} \int_{|y|\geq\varepsilon} \varphi(y)|K(x-y)|\big|\tfrac{1}{|x-y|^N} - \tfrac{1}{|x|^N}\big|\,dy + \int_{\mathbb{R}^N} \varphi(y)\,\tfrac{|K(x-y)-K(x)|}{|x|^N}\,dy.$$

If $|x| \geq 2|y|$ for all y in the support of φ, two estimates in the text are applicable; these appear in the proof that the hypotheses of Lemma 3.29 are satisfied:

$$\big|\tfrac{1}{|x-y|^N} - \tfrac{1}{|x|^N}\big| \leq N3^N \tfrac{|y|}{|x|^{N+1}} \qquad \text{and} \qquad |K(x-y) - K(x)| \leq \psi\big(\tfrac{2|y|}{|x|}\big).$$

The smoothness of K makes $\psi(t) \leq Ct$ for small positive t. Since the y's in question are all in the compact support of φ, both terms are bounded by multiples of $|x|^{-(N+1)}$.

Conclusion (c) is immediate from (a) and (b).

22. Part (a) is just a matter of tracking down the effects of dilations. Part (c) follows by dilating $\Phi = T\varphi - k$ to obtain $\Phi_\varepsilon = (T\varphi)_\varepsilon - k_\varepsilon$, by applying (a) to write $\Phi_\varepsilon = T\varphi_\varepsilon - k_\varepsilon$, by convolving with f, and by applying (b). Thus we have to prove (b).

For (b), we have $\varphi_\varepsilon * Tf = \varphi_\varepsilon * (\lim_\delta T_\delta f)$. The limit is in L^p, and convolution by the $L^{p'}$ function φ_ε is bounded from L^p to L^∞. Therefore $\varphi_\varepsilon * (\lim_\delta T_\delta f)$ equals $\lim_\delta(\varphi_\varepsilon * (T_\delta f)) = \lim_\delta(\varphi_\varepsilon * (k_\delta * f))$. This is equal to $\lim_\delta((\varphi_\varepsilon * k_\delta) * f) = \lim_\delta((T_\delta\varphi_\varepsilon) * f)$ since φ_ε is in L^1. Finally we can move the limit inside since $\lim_\delta T_\delta\varphi_\varepsilon$ can be considered as an $L^{p'}$ limit and f is in L^p.

23. From (c), we have $\sup_{\varepsilon>0} |T_\varepsilon f(x)| = \sup_{\varepsilon>0} |k_\varepsilon * f(x)| \leq \sup_{\varepsilon>0} |\Phi_\varepsilon * f(x)| + \sup_{\varepsilon>0} |\varphi_\varepsilon * (Tf)(x)| \leq C_\Phi f^*(x) + C_\varphi (Tf)^*(x)$, where C_Φ and C_φ are as in the given facts at the beginning of this group of problems.

24. Taking L^p norms in the previous problem and using Theorem 3.26 and the known behavior of Hardy–Littlewood maximal functions, we obtain

$$\begin{aligned}\big\|\sup_{\varepsilon>0} |T_\varepsilon f(x)|\big\|_p &\leq C_\Phi \|f^*\|_p + C_\varphi \|(Tf)^*\|_p \leq C_\Phi A_p \|f\|_p + C_\varphi A_p \|Tf\|_p \\ &\leq C_\Phi A_p \|f\|_p + C_\varphi A_p C_p \|f\|_p = C\|f\|_p,\end{aligned}$$

where A_p and C_p are constants such that $\|f^*\|_p \leq A_p\|f\|_p$ and $\|Tf\|_p \leq C_p\|f\|_p$. We know that $\lim_{\varepsilon>0} T_\varepsilon f(x)$ exists pointwise for f in the dense set $C^\infty_{\text{com}}(\mathbb{R}^N)$, and a familiar argument uses the above information to give the existence of the pointwise limit almost everywhere for all f in L^p.

25. This follows from the same argument as for Proposition 3.7.

26. Fix $\psi \geq 0$ in $C^\infty_{\text{com}}(\mathbb{R}^N)$ with integral 1, and define $\psi_\varepsilon(x) = \varepsilon^{-N}\psi(\varepsilon^{-1}x)$. If f is in $L^2_k(T^N)$, then $\psi_\varepsilon * f$ is smooth and periodic, hence is in $C^\infty(T^N)$. Suppose it is proved that

$$D^\alpha(\psi_\varepsilon * f) = \psi_\varepsilon * D^\alpha f \qquad \text{for } |\alpha| \leq k. \tag{$*$}$$

If we let η be the indicator function of $[-2\pi, 2\pi]^N$, then Proposition 3.5a shows that $\lim_{\varepsilon\downarrow 0} \|\eta(\psi_\varepsilon * D^\alpha f - D^\alpha f))\|_2 = 0$ for $|\alpha| \leq k$, and then $(*)$ shows that $\lim_{\varepsilon\downarrow 0} \|\eta(D^\alpha(\psi_\varepsilon * f) - D^\alpha f)\|_2 = 0$. Hence $\lim_{\varepsilon\downarrow 0} \|\psi_\varepsilon * f - f\|_{L^2_k(T^N)} = 0$.

For $(*)$, the critical fact is that the smooth function $\psi * f$ is periodic. If φ is periodic and ψ_ε is supported inside $[-\pi, \pi]^N$, then

$$\begin{aligned}
\int_{[-\pi,\pi]^N} \big(\psi_\varepsilon * D^\alpha f(x)\big)\varphi(x)\,dx &= \int_{[-\pi,\pi]^N}\int_{[-\pi,\pi]^N} \psi_\varepsilon(y) D^\alpha f(x-y)\varphi(x)\,dy\,dx \\
&= \int_{[-\pi,\pi]^N}\int_{[-\pi,\pi]^N} \psi_\varepsilon(y) D^\alpha f(x-y)\varphi(x)\,dx\,dy \\
&= (-1)^{|\alpha|}\int_{[-\pi,\pi]^N}\int_{[-\pi,\pi]^N} \psi_\varepsilon(y) f(x-y) D^\alpha\varphi(x)\,dx\,dy \\
&= (-1)^{|\alpha|}\int_{[-\pi,\pi]^N} (\psi_\varepsilon * f)(x) D^\alpha\varphi(x)\,dx \\
&= \int_{[-\pi,\pi]^N} (D^\alpha(\psi_\varepsilon * f))\varphi\,dx,
\end{aligned}$$

and $(*)$ follows.

27. We have

$$\begin{aligned}
\|D^\alpha f\|^2_{L^2_k(T^N)} &= \textstyle\sum_{|\beta|\leq k} (2\pi)^{-N} \int_{[-\pi,\pi]^N} |D^\beta D^\alpha f|^2\,dx \\
&= \textstyle\sum_{|\beta|\leq k} (2\pi)^{-N} \int_{[-\pi,\pi]^N} |D^{\alpha+\beta} f|^2\,dx \\
&\leq \textstyle\sum_{|\gamma|\leq k+|\alpha|} (2\pi)^{-N} \int_{[-\pi,\pi]^N} |D^\gamma f|^2\,dx \\
&= \|f\|^2_{L^2_{k+|\alpha|}(T^N)}.
\end{aligned}$$

Thus we can take $C_{\alpha,k} = 1$.

28. For each α, we have $(2\pi)^{-N}\int_{[-\pi,\pi]^N} |D^\alpha f|^2\,dx \leq (\sup_{x\in[-\pi,\pi]^N} |D^\alpha f(x)|)^2$. Summing for $|\alpha| \leq k$ gives

$$\|f\|^2_{L^2_k(T^N)} \leq \textstyle\sum_{|\alpha|\leq k} (\sup_{x\in[-\pi,\pi]^N} |D^\alpha f(x)|)^2,$$

and the right side is $\leq (\sum_{|\alpha|\leq k} \sup_{x\in[-\pi,\pi]^N} |D^\alpha f(x)|)^2$. Thus we can take $A_k = 1$.

29. Since $l_j^2 \leq |l|^2$, we have $l^{2\alpha} \leq (|l|^2)^{|\alpha|} \leq (1+|l|^2)^k$, and the left inequality of the problem follows with B_k equal to the reciprocal of the number of α's with $|\alpha| \leq k$. For the right inequality, we have $1 + |l|^2 = \sum_{|\alpha|\leq 1} l^{2\alpha}$. Raising both sides to the k^{th} power gives the desired result once the right side is expanded out since $l^{2\alpha} l^{2\beta} = l^{2(\alpha+\beta)}$.

30–31. For f in $C^\infty(T^N)$, let f have Fourier coefficients c_l. The l^{th} Fourier coefficient of $D^\alpha f$ is then $i^{|\alpha|}l^\alpha c_l$, and hence $\|D^\alpha f\|_2^2 = \sum_l |c_l|^2 l^{2\alpha}$. Consequently $\|f\|_{L^2_k(T^N)} = \sum_l |c_l|^2\big(\sum_{|\alpha|\le k} l^{2\alpha}\big)$. Then the estimate required for Problem 31 in the case of functions in $C^\infty(T^N)$ is immediate from the inequalities of Problem 29.

Problem 26 shows that $C^\infty(T^N)$ is dense in $L^2_k(T^N)$. Let f be given in $L^2_k(T^N)$, and choose $f^{(n)}$ in $C^\infty(T^N)$ convergent to f in $L^2_k(T^N)$. Since $f^{(n)}$ tends to f in L^2, the Fourier coefficients $c_l^{(n)}$ of $f^{(n)}$ tend to those c_l of f for each l. Applying Problem 29 to each $f^{(n)}$ and using Fatou's Lemma, we obtain $\sum_l |c_l|^2(1+|l|^2)^k \le C_k\|f\|^2_{L^2_k(T^N)}$. On the other hand, if f is given in $L^2_k(T^N)$ with Fourier coefficients c_l, then we can put $f^{(n)}(x) = \sum_{|l|\le n} c_l e^{il\cdot x}$. Since $f^{(n)}$ is given by a finite sum and since $D^\alpha f(x) = \sum_l c_l l^\alpha e^{il\cdot x}$ in the L^2 sense for $|\alpha| \le k$, we see that $f^{(n)}$ converges to f in $L^2_k(T^N)$. The left inequality of Problem 31 holds for each $f^{(n)}$ since $f^{(n)}$ is in $C^\infty(T^N)$, and the expression in the middle of that inequality for $f^{(n)}$ is $\le$ the corresponding expression for f. Passing to the limit, we obtain the left inequality of Problem 31 for f.

This settles Problem 31. It shows also that if f is in $L^2_k(T^N)$, then we have $\sum_l |c_l|^2(1+|l|^2)^k < \infty$. On the other hand, if this sum is finite, then we define $f^{(n)}$ to be $\sum_{|l|\le n} c_l e^{il\cdot x}$. Problem 31 gives us $B_k\|f^{(n)}\|^2_{L^2_k(T^N)} \le \sum_l |c_l|^2(1+|l|^2)^k$ for each n. Each $D^\alpha f^{(n)}$ for $|\alpha| \le k$ is convergent to something in L^2, and the completeness of $L^2_k(T^N)$ proved in Problem 25 shows that $f^{(n)}$ converges to something in $L^2_k(T^N)$. Consideration of Fourier coefficients shows that the limit function must be f. Hence f is in $L^2_k(T^N)$.

32. Put $c = K/N > 1/2$. Term by term we have $\sum_{l\in\mathbb{Z}^N}(1+|l|^2)^{-(N+1)/2} \le \sum_{l_1\in\mathbb{Z}}\cdots\sum_{l_N\in\mathbb{Z}}(1+l_1^2)^{-c}\cdots(1+l_N^2)^{-c} = \prod_{j=1}^N\big(\sum_{m\in\mathbb{Z}}(1+m^2)^{-c}\big)$, and the right side is finite since $c > 1/2$. This proves convergence of the sum.

Now suppose that f is in $L^2_K(T^N)$, and suppose that f has Fourier coefficients c_l. Problem 31 shows that $\sum_l |c_l|^2(1+|l|^2)^K < \infty$. The Schwarz inequality gives

$$\begin{aligned}\textstyle\sum_l |c_l| &= \textstyle\sum_l |c_l(1+|l|^2)^{K/2}|(1+|l|^2)^{-K/2}\\ &\le \big(\textstyle\sum_l |c_l|^2(1+|l|^2)^K\big)^{1/2}\big(\textstyle\sum_l (1+|l|^2)^{-K}\big)^{1/2},\end{aligned}$$

and we conclude that $\sum|c_l| < \infty$. Therefore the partial sums of the Fourier series of f converge to a continuous function. This continuous function has to match the L^2 limit almost everywhere, and the latter is f.

33. Let c_l be the Fourier coefficients of f. If f is in $L^2_K(T^N)$ with $K > N/2$, then Problem 32 shows that f is continuous and is given pointwise by the sum of its Fourier series. The inequalities in the solution for that problem show that $|f(x)| \le \sum_l |c_l| \le A_K\big(\sum_l |c_l|^2(1+|l|^2)^{-K}\big)^{1/2}$. In turn, Problem 31 shows that the right side is $\le A_K C_k^{1/2}\|f\|_{L^2_K(T^N)}$. This gives the desired estimate for $\alpha = 0$

with $m(0) = K$ for any integer K greater than $N/2$. Combining this estimate with the result of Problem 27, we obtain an inequality for all α, with $m(\alpha) = K + |\alpha|$ and $C_\alpha = A_K C_K^{1/2}$.

34. The comparisons of size are given in Problems 28 and 33. These comparisons establish the uniform continuity of the identity map in both directions, by the proof of Proposition 3.2. (The statement of the proposition asserts only continuity.)

Chapter IV

1. With the explicit definition of the norm topology on X/Y, we have $\|x + Y\| \le \|x\|$, and consequently the quotient mapping $q : X \to X/Y$ is continuous onto the normed X/Y. Because of completeness the Interior Mapping Theorem applies and shows that the quotient mapping carries open sets to open sets. Consequently a subset E of X/Y in the norm topology is open if and only if $q^{-1}(E)$ is open. This is the same as the defining condition for a subset of X/Y to be open in the quotient topology, and hence the topologies match.

2. Let $K = \ker(T)$, and let $q : X \to X/K$ be the quotient map. By linear algebra the map $T : X \to Y$ induces a one-one linear map $T' : X/K \to Y$, and then $T = T' \circ q$. Since K is closed in X, Proposition 4.4 shows that X/K is a topological vector space. Since $T(X)$ is finite dimensional and T' is one-one, X/K is finite dimensional. Proposition 4.5 implies that T' is continuous. Since T is the composition of continuous maps, it is continuous.

3. Let $T : X \to Y$ be a continuous linear map from one Banach space onto another, and let $K = \ker T$. As in Problem 2, write $T = T' \circ q$, where $q : X \to X/K$ is the quotient mapping. Here T' is one-one. Since a subset E of X/K is open if and only if $q^{-1}(E)$ is open, T' is continuous. Problem 1 shows that the topology on X/K comes from a Banach space structure. By the assumed special case of the Interior Mapping Theorem, T' carries open sets to open sets. Therefore the composition T carries open sets to open sets.

4. This follows from Proposition 4.5.

5. Take x_n to be the n^{th} member of an orthonormal basis. Then $\|x_n\| = 1$ for all n. Any u in H has an expansion $u = \sum_{n=1}^{\infty} c_n x_n$, convergent in H, with $c_n = (u, x_n)$ and $\sum |c_n|^2 < \infty$. Then $\{(u, x_n)\}$ tends to 0 for each u, and $\{x_n\}$ therefore tends to 0 weakly.

6. The weak convergence implies that $\lim_n (f_n, f) = (f, f) = \|f\|^2$. Therefore $\|f_n - f\|^2 = \|f_n\|^2 - 2\,\mathrm{Re}(f_n, f) + \|f\|^2$ tends to $\|f\|^2 - 2\|f\|^2 + \|f\|^2 = 0$.

7. Let the dense subset of X^* be D. For x^* in X^* and y^* in D, we have

$$\begin{aligned}|x^*(x_n) - x^*(x_0)| &\le |(x^* - y^*)(x_n)| + |y^*(x_n) - y^*(x_0)| + |(y^* - x^*)(x_0)| \\ &\le \|x^* - y^*\|\|x_n\| + |y^*(x_n) - y^*(x_0)| + \|x^* - y^*\|\|x_0\| \\ &\le (C + \|x_0\|)\|x^* - y^*\| + |y^*(x_n) - y^*(x_0)|,\end{aligned}$$

where $C = \sup_n \|x_n\|$. Given $x^* \in X^*$ and $\epsilon > 0$, choose y^* in D to make the first term on the right be $< \epsilon$, and then choose n large enough to make the second term $< \epsilon$.

8. For (a), let $D(f) = 1$. Then $t \mapsto \int_{[0,t]} |f|^p\,dx$ is a continuous nondecreasing function on $[0, 1]$ that is 0 at $t = 0$ and is 1 at $t = 1$. Therefore there exists a partition $0 = a_0 < a_1 < \cdots < a_n = 1$ of $[0, 1]$ such that $\int_{[0,a_j]} |f|^p\,dx = j/n$ for $0 \le j \le n$. If f_j for $j \ge 1$ is the product of n and the indicator function of $[a_{j-1}, a_j]$, then $D(f_j) = \frac{1}{n} n^p = n^{-(1-p)}$, and $f = \frac{1}{n}(f_1 + \cdots + f_n)$.

For (b), let $g_j = cf_j$ in (a), so that $D(g_j) = |c|^p D(f_j) = |c|^p n^{-(1-p)}$. If we put $c = n^{(1-p)/p}$, then $D(g_j) = 1$. Thus we obtain the expansion $n^{(1-p)/p} f = \frac{1}{n}(g_1 + \cdots + g_n)$ with $D(g_j) = 1$ for each j. Since $D(n^{(1-p)/p} f) = n^{1-p} D(f) = n^{1-p}$, the multiple $n^{(1-p)/p} f$ of f is a convex combination of functions h with $D(h) \le 1$. Taking a convex combination of 0 and this multiple of f shows that rf is a convex combination of functions h with $D(h) \le 1$ if $0 \le r \le n^{(1-p)/p}$. Since $\sup_n n^{(1-p)/p} = +\infty$, every nonnegative multiple of f is a convex combination of functions h with $D(h) \le 1$.

For (c), we scale the result of (b). The smallest convex set containing all functions $\varepsilon^{1/p} h$ with $D(h) \le 1$ contains all nonnegative multiples of f. Since $D(\varepsilon^{1/p} h) = \varepsilon D(h)$, the smallest convex set containing all functions k with $D(k) \le \varepsilon$ contains all nonnegative multiples of f. Since f is arbitrary, this convex set is all of $L^p([0, 1])$.

For (d), the sets where $D(f) \le \varepsilon$ form a local neighborhood base at 0. Thus if $L^p([0, 1])$ were locally convex, then any convex open set containing 0 would have to contain, for some $\varepsilon > 0$, the set of all f with $D(f) \le \varepsilon$. But the only convex set containing all f with $D(f) \le \varepsilon$ is all of $L^p([0, 1])$ by (c). Hence $L^p([0, 1])$ is not locally convex.

For (e), suppose that ℓ is a continuous linear functional on $L^p([0, 1])$. Then we can find some $\varepsilon > 0$ such that $D(f) < \varepsilon$ implies $\operatorname{Re} \ell(f) < 1$. The set of all f where $\operatorname{Re} \ell(f) < 1$ is a convex set, and it contains the set of all f with $D(f) < \varepsilon$. But we saw in (c) that the only such convex set is $L^p([0, 1])$ itself. Therefore $\operatorname{Re} \ell(f) < 1$ for all f in $L^p([0, 1])$. Using scalar multiples, we see that $\operatorname{Re} \ell(f) = 0$ for all f. Therefore $\ell(f) = 0$, and the only continuous linear functional ℓ on $L^p([0, 1])$ is $\ell = 0$.

9. In (a), if φ is compactly supported in K_{p_0}, then $\varepsilon_p^{-1} \sup_{x \notin K_p} \sup_{|\alpha| \le m_p} |D^\alpha \varphi(x)|$ is 0 for $p \ge p_0$. Thus $\|\varphi\|_{m,\varepsilon}$ is a supremum for $p < p_0$ of finitely many expressions that are each finite for any smooth function on U. Hence $\|\varphi\|_{m,\varepsilon}$ is finite. Conversely if φ is not compactly supported, then the expressions $s_p = \sup_{x \notin K_p} |\varphi(x)|$ have $0 < s_p \le \infty$ for all p. If we define the sequence ε by $\varepsilon_p = \min(p^{-1}, s_p)$, then ε_p decreases to 0 and every sequence m has $\|\varphi\|_{m,\varepsilon} \ge \varepsilon_p^{-1} \sup_{x \notin K_p} |\varphi(x)| \ge p$ for all p. Since p is arbitrary, $\|\varphi\|_{m,\varepsilon} = \infty$.

For (b), we have only to show that the inclusion of $C^\infty_{K_p}$ into $(C^\infty_{\text{com}}(U), \mathcal{T}')$ is continuous for every p. If (m, ε) is given, we are to find an open neighborhood of 0 in

$C^\infty_{K_p}$ such that $\|\varphi\|_{m,\varepsilon} < 1$ for all φ in this neighborhood. Put $M = \max(m_1, \ldots, m_p)$ and $\delta = \min(\varepsilon_1, \ldots, \varphi_p)$. If φ is supported in K_p and $\sup_{x\in K_p} \sup_{|\alpha|\le M} |D^\alpha \varphi(x)| < \delta$, then $\varepsilon_r^{-1} \sup_{x\notin K_r} \sup_{|\alpha|\le m_r} |D^\alpha \varphi(x)|$ is 0 for $r \ge p$ and is < 1 for $r < p$. Therefore its supremum on r, which is $\|\varphi\|_{m,\varepsilon}$, is < 1.

For (c), define $m_p = \max\{p, n_1, \ldots, n_p\}$ for each p, and then $\{m_p\}$ is monotone increasing and tends to infinity. Next choose C_p for each p by the compactness of the support of ψ_p and the use of the Leibniz rule on $\psi_p\eta$ so that whenever $|D^\alpha\eta(x)| \le c$ for some $\eta \in C^\infty(U)$, all $x \notin K_p$, and all α with $|\alpha| \le m_p$, then $2^{p+1}|D^\alpha(\psi_p\eta)(x)| \le C_p c$ for that η, all $x \in U$, and all α with $|\alpha| \le m_p$. Choose ε_p to be $< \delta_p/C_p$ and to be such that $\{\varepsilon_p\}$ is monotone decreasing and has limit 0. If $\|\varphi\|_{m,\varepsilon} < 1$, then $\sup_{x\notin K_p} \sup_{|\alpha|\le m_p} |D^\alpha\varphi(x)| < \varepsilon_p$ for all p. Taking $\eta = \varphi$ in the definition of C_p, we see that $\sup_{x\in U} \sup_{|\alpha|\le m_p} 2^{p+1}|D^\alpha(\psi_p\varphi)(x)| \le C_p\varepsilon_p < \delta_p$. Since $\psi_p\varphi$ is in $C^\infty_{K_{p+3}}$ and $m_p \ge n_p$, we see that $2^{p+1}\psi_p\varphi$ meets the condition for being in $N \cap C^\infty_{K_{p+3}}$.

For (d), we see from (c) that $2^{p+1}\psi_p\varphi$ is in N for all $p \ge 0$. The expansion $\varphi = \sum_{p\ge 0} 2^{-(p+1)}(2^{p+1}\psi_p\varphi)$ is a finite sum since φ has compact support, and it therefore exhibits φ as a convex combination of the 0 function and finitely many functions $2^{p+1}\psi_p\varphi$, each of which is in N. Since N is convex, φ is in N. This proves the asserted continuity.

For (e), each vector subspace $C^\infty_{K_p}$ is closed nowhere dense, and the union of these subspaces is all of $C^\infty_{\text{com}}(U)$.

10. Disproof: The answer is certainly independent of H, and we can therefore specialize to $H = L^2([0, 1])$. The multiplication algebra by $L^\infty([0, 1])$ is isometric to a subalgebra of $\mathcal{B}(H, H)$ and is not separable. Therefore $\mathcal{B}(H, H)$ is not separable.

11. Certainly $\mathcal{A}' \supseteq \mathcal{M}(L^2(S, \mu))$. Let T be in $\mathcal{A}'$, and put $g = T(1)$. For f continuous, $Tf = T(f1) = TM_f1 = M_fT1 = M_fg = fg = gf$. If we can prove that g is in $L^\infty(S, \mu)$, then T and M_g will be bounded operators equal on the dense subset $C(S)$ of $L^2(S, \mu)$ and therefore equal everywhere. Let $E_N = \{x \mid N \le |g(x)| \le N + 1\}$, and suppose that $\mu(E_N) > 0$. We shall derive an upper bound for N. Choose a compact set $K_N \subseteq E_N$ with $\mu(K_N) > 0$. Then choose f in $C(S)$ with values in $[0, 1]$ such that $f \ge 1$ on K_N and $\int_S f\,d\mu \le 2\mu(K_N)$. Then $\int_S |gf|^2\,d\mu \ge \int_{K_N} |gf|^2\,d\mu = \int_{K_N} |g|^2\,d\mu \ge N^2\mu(K_N)$. Also, $\int_S |f|^2\,d\mu \le \int_S f\,d\mu \le 2\mu(K_N)$ since $0 \le f \le 1$. Therefore $N\mu(K_N)^{1/2} \le \|gf\|_2 \le \|T\|\|f\|_2 \le \sqrt{2}\,\|T\|\mu(K_N)^{1/2}$, and we obtain $N \le \sqrt{2}\,\|T\|$. This gives an upper bound for N and shows that g is in $L^\infty(S, \mu)$.

12. The Spectral Theorem shows that we may assume that A is of the form M_g and acts on $H = L^2(S, \mu)$, with g in $L^\infty(S, \mu)$. Certainly we have $\sup_{\|f\|_2\le 1} |(M_gf, f)| \le \|g\|_\infty$. Let us prove the reverse inequality. Lemma 4.55 and Proposition 4.43 show that $\|g\|_\infty$ is the supremum of the numbers $|\lambda_0|$ such that λ_0 is in the essential image of M_g. For λ_0 in the essential image, fix $\epsilon > 0$ and let f_1 be the indicator function of

$g^{-1}(\{|\lambda - \lambda_0| < \epsilon\})$. Then

$$\int_S g|f_1|^2\,d\mu = \int_{|g(x)-\lambda_0|<\epsilon} g\,d\mu = \lambda_0\mu\big(|g(x)-\lambda_0| < \epsilon\big) + \int_{|g(x)-\lambda_0|<\epsilon} (g-\lambda_0)\,d\mu.$$

The last term on the right is $\leq \epsilon\mu\big(|g(x) - \lambda_0| < \epsilon\big)$ in absolute value. Hence $\int_S g|f_1|^2\,d\mu = (\lambda_0 + \zeta)\mu\big(|g(x) - \lambda_0| < \epsilon\big)$ with $|\zeta| \leq \epsilon$. Dividing by $\|f_1\|_2^2 = \mu\big(|g(x) - \lambda_0| < \epsilon\big)$ and setting $f = f_1/\|f_1\|_2$, we obtain $\big|\int_S g|f|^2\,d\mu - \lambda_0\big| \leq \epsilon$. Since ϵ is arbitrary, λ_0 is in the closure of $\big\{(M_g f, f) \bigm| \|f\|_2 = 1\big\}$. Taking the supremum over λ_0 in the essential image, we obtain $\sup_{\|f\|_2 \leq 1} |(M_g f, f)| \geq \|g\|_\infty$.

13. This is what the proof of Theorem 4.53 gives when the assumption that $\mathcal{A}$ is maximal is dropped and the cyclic vector is produced by a hypothesis rather than by Proposition 4.52.

14. Apply the previous problem. Proposition 4.63 shows that $\mathcal{A}_{\mathrm{m}}^*$ is canonically homeomorphic to $\sigma(A)$. Under this identification we want to see that UAU^{-1} is multiplication by z. Thus let $\psi : \sigma(A) \to \mathcal{A}_{\mathrm{m}}^*$ be the homeomorphism obtained from the proposition. The solution of the previous problem and the proof of Theorem 4.53 show that UAU^{-1} is multiplication by $\widehat{A}$ when we work with $\mathcal{A}_{\mathrm{m}}^*$, and it is therefore $\widehat{A}\circ\psi$ when we work with $\sigma(A)$. The defining property of ψ is that $f(z) = f\circ\widehat{A}(\psi(z))$ for $f \in C(\sigma(A))$ and $z \in \sigma(A)$. This equation for the function $f(z) = z$ says that $\widehat{A} \circ \psi(z) = z$, and hence UAU^{-1} is multiplication by z on $\sigma(A)$.

15. For (a), $\mathcal{A}$ immediately contains all M_P for arbitrary polynomials P with complex coefficients on $[0, 1]$. By the Stone–Weierstrass Theorem, $\mathcal{A}$ contains all operators M_f with f continuous on $[0, 1]$. This collection of operators is an algebra closed under adjoints and operator limits (which are the same as essentially uniform limits of the functions), and hence it exhausts $\mathcal{A}$. If we then form $\mathcal{A}1$, we obtain all continuous functions in $L^2([0, 1])$, and these are dense. Hence 1 is cyclic.

For (b), Proposition 4.63 says that the spectrum may be identified with $\sigma(M_x)$, and Lemma 4.55 shows that this is $[0, 1]$.

In (c), the system of operators M_φ satisfies conditions (a) through (d) for the system $\varphi(M_x)$ of Theorem 4.57. By uniqueness, $\varphi(M_x) = M_\varphi$ for every bounded Borel function on $[0, 1]$.

17. If $0 < \mu(S) < 1$, then μ is a nontrivial convex combination of 0 and a measure with total mass 1 and is therefore not extreme. Since 0 is evidently extreme, the problem is to identify the extreme measures among those with total mass 1. If μ is given with $\mu(S) = 1$ and if some Borel set E has $0 < \mu(E) < 1$, define $\mu_1(A) = \mu(E)^{-1}\mu(E \cap A)$ and $\mu_2 = \mu(E^c)^{-1}\mu(E^c \cap A)$. Then μ_1 and μ_2 have total mass 1, and the equality $\mu = \mu(E)\mu_1 + \mu(E^c)\mu_2$ shows that μ is not extreme.

Thus we may assume that μ takes on only the values 0 and 1. In this case the regularity of μ implies that μ is a point mass, as is shown in Problem 6 of Chapter XI of *Basic*.

18. For (a), we have $f = (1 - t)\|f_1\|_1^{-1} f_1 + t\|f_2\|_1^{-1} f_2$ with $t = \|f_2\|_1$. For (b), we observe for any f in $L^1([0, 1])$ with $\|f\|_1 = 1$ that $t \mapsto \int_{[0,t]} |f|\,dx$ is

continuous on $[0,1]$, is 0 at $t=0$, and is 1 at $t=1$. Therefore there exists some t_0 with $\int_{[0,t_0]} |f|\,dx = \frac{1}{2}$. The set $E = [0,t_0]$ is then a set to which we can apply (a) to see that f is not an extreme point of the closed unit ball.

19. For the compactness of K in (a), we are to show that the set of invariant measures is closed. Such measures μ have $\int_S f\,d\mu = \int_S (f\circ F)\,d\mu$ for all $f \in C(S)$. If we have a net $\{\mu_n\}$ of such measures convergent weak-star to μ, then we can pass to the limit in the equality for each μ_n and obtain $\int_S f\,d\mu = \int_S (f\circ F)\,d\mu$ for the limit μ since f and $f\circ F$ are both continuous. If we define $\nu(E) = \mu(F^{-1}(E))$, this equality says that $\int_S f\,d\mu = \int_S f\,d\nu$ for every $f\in C(S)$. By the uniqueness in the Riesz Representation Theorem, $\mu=\nu$. Therefore the limit μ is invariant under F.

In (b), if μ could be extreme but not ergodic, we could find a Borel set E with $0<\mu(E)<1$ such that $F(E)=E$. Put $\mu_1(A) = \mu(E)^{-1}\mu(A\cap E)$ and $\mu_2(A) = \mu(E^c)^{-1}\mu(A\cap E^c)$. The invariance of the set E implies that μ_1 and μ_2 are invariant. Since $\mu = \mu(E)\mu_1 + \mu(E^c)\mu_2$, μ is exhibited as a nontrivial convex combination of invariant measures and cannot be extreme.

For (c), the answer is "no." Take S to be a two-point set with the discrete topology, and let F interchange the two points. Then every measure μ on S with $\mu(S)=1$ is ergodic, but only the two point masses are extreme points.

20. For (a) the assumed condition on f for the function $c(n)$ that is nonzero at $n=0$ and is 0 elsewhere shows that $f(0)\geq 0$. The condition on f for the function $c(n)$ that is nonzero at 0 and k and is 0 elsewhere is that the matrix $\begin{pmatrix} f(0) & f(k) \\ f(-k) & f(0)\end{pmatrix}$ is Hermitian and positive semidefinite. The Hermitian condition forces $f(-k)=\overline{f(k)}$, and the condition determinant ≥ 0 then says that $|f(k)|^2 \leq f(0)^2$.

For (b), Example 2 of weak-star convergence in Section 3 says that a necessary and sufficient condition for a sequence $\{f_n\}$ in L^∞ to converge to f weak-star is that $\{\|f_m\|_\infty\}$ be bounded, which we are assuming, and that $\int_E f_n\,d\mu \to \int_E f\,d\mu$ for every E of finite measure. Here the sets of finite measure in $\mathbb{Z}$ are the finite sets, and thus the relevant convergence is pointwise convergence.

For (c), Theorem 4.14 shows that the weak-star topology on the closed unit ball of $L^\infty(\mathbb{Z})$ is compact metric, and therefore the topology is specified by sequences. The convexity of K is routine, and we just have to see that K is closed. We can do this by assuming that we have a pointwise convergent sequence whose members are in K and by proving that the limit is in K. This too is routine.

For (d), suppose that $e^{in\theta} = (1-t)F_1(n) + tF_2(n)$ nontrivially. Taking the absolute value and using (a), we have $1 \leq (1-t)|F_1(n)| + t|F_2(n)| \leq (1-t)+t = 1$, and equality must hold throughout. Therefore $|F_1(n)| = |F_2(n)| = 1$. Suppressing the parameter n, suppose that we have $e^{i\psi} = (1-t)e^{i\varphi_1} + te^{i\varphi_2}$ nontrivially. Multiplying through by $e^{-i\psi}$, we reduce to the case that $\psi=0$. So we have $1 = (1-t)e^{i\varphi_1'} + te^{i\varphi_2'}$. The real part is $1 = (1-t)\cos\varphi_1' + t\cos\varphi_2'$, and we must have $\cos\varphi_1' = \cos\varphi_2' = 1$ and $e^{i\varphi_1'} = e^{i\varphi_2'} = 1$. Hence $F_1(n) = e^{in\theta} = F_2(n)$, and $n\mapsto e^{in\theta}$ is an extreme point.

For (e), the Fourier coefficient mapping from complex Borel measures on the circle to doubly infinite sequences is linear and one-one, and we are told to assume that the mapping carries the set of Borel measures onto the set of positive definite functions. The value of the positive definite function at 0 is then the total measure of the circle. Hence the question translates into identifying the extreme Borel measures of total mass 1 on the circle. Problem 17 shows that these are the point masses.

21. For (a), the convergence is proved by showing that the partial sums form a Cauchy sequence. For $m \leq n$, we have $\big\| \sum_{k=0}^{n}(f/C)^k - \sum_{k=0}^{m}(f/C)^k \big\|_{\sup} = \big\| \sum_{k=m+1}^{n}(f/C)^k \big\|_{\sup} \leq \sum_{k=m+1}^{n} \|f/C\|_{\sup}^k$, and the right side tends to 0 as m and n tend to infinity because $\|f/C\|_{\sup} = |C|^{-1}\|f\|_{\sup} < 1$. So the series converges to some x. Since $\big(\sum_{k=0}^{n}(f/C)^k\big)(1 - f/C) = 1 - (f/C)^{n+1}$ and since multiplication is continuous, the element x is a multiplicative inverse to $1 - f/C$.

In (b), $\ell(f) = C$ would imply $\ell(1 - f/C) = \ell(1) - \ell(f)/C = 0$. But then $0 = 0 \cdot \ell(x) = \ell(1 - f/C)\ell(x) = \ell(1) = 1$ would give a contradiction.

From (b) we obtain $|\ell(f)| \leq 1$. Taking the supremum over all f with $\|f\|_{\sup} \leq 1$, we find that $\|\ell\| \leq 1$. Thus ℓ is bounded. This proves (c).

22. Problem 21 shows that ℓ is bounded. The result follows by using the Stone Representation Theorem and the first example after its proof.

23. If t is in T, define $\ell_{u(t)}(f) = (Uf)(t)$ for f in $C(S)$. It is routine to check that ℓ_t satisfies the hypotheses of Problem 22 and is therefore given by evaluation at some s in S. Define this s to be $u(t)$. The proofs of (a), (b), and (c) are then straightforward.

24. This is just a matter of applying Problem 23 and tracking down the isomorphisms.

25. Let S be a nonempty set, and let $\mathcal{A}$ be a uniformly closed subalgebra of $B(S)$ with the properties that $\mathcal{A}$ is stable under complex conjugation and contains 1. If S_2 is a compact Hausdorff space and $V : \mathcal{A} \to C(S_2)$ is an algebra isomorphism mapping 1 to 1 and respecting conjugation and if S_1, p, and U are as in Theorem 4.15, then there exists a unique homeomorphism $\Phi : S_2 \to S_1$ such that $(Uf)(\Phi(s_2)) = (Vf)(s_2)$ for all f in $\mathcal{A}$. Then one has to give a proof.

26. For (a), the reflexive and symmetric properties are immediate from the definition. For the transitive property let $x_i \sim x_j$ and $x_j \sim x_l$. Say that $i \leq k$, $j \leq k$, $\psi_{ki}(x_i) = \psi_{kj}(x_j)$, $j \leq m$, $l \leq m$, $\psi_{mj}(x_j) = \psi_{ml}(x_l)$. Choose n with $k \leq n$ and $m \leq n$. Application of ψ_{nk} to $\psi_{ki}(x_i) = \psi_{kj}(x_j)$ gives $\psi_{ni}(x_i) = \psi_{nj}(x_j)$, and application of ψ_{nm} to $\psi_{mj}(x_j) = \psi_{ml}(x_l)$ gives $\psi_{nj}(x_j) = \psi_{nl}(x_l)$. Therefore $\psi_{ni}(x_i) = \psi_{nl}(x_l)$, and $\sim$ is transitive.

For (b), suppose that $\psi_{ki}(x_i) = \psi_{kj}(x_j)$. We are to show that $\psi_{li}(x_i) = \psi_{lj}(x_j)$ whenever $i \leq l$ and $j \leq l$. Assume the contrary for some l. Choose m with $k \leq m$ and $l \leq m$. Application of ψ_{mk} to $\psi_{ki}(x_i) = \psi_{kj}(x_j)$ gives $\psi_{mi}(x_i) = \psi_{mj}(x_j)$. On the other hand, application of ψ_{ml} to $\psi_{li}(x_i) \neq \psi_{lj}(x_j)$ gives $\psi_{mi}(x_i) \neq \psi_{mj}(x_j)$ since ψ_{ml} is by assumption one-one. Thus we have a contradiction.

27. Suppose that we are given maps $\varphi_i : W_i \to Z$ with $\varphi_j \circ \psi_{ji} = \varphi_i$ whenever $i \leq j$. Define $\widetilde{\Phi} : \coprod W_i \to Z$ by $\widetilde{\Phi}(x_j) = \varphi_j(x)$ if x_j is in W_j. The map $\widetilde{\Phi}$ is continuous, and the claim is that it descends to the quotient to give a map Φ satisfying $\Phi(q(x_j)) = \widetilde{\Phi}(x_j)$. To see the necessary consistency, suppose $x_j \sim x_l$ with x_l in W_l. Say that $j \leq k$, $l \leq k$, and $\psi_{kj}(x_j) = \psi_{kl}(x_l)$. Then we have $\widetilde{\Phi}(x_j) = \varphi_j(x_j) = \varphi_k\psi_{kj}(x_j) = \varphi_k\psi_{kl}(x_l) = \varphi_l(x_l) = \widetilde{\Phi}(x_l)$, and the consistency is proved. The definition of Φ is complete, and we have arranged that $\Phi \circ (q\big|_{W_j}) = \varphi_j$ for each j. This establishes existence of the map Φ in the universal mapping property. Since q carries $\coprod_i W_i$ onto W, the formulas $\Phi \circ (q\big|_{W_j}) = \varphi_j$ force the definition we have used for Φ. This establishes the uniqueness of the map Φ in the universal mapping property.

28. With $(V, \{p_i\})$ as a direct limit, take $Z = W$ and $\varphi_i = q_i$. Each map φ_i carries W_i into Z, and the universal mapping property of $(V, \{p_i\})$ yields a mapping $F : V \to W$ with $q_i = F \circ p_i$ for all i. Reversing the roles of $(V, \{p_i\})$ and $(W, \{q_i\})$, we obtain a mapping $G : V \to W$ with $p_i = G \circ q_i$ for all i.

With $(V, \{p_i\})$ as a direct limit, take $Z = V$ and $\varphi_i = p_i$. Then the identity $1\big|_V$ meets the condition of the universal mapping property for this situation. On the other hand, so does $G \circ F$, which carries V to itself and has $p_i = G \circ q_i = (G \circ F) \circ p_i$. By the uniqueness that is part of the universal mapping property, $G \circ F = 1\big|_V$. Similarly $F \circ G = 1\big|_W$. Thus F is a homeomorphism.

The homeomorphism F is unique because any such mapping $F^\#$ must similarly have $G \circ F^\# = 1\big|_V$ and $F^\# \circ G = 1\big|_W$. Thus $F^\#$ must be a two-sided inverse to G, and there can be only one such function.

29. For (a), let U be an open set in $\coprod_i W_i$. We are to prove that $q(U)$ is open. Since each W_i is open in the disjoint union, we may assume that $U \subseteq W_i$ for some i. We are to prove that $q^{-1}(q(U))$ is open, hence that $q^{-1}(q(U)) \cap W_j$ is open for each j. Thus we are to show that the set V of all x_j in W_j such that $x_j \sim x_i$ for some x_i in U is open in W_j. Choose k with $i \leq k$ and $j \leq k$. Then we have $V = \psi_{kj}^{-1}(\psi_{ki}(U))$. The hypothesis for this problem makes $\psi_{ki}(U)$ open in W_k, and then $\psi_{kj}^{-1}(\psi_{ki}(U))$ is open since ψ_{kj} is continuous.

For (b), we are to separate $q(x_i)$ and $q(x_j)$ by disjoint open sets if x_i and x_j are not equivalent. Choose k with $i \leq k$ and $j \leq k$, so that $\psi_{ki}(x_i)$ and $\psi_{kj}(x_j)$ are both in W_k. They are distinct in W_k by Problem 26b. Since W_k is Hausdorff, we can choose disjoint open sets A and B in W_k with $\psi_{ki}(x_i)$ in A and $\psi_{kj}(x_j)$ in B. Then $q(A)$ and $q(B)$ are disjoint since q is one-one on W_k, and they are open by (a).

For (c), the mapping into the direct limit is continuous and open and therefore carries compact neighborhoods to compact neighborhoods. Since the quotient map is onto the direct limit, every point of the direct limit has a compact neighborhood.

For an example in (d), take $W_i = \{1, \dots, i\}$ for each i, with ψ_{ji} equal to the inclusion if $i \leq j$. Each W_i is finite, hence compact, and the direct limit is the set of positive integers with the discrete topology.

30. Each $X(S)$ is Hausdorff as the product of Hausdorff spaces. The space $\big(\bigtimes_{i\notin S} K_i\big)$ is compact by the Tychonoff Product Theorem, and then $X(S)$ is the product of finitely many locally compact spaces, which is locally compact. The Hausdorff property is handled by Problem 29b, and the final assertion is clear from the definition.

Chapter V

1. If K is compact in U, then K is compact in V, and hence the inclusion of C_K^∞ into $C_{\text{com}}^\infty(V)$ is continuous. By Proposition 4.29 the inclusion of $C_{\text{com}}^\infty(U)$ into $C_{\text{com}}^\infty(V)$ is continuous.

2. Fix K compact large enough to contain support(φ). Then the map $\psi \mapsto \psi\varphi$ is continuous from $C^\infty(U)$ into C_K^∞. The inclusion of C_K^∞ into $C_{\text{com}}^\infty(U)$ is continuous, and hence $\psi \mapsto \psi\varphi$, being a composition of continuous functions, is continuous from $C^\infty(U)$ into $C_{\text{com}}^\infty(U)$.

3. Let $\{K_j\}$ be an exhausting sequence of compact subsets of U, and choose $\psi_j \in C_{\text{com}}^\infty(U)$ that is 1 on K_j and is 0 off K_{j+1}. For each j, the product $(\varphi\big|_U - \varphi_1)\psi_j$ is in $C_{\text{com}}^\infty(U)$ with support contained in the open set $U \cap (\text{support}(T_U))^c$. Therefore $T_U((\varphi\big|_U - \varphi_1)\psi_j) = 0$ for each j. The functions $(\varphi\big|_U - \varphi_1)\psi_j$ tend to $\varphi\big|_U - \varphi_1$ in the topology of $C^\infty(U)$, and therefore $T_U(\varphi\big|_U - \varphi_1) = 0$. Hence $T_U(\varphi\big|_U) = T_U(\varphi_1)$ as required.

4. An adjustment is needed to the proof of Theorem 5.1. The proof in the text in effect used the expressions $\|f\|_{K',\alpha} = \sup_{x\in K'} |(D^\alpha f)(x)|$ as seminorms together describing the relative topology of $C_{K'}^\infty$ as a subspace of $C^\infty(\mathbb{R}^n)$. To modify the proof of the theorem, we need to see that the same relative topology results from using the expressions $\|f\|_{K',\alpha,\text{new}} = \|(D^\alpha f)\|_{L^2(K')}$. In one direction we have $\|(D^\alpha f)\|_{L^2(K')} \le C \sup_{x\in K'} |(D^\alpha f)(x)|$, the constant C being the L^2 norm of the function 1 on K'. In the reverse direction we apply Sobolev's inequality (Theorem 3.11) with U equal to the interior of K'. This open set satisfies the cone condition. Sobolev's inequality shows for $k > N/2$ that $\sup_{x\in K'} |(D^\alpha f)(x)| \le C\big(\sum_{|\beta|\le k} \|(D^{\alpha+\beta} f)\|^2_{L^2(K')}\big)^{1/2}$. We follow the lines of the proof of Theorem 5.1, using these new seminorms and using linear functionals on spaces of L^2 functions instead of spaces of continuous functions, and the desired result follows.

5. For (a), we write $\langle T, \varphi\rangle = \sum_\alpha \int_{\mathbb{R}^N} D^\alpha \varphi \, d\rho_\alpha(x)$ by means of Theorem 5.1. Substitution and use of Lemma 5.6 gives

$$\begin{aligned}\langle T, F\rangle &= \textstyle\sum_\alpha \int_{\mathbb{R}^N} D_x^\alpha \int_K \Phi(x,y)\, d\mu(y)\, d\rho_\alpha(x)\\ &= \textstyle\sum_\alpha \int_{\mathbb{R}^N} \int_K D_x^\alpha \Phi(x,y)\, d\mu(y)\, d\rho_\alpha(x).\end{aligned}$$

On the other hand, $\int_K \langle T, \Phi(\,\cdot\,, y)\rangle\, d\mu(y) = \int_K \sum_\alpha \int_{\mathbb{R}^N} D_x^\alpha \Phi(x,y)\, d\rho_\alpha(x)\, d\mu(y)$, and the two expressions are equal by Fubini's Theorem.

For (b), choose a compact subset L of $\mathbb{R}^N$ such that $L \times K$ contains support(Φ), and choose η in $C^\infty_{\text{com}}(\mathbb{R}^N)$ that is identically 1 on L. Part (a) shows that

$$\langle \eta S, F\rangle = \textstyle\int_K \langle \eta S, \Phi(\,\cdot\,, y)\rangle\, d\mu(y).$$

On the other hand, we have $\langle \eta S, F\rangle = \langle S, \eta F\rangle = \langle S, F\rangle$, and $\langle \eta S, \Phi(\,\cdot\,, y)\rangle = \langle S, \eta(\,\cdot\,)\Phi(\,\cdot\,, y)\rangle = \langle S, \Phi(\,\cdot\,, y)\rangle$, and the result follows.

6. For any member η of $C^\infty_{\text{com}}(U)$ with values in $[0, 1]$, ηT is a member of $\mathcal{E}'(U)$. If φ is a real-valued member of $C^\infty_{\text{com}}(U)$, then for both choices of the sign $\pm$, $\eta(\|\varphi\|_{\text{sup}} \pm \varphi)$ is a member of $C^\infty_{\text{com}}(U)$ that is ≥ 0. Hence $\langle T, \eta(\|\varphi\|_{\text{sup}} \pm \varphi)\rangle \geq 0$, and $\langle T, \eta\rangle\|\varphi\|_{\text{sup}} = \langle T, \eta\|\varphi\|_{\text{sup}}\rangle \geq \mp\langle T, \eta\varphi\rangle = \mp\langle \eta T, \varphi\rangle$, i.e., $|\langle \eta T, \varphi\rangle| \leq \langle T, \eta\rangle\|\varphi\|_{\text{sup}}$. For complex-valued φ, such an estimate is valid for the real and imaginary parts separately, and we conclude that $\varphi \mapsto \langle \eta T, \varphi\rangle$ is a bounded linear functional on $C^\infty_{\text{com}}(U)$ relative to the supremum norm. Corollary 3.6a shows that $C^\infty_{\text{com}}(U)$ is dense in $C_{\text{com}}(U)$ and that the approximating functions to a function ≥ 0 can be taken to be ≥ 0. Consequently the continuous extension of ηT is a positive linear functional on $C_{\text{com}}(U)$. By the Riesz Representation Theorem the extension is given by a Borel measure μ_η. The boundedness of the linear functional forces $\mu_\eta(U)$ to be finite.

Let $\{K_p\}$ be an exhausting sequence. Define η_p to be a member of $C^\infty_{\text{com}}(U)$ with values in $[0, 1]$ that is 1 on K_{2p} and is 0 off K^o_{2p+1}, and write μ_p for the corresponding Borel measure μ_{η_p}. Then the sequence $\{\eta_p(x)\}$ is nondecreasing for each x and has limit 1. The measures μ_p have to be nondecreasing on each set, and we define $\mu(E) = \lim_p \mu_p(E)$ for each Borel set E. The nondecreasing limit of measures is a measure, with the complete additivity holding by monotone convergence. We show that $\langle T, \varphi\rangle = \int_U \varphi\, d\mu$ for every φ in $C^\infty_{\text{com}}(U)$.

For any φ in $C^\infty_{\text{com}}(U)$, as soon as p_0 is large enough so that K_{2p_0} contains support(φ), we have $\langle \eta_p T, \varphi\rangle = \langle T, \varphi\rangle$ for $p \geq p_0$. Also, $\mu_p(E)$ remains constant for each Borel subset of K_{2p} when $p \geq p_0$, and hence $\mu(E) = \mu_p(E)$ for such subsets. Thus $\langle T, \varphi\rangle = \langle \eta_p T, \varphi\rangle = \int_U \varphi\, d\mu_p = \int_U \varphi\, d\mu$, as asserted.

7. Computation gives $\Delta(e^{-\pi|x|^2}) = 4\pi^2|x|^2 e^{-\pi|x|^2} - 2\pi N e^{-\pi|x|^2}$. What needs computing is $\int_{\mathbb{R}^N} |x|^{-(N-2)}|x|^{2p} e^{-\pi|x|^2}\, dx$ for $p = 1$ and $p = 0$, and then one has to sort out the result. This integral equals $\Omega_{N-1}\int_0^\infty r^{2p+1}e^{-\pi r^2}\, dr$. For $p = 1$ and $p = 0$, the integral is elementary. Alternatively, it can be converted into a value of the gamma function by the change of variables $\pi r^2 \mapsto s$. In neither case does the value of Ω_{N-1} need to be computed.

8. Part (a) follows from the chain rule and the boundedness of each derivative of η since $(\eta_\varepsilon)^{(k)}(x) = \varepsilon^{-k}\eta^{(k)}(\varepsilon^{-1}x)$.

For (b), if φ has compact support, then $(1 - \eta_\varepsilon)\varphi$ has compact support away from $\{0\}$. Therefore $\langle T, (1 - \eta_\varepsilon)\varphi\rangle = 0$, and $\langle T, \varphi\rangle = \langle T, (1 - \eta_\varepsilon)\varphi\rangle + \langle T, \eta_\varepsilon\varphi\rangle = \langle T, \eta_\varepsilon\varphi\rangle$. Since $\varphi \mapsto \langle T, \varphi\rangle$ and $\varphi \mapsto \langle T, \eta_\varepsilon\varphi\rangle$ are continuous and $C^\infty_{\text{com}}(\mathbb{R}^N)$ is dense in $C^\infty(\mathbb{R}^N)$, $\langle T, \varphi\rangle = \langle T, \eta_\varepsilon\varphi\rangle$ for all φ in $C^\infty(\mathbb{R}^N)$.

In (c), we apply (a) and obtain

$$\begin{aligned}|\langle T, \eta_\varepsilon\varphi\rangle| &\le C \textstyle\sum_{k=0}^n \sup_{|x|\le M} |D^k(\eta_\varepsilon\varphi)(x)| = C \sum_{k=0}^n \sup_{|x|\le\varepsilon} |D^k(\eta_\varepsilon\varphi)(x)| \\ &\le C' \textstyle\sum_{k=0}^n \sum_{l=0}^k \sup_{|x|\le\varepsilon} |D^{k-l}(\eta_\varepsilon)(x)(D^l\varphi)(x)| \\ &\le C'' \textstyle\sum_{k=0}^n \sum_{l=0}^k \varepsilon^{l-k} \sup_{|x|\le\varepsilon} |(D^l\varphi)(x)| \\ &\le C''' \textstyle\sum_{l=0}^n \varepsilon^{l-n} \sup_{|x|\le\varepsilon} |(D^l\varphi)(x)|.\end{aligned}$$

When $\varphi(x) = \psi(x)x^{n+1}$, $|D^l\varphi(x)| \le c\sum_{r=0}^l |D^{l-r}\psi(x)||x^{n+1-r}|$, and the supremum for $|x| \le \varepsilon$ is $\le c'\varepsilon^{n+1-l}$. Therefore

$$|\langle T, \varphi\rangle| = |\langle T, \eta_\varepsilon\varphi\rangle| \le c'C''' \textstyle\sum_{l=0}^n \varepsilon^{l-n}\varepsilon^{n+1-l} = c'C'''(n+1)\varepsilon.$$

The right side tends to 0 as ε decreases to 0, and thus $\langle T, \varphi\rangle = 0$.

In (d), the Taylor expansion of a general φ is $\varphi(x) = \sum_{k=0}^n \frac{1}{k!}\varphi^{(k)}(0)x^k + \psi(x)x^{n+1}$ with ψ in $C^\infty(\mathbb{R}^1)$. Applying $\langle T, \cdot\,\rangle$ to both sides and using (c), we obtain $\langle T, \varphi\rangle = \sum_{k=0}^n \frac{1}{k!}\varphi^{(k)}(0)\langle T, x^k\rangle$.

9. The adjustments in the argument are to (a) and (c). For (a), the estimate is $|(D^\alpha\eta_\varepsilon(x)| \le C_{|\alpha|}\varepsilon^{-|\alpha|}$ and is again proved by the chain rule. Each differentiation introduces a factor of ε^{-1}. For (c), Taylor's Theorem says that the remainder term in computing a smooth function $\varphi(x)$ about the point 0 is

$$\sum_{\substack{l_1+\cdots+l_N=n+1,\\ \text{all } l_j\ge 0}} \tfrac{n+1}{l_1!\cdots l_N!}\, x_1^{l_1}\cdots x_N^{l_N} \int_0^1 (1-s)^n \,\tfrac{\partial^{n+1}\varphi}{\partial x_1^{l_1}\cdots x_N^{l_N}}(sx)\,ds,$$

hence is of the form

$$\sum_{\substack{l_1+\cdots+l_N=n+1,\\ \text{all } l_j\ge 0}} \psi_{l_1,\dots,l_N}(x)x_1^{l_1}\cdots x_N^{l_N}.$$

Thus one works with a function $\psi(x)x_1^{l_1}\cdots x_N^{l_N}$ with ψ smooth and with $\sum_j l_j = n+1$. The argument for (c) in Problem 8 now can be used.

10. As with Problem 9, the arguments for (a) and (c) in Problem 8 need adjustment, and this time we need to change (d) completely. For (a), we use the above function η for $\mathbb{R}^{N-L}$, and we define $\eta_\varepsilon(x', x'') = \eta(\varepsilon^{-1}x'')$. Then (a) causes no difficulties. For (c), we need a new form of Taylor's Theorem. The point is to treat $\varphi(x', x'')$ as a function of x'' alone, form a Taylor expansion with remainder, and carry along x' as a parameter. The result is that the remainder term is a sum of terms of the form $\psi(x', x'')M(x'')$, where ψ is in $C^\infty(\mathbb{R}^N)$ and M is a homogeneous monomial in the x'' variables of total degree $n+1$. Then (c) causes no difficulties and again gives 0. In (d), the main terms of the Taylor expansion are of the form $c_\alpha D^\alpha\varphi(x', 0)(x'')^\alpha$, where α is a multi-index that is nonzero only in the positions corresponding to x'' and has total degree $\le n$. We introduce a linear functional T_α on $C^\infty(\mathbb{R}^L)$ by the definition $\langle T_\alpha, \psi(x')\rangle = c_\alpha\langle T, \psi(x')(x'')^\alpha\rangle$. Then T_α is continuous, and the expansion $\langle T, \varphi\rangle = \sum_{|\alpha|\le n}\langle T_\alpha, (D^\alpha\varphi)\big|_{\mathbb{R}^L}\rangle$ has the required form.

11. Subtracting two tempered distributions solving $\Delta u = f$, we obtain a tempered distribution u with $\Delta u = 0$. From $\mathcal{F}(D^\alpha u) = (2\pi i)^{|\alpha|}\xi^\alpha \mathcal{F}(u)$ and $\mathcal{F}(\Delta u) = 0$, we obtain $|\xi|^2\mathcal{F}(u) = 0$. It follows that $\mathcal{F}(u)$ is supported at $\{0\}$. Problem 9 then shows that $\mathcal{F}(u)$ is a finite sum of the form $\sum_\alpha c_\alpha D^\alpha \delta$. Taking the inverse Fourier transform of both sides, we see that the distribution u equals a polynomial function.

12. Apply Theorem 5.1 to a member S of $\mathcal{E}'((-2\pi, 2\pi)^N)$, writing it as a sum of finitely many derivatives of complex Borel measures ρ_α of compact support: $\langle S, \varphi\rangle = \sum_{|\alpha|\le m} \int_K D^\alpha\varphi \, d\rho_\alpha$, where K is a compact subcube of $(-2\pi, 2\pi)^N$. For $\varphi(x) = e^{-ik\cdot x}$, we have $\sup_{x\in K} |D^\alpha(e^{-ik\cdot x})| \le |k^\alpha|$, and therefore $|\langle S, e^{-ik\cdot x}\rangle| \le \sum_{|\alpha|\le m} |k^\alpha| \|\rho_\alpha\| \le C(1+|k|^2)^{m/2}$, where $C = \sum_{|\alpha|\le m} \|\rho_\alpha\|$.

13. Change notation and suppose that $|c_k| \le C(1+|k|^2)^m$ for all k. The series $f(x) = \displaystyle\sum_k \frac{c_k e^{ik\cdot x}}{(1+|k|^2)^{m+N+1}}$ is then absolutely uniformly convergent, and $f(x)$ is continuous periodic. Define $S' \in \mathcal{E}'((-2\pi, 2\pi)^N)$ by

$$\langle S', \varphi\rangle = (2\pi)^{-N} \int_{[-\pi,\pi]^N} f(x)\varphi(x)\,dx.$$

Let $\mathcal{D} = 1 - \Delta$, and define $S = \mathcal{D}^{m+N+1}S'$. Then

$$\begin{aligned}\langle S, e^{-ik\cdot x}\rangle &= \langle S', \mathcal{D}^{m+N+1}(e^{-ik\cdot x})\rangle = (1+|k|^2)^{m+N+1}\langle S', e^{-ik\cdot x}\rangle \\ &= (1+|k|^2)^{m+N+1}(2\pi)^{-N}\textstyle\int_{[-\pi,\pi]^N} f(x)e^{-ik\cdot x}\,dx \\ &= (1+|k|^2)^{m+N+1}\tfrac{c_k}{(1+|k|^2)^{m+N+1}} = c_k,\end{aligned}$$

as required.

14. For each φ, the set of ψ with $|B(\varphi, \psi)| \le \|\varphi\|_{L^2_k(T^N)}$ is the set where the continuous function $|B(\varphi, \cdot\,)|$ is $\le$ some constant, and this is closed. The set $E_{k,M}$ is the intersection of such sets and is therefore closed. For each ψ, the function $B(\,\cdot\,, \psi)$ is linear and continuous, and therefore there exists an integer k and a constant M for which $|B(\varphi, \psi)| \le M\|\varphi\|_{L^2_k(T^N)}$ for all φ. This proves (a).

Since $C^\infty(T^N)$ is complete, the Baire Category Theorem shows that some $E_{k,M}$ has nonempty interior, hence contains a basic open set, i.e., some set of the form $U = \{\psi' \mid \|\psi' - \psi_0\|_{L^2_s(T^N)} < \epsilon\}$. If ψ has $\|\psi\|_{L^2_s(T^N)} < \epsilon$, then $\psi_0 + \psi$ is in U and thus has $|B(\varphi, \psi_0+\psi)| \le M\|\varphi\|_{L^2_k(T^N)}$ for all φ in $C^\infty(T^N)$. Then

$$|B(\varphi, \psi)| \le |B(\varphi, \psi_0+\psi)| + |B(\varphi, \psi_0)| \le M\|\varphi\|_{L^2_k(T^N)} + C_{\psi_0, k(\psi_0)}\|\varphi\|_{L^2_{k(\psi_0)}(T^N)}.$$

The right side is $\le M'\|\varphi\|_{L^2_{k_1}(T^N)}$ for $k_1 = \max\{k, k(\psi_0)\}$ and $M' = M + C_{\psi_0,k(\psi_0)}$. Hence $|B(\varphi, \psi)| \le M'\epsilon^{-1}\|\varphi\|_{L^2_{k_1}(T^N)}\|\psi\|_{L^2_s(T^N)} \le M'\epsilon^{-1}\|\varphi\|_{L^2_{k_2}(T^N)}\|\psi\|_{L^2_{k_2}(T^N)}$, where $k_2 = \max\{k_1, s\}$.

15. We apply the inequality of Problem 14b to $D^\alpha\varphi$ and $D^\beta\psi$, and then the result follows by applying the norm inequality of Problem 27 in Chapter III to $\|D^\alpha\varphi\|_{L^2_k(T^N)}$ and $\|D^\beta\psi\|_{L^2_k(T^N)}$.

16. The functions $e^{il\cdot x}e^{im\cdot y}$ are orthonormal in $L^2(T^N\times T^N)$, and it is therefore enough to show that the sum of the absolute-value squared of the coefficients is finite. That is, we are to show that

$$\sum_{l,m\in\mathbb{Z}^N}\frac{|b_{lm}|^2 l^{2\alpha}m^{2\beta}}{\big(\sum_{|\alpha'|\le k'}l^{2\alpha'}\big)^2\big(\sum_{|\beta'|\le k'}m^{2\beta'}\big)^2}<\infty$$

whenever $|\alpha|\le k'$ and $|\beta|\le k'$. Since $l^{2\alpha}\le\sum_{|\alpha'|\le k'}l^{2\alpha'}$ and $m^{2\beta}\le\sum_{|\beta'|\le k'}m^{2\beta'}$, it is enough to prove that

$$\sum_{l,m\in\mathbb{Z}^N}\frac{|b_{lm}|^2}{\big(\sum_{|\alpha'|\le k'}l^{2\alpha'}\big)\big(\sum_{|\beta'|\le k'}m^{2\beta'}\big)}<\infty. \tag{$*$}$$

If we use the estimate of Problem 15 for $\varphi=e^{il\cdot(\,\cdot\,)}$ and $\psi=e^{im\cdot(\,\cdot\,)}$, we have

$$l^{2\alpha}m^{2\beta}|b_{lm}|^2=|B(D^\alpha e^{il\cdot(\,\cdot\,)},D^\beta e^{im\cdot(\,\cdot\,)})|^2\le C^2\|e^{il\cdot(\,\cdot\,)}\|^2_{L^2_{k'}(T^N)}\|e^{im\cdot(\,\cdot\,)}\|^2_{L^2_{k'}(T^N)}$$

for $|\alpha|\le K$ and $|\beta|\le K$. Hence

$$l^{2\alpha}m^{2\beta}|b_{lm}|^2\le C^2\Big(\sum_{|\alpha'|\le k'}l^{2\alpha'}\Big)\Big(\sum_{|\beta'|\le k'}m^{2\beta'}\Big).$$

Summing over α and β for $|\alpha|\le K$ and $|\beta|\le K$ and taking into account Problem 29 in Chapter III, we obtain

$$(1+|l|^2)^K(1+|m|^2)^K|b_{lm}|^2\le C'\Big(\sum_{|\alpha'|\le k'}l^{2\alpha'}\Big)\Big(\sum_{|\beta'|\le k'}m^{2\beta'}\Big)$$

for a constant C'. Thus the left side of $(*)$ is $\le C'\sum_{l,m\in\mathbb{Z}^N}(1+|l|^2)^{-K}(1+|m|^2)^{-K}$, and Problem 32 of Chapter III shows that this is finite.

17. Since $F_{\alpha,\beta}$ is in $L^2(T^N\times T^N)$, B' is a continuous function of two $L^2(T^N)$ variables $D^\alpha\varphi$ and $D^\beta\psi$. In particular it is well defined for φ and ψ in $C^\infty(T^N)$. Because of continuity in L^2 and orthogonality, we have

$$\begin{aligned}(2\pi)^{-2N}&\int_{[-\pi,\pi]^{2N}}F_{\alpha,\beta}(x,y)D^\alpha e^{il\cdot x}D^\beta e^{im\cdot y}\,dx\,dy\\ &=(2\pi)^{-2N}\int_{[-\pi,\pi]^{2N}}\frac{b_{lm}(-i)^{|\alpha|+|\beta|}l^\alpha m^\beta i^{|\alpha|+|\beta|}l^\alpha m^\beta}{\big(\sum\limits_{|\alpha'|\le k'}l^{2\alpha'}\big)\big(\sum\limits_{|\beta'|\le k'}m^{2\beta'}\big)}\,dx\,dy\\ &=\frac{b_{lm}l^{2\alpha}m^{2\beta}}{\big(\sum\limits_{|\alpha'|\le k'}l^{2\alpha'}\big)\big(\sum\limits_{|\beta'|\le k'}m^{2\beta'}\big)}.\end{aligned}$$

Summing for α and β with $|\alpha|\le k'$ and $|\beta|\le k'$, we obtain $B'(e^{il\cdot(\,\cdot\,)},e^{im\cdot(\,\cdot\,)})=B(e^{il\cdot(\,\cdot\,)},e^{im\cdot(\,\cdot\,)})$.

18. The result of Problem 17 implies that $B'(\varphi, \psi) = B(\varphi, \psi)$ if φ and ψ are trigonometric polynomials. It shows also that convergence in L^2 of either variable and its derivatives through order k' implies convergence of B'. Since convergence in $C^\infty(T^N)$ implies convergence in $L^2_{k'}(T^N)$ and since B is separately continuous, $B' = B$ on $C^\infty(T^N)$. The expression on the right side of the display in the statement of Problem 17 is the action of a distribution on $T^N \times T^N$ upon the function $\varphi \otimes \psi$, and thus $B(\varphi, \psi) = \langle S, \varphi \otimes \psi \rangle$ for a suitable distribution S.

19. By the Schwarz inequality, $|B(f, g)| \le \|H(\eta f)\|_2 \|\eta g\|_2 = \|\eta f\|_2 \|\eta g\|_2 \le \|f\|_2 \|g\|_2 \le \|f\|_{\text{sup}} \|g\|_{\text{sup}}$. This proves (a).

For (b), we argue by contradiction. Using continuous functions f and g with disjoint supports, we see near (0, 0) that we must have $d\rho(x, y) = \frac{1}{\pi} \frac{dx\,dy}{x-y}$. However, the function $\frac{1}{x-y}$ is not locally integrable, and there can be no such signed measure ρ.

Chapter VI

1. For (a), let C be the connected component of 1. Since multiplication is continuous, it carries the connected set $C \times C$ to a connected set containing 1, hence to a subset of C. Thus C is closed under products. Similarly it is closed under inverses. It is topologically closed since the closure of a connected set is connected. If x is in G, then the map $x \mapsto gxg^{-1}$ is continuous and therefore carries the connected set C to a connected set containing 1, hence to a subset of C. Thus $gCg^{-1} \subseteq C$ for all g, and C is normal. For (b), one can take the additive rationals or the countable product of two-element groups; for each the identity component contains only the identity element.

2. In (a), if g fixes the first standard basis vector, then the first column of g is the first standard basis vector. Since g is a rotation, $g^{\text{tr}} g = 1$. In particular $\sum_j (g^{\text{tr}})_{ij} g_{j1} = \delta_{i1}$. Thus $(g^{\text{tr}})_{i1} = \delta_{i1}$ for all i, and $g_{1i} = \delta_{i1}$. In other words, the first row of g is 0 except in the first entry.

In (b), let v be any unit vector in $\mathbb{R}^N$, and extend v to a basis $v, v_2, \dots, v_N$. The Gram–Schmidt orthogonalization process replaces this basis by an orthonormal basis such that the first member is still v. We form a matrix with this orthonormal basis as its columns. If it has determinant -1, we multiply the last column by -1, and then the determinant will be 1. The resulting matrix is in $SO(N)$ and carries the first standard basis vector to v.

For (c), we obtain a continuous function $SO(N) \to S^{N-1}$ given by $g \mapsto ge_1$, where e_1 is the first standard basis vector. This function descends to a function $SO(N)/SO(N-1) \to S^{N-1}$ that is necessarily continuous. It is one-one onto, its domain is compact, and the image is Hausdorff. Hence it is a homeomorphism.

3. What needs to be shown is that every sufficiently small open neighborhood N of $1 \cdot H$ in G/H is mapped to an open set by π. Since G/H is locally compact

and has a countable base, there exist open neighborhoods U_k of $1 \cdot H$ such that U_k^{cl} is compact, $U_k^{\text{cl}} \subseteq U_{k+1}$, and $G/H = \bigcup_k U_k$. The Baire Category Theorem for X shows that $\pi(U_n)$ has nonempty interior V for some n. Let y be a member of G such that $\pi(yH)$ is in V, and put $U = \pi^{-1}(y^{-1}V)$. Then U is an open neighborhood of $1 \cdot H$ in G/H and $\pi(U) = y^{-1}V$ is open in X. Also, U^{cl} is compact as a closed subset of U_n^{cl}. Let N be any open neighborhood of $1 \cdot H$ in G/H that is contained in U. Since U^{cl} is compact, π is a homeomorphism from U^{cl} with the relative topology to $\pi(U^{\text{cl}})$ with the relative topology. Thus $\pi(N)$ is relatively open in $\pi(U^{\text{cl}})$. Hence $\pi(N) = \pi(U^{\text{cl}}) \cap W$ for some open set W in X. Since $\pi(N) \subseteq \pi(U)$, we can intersect both sides with $\pi(U)$ and get $\pi(N) = \pi(U^{\text{cl}}) \cap W \cap \pi(U) = W \cap \pi(U)$. Since $W \cap \pi(U)$ is open in X, $\pi(N)$ is open in X.

4. This is a special case of the previous problem.

5. No. The reason is that the subset $\mathbb{R}^1 p$ cannot be locally compact. In fact, if it were locally compact, then it would be open in its closure, by Problem 4 in Chapter X of *Basic*. Since T^2 is a group and $\mathbb{R}^1 p$ is a subgroup, $(\mathbb{R}^1 p)^{\text{cl}}$ is a group, and $\mathbb{R}^1 p$ would be an open dense subgroup. An open subgroup is closed, and hence $\mathbb{R}^1 p$ would be equal to $(\mathbb{R}^1 p)^{\text{cl}}$, i.e., $\mathbb{R}^1 p$ would have to be closed in T^2. Then $\mathbb{R}^1 \cap \{(e^{i\theta}, 1)\}$ would be a closed subgroup of the circle group $\{(e^{i\theta}, 1)\}$ and would have to be a finite subgroup or the entire circle. On the other hand, we readily check that $\mathbb{R}^1 p \cap \{(e^{i\theta}, 1)\}$ is countably infinite. It therefore cannot be closed.

6. Take V to be any bounded open neighborhood of 1. Inductively for $n \geq 1$, choose g_n such that $g_n \notin \bigcup_{k=1}^{n-1} g_k V$. Then choose an open neighborhood U of 1 with $U = U^{-1}$ and $UU \subseteq V$. Let us see that $g_k U \cap g_n U = \varnothing$ if $k < n$. If g is in $g_k U \cap g_n U$, then $g_k u = g_n u'$ with u and u' in U, and hence g_n is in $g_k U U^{-1} \subseteq g_k V$. This contradicts the defining property of g_n. Thus the sets $g_n U$ are disjoint. The left Haar measure of their union therefore equals the sum of their left Haar measures, and their left Haar measures are equal to some positive number, U being a nonempty open set. Consequently the left Haar measure of G is infinite.

7. For (a), we have

$$\begin{aligned}
\lambda(E)\rho(G) &= \int_G \int_G I_E(y)\,d\lambda(y)\,d\rho(x) = \int_G \int_G I_E(xy)\,d\lambda(y)\,d\rho(x) \\
&= \int_G \int_G I_E(xy)\,d\rho(x)\,d\lambda(y) = \int_G \int_G I_E(x)\,d\rho(x)\,d\lambda(y) \\
&= \lambda(G)\rho(E).
\end{aligned}$$

Therefore $\lambda(E)\rho(G) = \lambda(G)\rho(E)$ as asserted.

For (b), let λ_1 and λ_2 be two left Haar measures. Without loss of generality, we may assume that $\lambda_1(G) = \lambda_2(G) = 1$. Let ρ be a right Haar measure (existence by Theorem 12.1). Applying (a) twice, we obtain $\lambda_1(E)\rho(G) = \lambda_1(G)\rho(E) = \rho(E) = \lambda_2(G)\rho(E) = \lambda_2(E)\rho(G)$, and hence $\lambda_1(E) = \lambda_2(E)$ on Baire sets. Consequently $\lambda_1 = \lambda_2$ as regular Borel measures.

8. In (a), both are Haar measures on $G^{(n)}$ of total measure one. Parts (b) and (c) are special cases of Problems 15–19 of Chapter XI of *Basic*.

9. For fixed g in G, we have $d_l(\Phi(gx)) = d_l(\Phi(g)\Phi(x)) = d_l(\Phi(x))$, and hence $d_l(\Phi(\cdot))$ and $d_l(\cdot)$ are left Haar measures on G. The uniqueness in Theorem 6.8 shows that they are multiples of one another.

10. Under left translation we have $(s_0, t_0)(s, t) = (s_0 s)((s^{-1}t_0 s)t)$. If φ is left translation by (s_0, t_0), then $(ds\,dt)_{\varphi^{-1}} = d(s_0 s)\,d((s^{-1}t_0 s)t) = ds\,dt$, and $ds\,dt$ is a left Haar measure. Under right translation we have $(s, t)(s_0, t_0) = (ss_0)((s_0^{-1}ts_0)t_0)$. Thus $ds\,dt$ goes to $d(ss_0)\,d((s_0^{-1}ts_0)t_0) = ds\,d(s_0^{-1}ts_0) = \delta(s_0^{-1})\,ds\,dt$, and $\delta(s)\,ds\,dt$ goes to $\delta(ss_0)\delta(s_0^{-1})\,ds\,dt = \delta(s)\,ds\,dt$. In other words, $\delta(s)\,ds\,dt$ is a right Haar measure.

11. In (a), we have $\int_V f(c^{-1}x)\,dx = \int_V f(x)\,d(cx) = |c|_V \int_V f(x)\,dx$ for f in $C_{\text{com}}(V)$. Hence $|c_1c_2|_V \int_V f(x)\,dx = \int_V f((c_1c_2)^{-1}x)\,dx = \int_V f(c_2^{-1}x)\,d(c_1x) = |c_1|_V \int_V f(c_2^{-1}x)\,dx = |c_1|_V|c_2|_V \int_V f(x)\,dx$. Choosing f with $\int_V f(x)\,dx \neq 0$, we obtain $|c_1c_2|_V = |c_1|_V|c_2|_V$ when $c_1 \neq 0$ and $c_2 \neq 0$. The equality is trivial when one or both of c_1 and c_2 are 0, and hence we have $|c_1c_2|_V = |c_1|_V|c_2|_V$ in all cases.

To prove continuity, we first check continuity at each $c_0 \neq 0$. Let $S = \text{support}(f)$, and let N be a compact neighborhood of c_0 not containing 0. If c is in N, then $f(c^{-1}x)$ is nonzero only for x in the compact set NS. Let $\epsilon > 0$ be given. Continuity of $(c, x) \mapsto f(c^{-1}x)$ allows us to find, for each x in NS, an open subneighborhood N_x of c_0 and an open neighborhood U_x of x such that $|f(c^{-1}y) - f(c_0^{-1}x)| < \epsilon$ for $c \in N_x$ and $y \in U_x$. Then $|f(c^{-1}y) - f(c_0^{-1}y)| < 2\epsilon$ for $c \in N_x$ and $y \in U_x$. The open sets U_x cover NS. Forming a finite subcover and intersecting the corresponding finitely many sets N_x, we obtain an open neighborhood N' of c_0 such that $|f(c^{-1}y) - f(c_0^{-1}y)| < 2\epsilon$ for $c \in N'$ whenever y is in NS. As a result, $c \mapsto \int_V f(c^{-1}x)\,dx$ is continuous at $c = c_0$. Therefore $c \mapsto |c|_V \int_V f(x)\,dx$ is continuous at c_0, and so is $c \mapsto |c|_V$.

To prove continuity at $c = 0$, we are to show that $\lim_{c\to 0} \int_V f(c^{-1}x)\,dx = 0$. Let U be any compact neighborhood of 0 in V. Find a sufficiently small neighborhood N of 0 in V such that $c \in V$ implies that $c\,\text{support}(f)$ does not meet U^c. Then $c^{-1}U^c \cap \text{support}(f) = \varnothing$. For such c's, we have $\left|\int_V f(c^{-1}x)\,dx\right| = \left|\int_U f(c^{-1}x)\,dx\right| \leq \|f\|_{\sup}\,(dx(U))$. The desired limit relation follows.

Finally, even without the continuity at $c = 0$, these properties imply that $|c|_V = |c|^t$ for some real t. The continuity at $c = 0$ forces $t \geq 0$. Then it follows that $|c_1|_V \leq |c_2|_V$ if $|c_1| \leq |c_2|$.

In (b), V/W is itself a locally compact topological vector space, and its group operation is addition. With the normalization of Haar measures as in Theorem 6.18, μ becomes a Haar measure on V/W, and we write it as $d(v+W)$. Then $\int_V f(v)\,dv = \int_{V/W} \left(\int_W f(v+w)\,dw\right) d(v+W)$. If we replace f by $f(c^{-1}\cdot)$ and move the c into the measures, we obtain $\int_V f(v)\,d(cv) = \int_{V/W} \left(\int_W f(v+w)\,d(cw)\right) d(c(v+W))$ and therefore $|c|_V \int_V f(v)\,dv = |c|_{V/W} \int_{V/W} \left(|c|_W \int_W f(v+w)\,dw\right) d(v+W)$. Hence $|c|_V = |c|_{V/W}|c|_W$.

In (c), choose N such that $|2|_V < 2^N$. If V has an N-dimensional subspace W, then Proposition 4.5 and Corollary 4.6 show that this subspace is closed and is Euclidean. Therefore $|2|_W = 2^N$. Then (b) shows that $|2|_{V/W} = |2|_V/|2|_W = 2^{-N}|2|_V < 1$. But this conclusion contradicts the fact that $|c|_{V/W} \geq 1$ if $|c| \geq 1$. We conclude that $\dim V < N$.

12. By inspection, $(\ell_{v_1}, \ell_{v_2}) = (v_2, v_1)$ has the properties of an inner product. The Banach-space norm of ℓ_v is $\sup_{\|v'\|\leq 1} |\ell_v(v')| = \sup_{\|v'\|\leq 1} |(v', v)|$. This is $\leq \|v\| = \|\ell_v\|$ by the Schwarz inequality, and it is $\geq \|v\| = \|\ell_v\|$ because we can choose $v' = v/\|v\|$.

The contragredient has $(\Phi^c(x)\ell_v)(v') = \ell_v(\Phi(x^{-1})v') = (\Phi(x^{-1})v', v) = (v', \Phi(x)v) = \ell_{\Phi(x)v}(v')$. Hence $\Phi^c(x)\ell_v = \ell_{\Phi(x)v}$, and $(\Phi^c(x)\ell_v, \Phi^c(x)\ell'_v) = (\Phi(x)v', \Phi(x)v) = (v', v) = (\ell_v, \ell_{v'})$.

13. Taking the adjoint of $E\Phi(g) = \Phi'(g)E$ gives $\Phi(g)^*E^* = E^*\Phi'(g)^*$ for all g. Since Φ is unitary, $\Phi(g)^{-1}E^* = E^*\Phi'(g)^{-1}$ for all g, and thus $\Phi(g)E^* = E^*\Phi'(g)$. Then $E^*E\Phi(g) = E^*\Phi'(g)E = \Phi(g)E^*E$. By Schur's Lemma, E^*E is scalar, say equal to cI. Since E is invertible, c is not zero. If $v \neq 0$, then $c\|v\|^2 = (cI(v), v) = (E^*E(v), v) = (E(v), E(v)) \geq 0$. So $c > 0$. If $\sqrt{c}$ denotes the positive square root of c, then $F = (\sqrt{c})^{-1}E$ exhibits Φ and Φ' as equivalent, and F is unitary because $F^*F = (\sqrt{c})^{-2}E^*E = c^{-1}cI = I$.

14. The operator $\Phi(\rho)$, for ρ in $O(N)$, makes sense on all of $L^2(\mathbb{R}^N)$, as well as on the vector space H_k. It was observed in the example toward the end of Section 8 that the Fourier transform $\mathcal{F}$ commutes with the action by members of $O(N)$. Thus we have $\mathcal{F}(\Phi(\rho)(h_j(x)f(|x|))) = \Phi(\rho)\mathcal{F}(h_j(x)f(|x|))$. The left side at y equals the expression $\sum_i \Phi(\rho)_{ij}\mathcal{F}((h_i(x)f(|x|)))(y) = \sum_i \Phi(\rho)_{ij}\sum_s h_s(y)f_{si}(|y|) = \sum_s \big(\sum_i \Phi(\rho)_{ij}f_{si}(|y|)\big)h_s(y)$, and the right side is $\Phi(\rho)\big(\sum_t h_t(y)f_{ti}(|y|)\big) = = \sum_t\sum_s \Phi(\rho)_{st}h_s(y)f_{ti}(|y|) = \sum_s \big(\sum_t \Phi(\rho)_{st}f_{ti}(|y|)\big)h_s(y)$. The equality of the two sides gives us, for each $|y|$, the matrix equality asserted in (a).

Corollary 6.27, the formula of part (a), and the irreducibility of H_k together imply that $F(|y|)$ is a scalar matrix for each $|y|$. In other words, $f_{ij}(|y|) = g(|y|)\delta_{ij}$ for some scalar-valued function g. Then $\mathcal{F}(h_j(x)f(|x|))(y) = \sum_i h_i(y)f_{ij}(|y|) = \sum_i h_i(y)g(|y|)\delta_{ij} = h_j(y)g(|y|)$ for all j. Since h is a linear combination of the h_j's, $\mathcal{F}(h(x)f(|x|))(y) = h(y)g(|y|)$. This proves (b).

For (c), we observe that $F(|y|)$ is continuous if $f(|x|)$ is continuous of compact support. In fact, the inner product on H_k can be taken to be integration with $d\omega$ over the unit sphere S^{N-1}. By homogeneity this is the same as the inner product relative to $r^{-2k}\,d\omega$ over the sphere of radius r centered at 0. Then the formula for f_{ij} is

$$\begin{aligned} f_{ij}(r) &= \textstyle\int_{S^{N-1}} \mathcal{F}(h_j(x)f(|x|))(r\omega)\overline{h_i(r\omega)}r^{-2k}\,d\omega \\ &= \textstyle\int_{S^{N-1}} \mathcal{F}(h_j(x)f(|x|))(r\omega)\overline{h_i(\omega)}r^{-k}\,d\omega \end{aligned}$$

for $r > 0$, and this is continuous in r since $\mathcal{F}(h_j(x)f(|x|))$ is continuous on $\mathbb{R}^N$. Thus the vector subspace of all f in $L^2((0,\infty), r^{N-1+2k}\,dr)$ for which $\mathcal{F}(h(x)f(|x|))$

is of the form $h(y)g(|y|)$ contains the dense subspace $C_{\text{com}}((0,\infty))$. Let $f^{(n)}$ in $C_{\text{com}}((0,\infty))$ tend to f in $L^2((0,\infty), r^{N-1+2k}\,dr)$. Then $h(x)f^{(n)}(|x|)$ tends to $h(x)f(|x|)$ in $L^2(\mathbb{R}^N)$, and $\mathcal{F}(h(x)f^{(n)}(|x|))$ tends to $\mathcal{F}(h(x)f(|x|))$ in norm. A subsequence therefore converges almost everywhere. Since $\mathcal{F}(h(x)f^{(n)}(|x|))(y) = h(y)g^{(n)}(|y|)$ almost everywhere, the limit function must be of the form $h(y)g(|y|)$ almost everywhere.

15. If $\{v_j\}$ is an orthonormal basis of V, then $\{\ell_{v_j}\}$ is an orthonormal basis of V^*, and $(\Phi^c(x)\ell_{v_j}, \ell_{v_j}) = (\ell_{\Phi(x)v_j}, \ell_{v_j}) = (v_j, \Phi(x)v_j) = \overline{(\Phi(x)v_j, v_j)}$. Summing on j gives the desired equality of group characters.

16. Following the notation of that example, let $\tau_{ij}(x) = (\tau(x)u_j, u_i)$, let l be the left-regular representation, and let ℓ_v be as in Problem 12. Consider, for fixed j_0, the image of $\tau^c(g)\ell_{u_i}$ under the linear extension of the map $E'(\ell_{u_k})(x) = (\tau(x)u_{j_0}, u_k)$. This is $E'(\ell_{\sum_k c_k u_k})(x) = E'\big(\sum_k \bar c_k \ell_{u_k}\big)(x) = \sum_k \bar c_k E'(\ell_{u_k})(x) = \sum_k \bar c_k(\tau(x)u_{j_0}, u_k) = (\tau(x)u_{j_0}, \sum_k c_k u_k)$, and hence $E'(\ell_v)(x) = (\tau(x)u_{j_0}, v)$. Then the image of interest is

$$\begin{aligned} E'(\tau^c(g)\ell_{u_i})(x) &= E'(\ell_{\tau(g)u_i})(x) = (\tau(x)u_{j_0}, \tau(g)u_i) \\ &= (\tau(g^{-1}x)u_{j_0}, u_i) = (l(g)\tau_{ij_0})(x). \end{aligned}$$

Hence l carries a column of matrix coefficients to itself and is equivalent on such a column to τ^c.

17. In (a), the left-regular representation on $G = \mathbb{R}/2\pi\mathbb{Z}$ is given by $(l(\theta)f)(e^{i\varphi}) = f(e^{i(\varphi-\theta)})$. Assuming on the contrary that l is continuous in the operator norm topology, choose $\delta > 0$ such that $|\theta| < \delta$ implies $\|l(\theta) - 1\| < 1$. Since $\|e^{in\varphi}\|_2 = 1$, we must have $\|l(\theta)(e^{in\varphi}) - e^{in\varphi}\|_2 < 1$ for $|\theta| < \delta$. Then

$$|e^{-in\theta} - 1|^2 = \tfrac{1}{2\pi}\int_\pi^\pi |e^{-in\theta} - 1|^2\,d\varphi = \tfrac{1}{2\pi}\int_{-\pi}^\pi |e^{in(\varphi-\theta)} - e^{in\varphi}|^2\,d\varphi < 1$$

for all θ with $|\theta| < \delta$ and for all n. For large N, $\theta = \frac{\pi}{2N}$ satisfies the condition on θ, and $n = N$ has $|e^{-in\theta} - 1|^2 = |-i-1|^2 = 2$, contradiction.

In (b), $\|\Phi(g)v - v\|^2 = (\Phi(g)v - v, \Phi(g)v - v) = \|\Phi(g)\|^2 - 2\,\text{Re}(\Phi(g)v, v) + \|v\|^2 = 2\|v\|^2 - 2\,\text{Re}(\Phi(g)v, v)$. The weak continuity shows that the right side tends to 0 as g tends to 1, and hence the left side tends to 0, i.e., Φ is strongly continuous.

18. In (a), we apply Problem 15. Let $\{u_i\}$ be an orthonormal basis of the space of Φ. In the formula $(\Phi(f)u_k, u_k) = \int_G (\Phi(x)u_k, u_k)f(x)\,dx$, we take f to be of the form $f(x) = \overline{(\Phi(x)u_j, u_i)}$. Substituting and using Schur orthogonality gives $(\Phi(f)u_k, u_k) = d^{-1}(u_k, u_j)\overline{(u_k, u_i)}$. Summing on k shows that $\text{Tr}\,\Phi(f) = d^{-1}\delta_{ij}$, and the right side is $d^{-1}f(1)$ for this f. Thus $f(1) = d\Phi(f)$. Passing to a linear combination of such f's, we obtain the asserted formula.

Part (b) follows by taking linear combinations of results from (a), and part (c) follows by applying (b) to a function $f^* * f$, where $f^*(x) = \overline{f(x^{-1})}$. Part (d) follows by decomposing the right-regular representation on $L^2(G)$ into irreducible representations and using the identification in Section 8 of the isotypic subspaces.

19. For (a), $h * f(x) = \int_G h(xy^{-1}) f(y)\,dy = \int_G h(y^{-1}x) f(y)\,dy = f * h(x)$.

For (b), it is enough to check the assertion for f equal to a matrix coefficient $x \mapsto (\Phi(x)u_j, u_i) = \Phi_{ij}(x)$ of an irreducible unitary representation Φ. If Φ has degree d, then we have

$$\int_G f(gxg^{-1})\,dg = \int_G \Phi_{ij}(gxg^{-1})\,dg = \sum_{k,l} \int_G \Phi_{ik}(g)\Phi_{kl}(x)\Phi_{lj}(g^{-1})\,dg$$
$$= \sum_{k,l} \Phi_{kl}(x) \int_G \Phi_{ik}(g)\overline{\Phi_{jl}(g)}\,dg = \sum_{k,l} \Phi_{kl}(x) d^{-1}\delta_{ij}\delta_{kl} = \delta_{ij} d^{-1} \sum_k \Phi_{kk}(x),$$

as required.

In (c), Corollary 6.33 shows that h is the uniform limit of a net of trigonometric polynomials. Since $C(G)$ is metrizable, h is the uniform limit of a sequence of trigonometric polynomials h_n. If $\epsilon > 0$ is given, we can find N such that $n \geq N$ implies $|h_n(y) - h(y)| \leq \epsilon$ for all y. Then $|h_n(gxg^{-1}) - h(gxg^{-1})| \leq \epsilon$ and so $\left| \int_G h_n(gxg^{-1})\,dg - \int_G h(gxg^{-1})\,dg \right| \leq \epsilon$. The function $H_n(x) = \int_G h_n(gxg^{-1})\,dg$ is a linear combination of irreducible characters by (b), and $\int_G h(gxg^{-1})\,dg$ is just h. Thus h is the uniform limit of the sequence of functions H_n, each of which is a linear combination of characters.

In (d), it is enough to prove that the space of linear combinations of irreducible characters is dense in the vector subspace of L^2 in question. If h is in this subspace, choose a sequence of functions h_n in $C(G)$ converging to h in L^2. Then $H_n(x) = \int_G h_n(gxg^{-1})\,dg$ converges to h in L^2, and each H_n is continuous and has the invariance property that $H_n(gxg^{-1}) = H_n(x)$. Hence the vector subspace of members of $C(G)$ with this invariance property is L^2 dense in the subspace of L^2 in question. By (c), any member of $C(G)$ with the invariance property is the uniform limit of a sequence of functions, each of which is a finite linear combination of characters. Since uniform convergence implies L^2 convergence on a space of finite measure, the space of linear combinations of irreducible characters is L^2 dense in the space in question.

20. In (a), the sum $\sum_\alpha (d^{(\alpha)})^2$ counts the number of elements in the basis of $L^2(G)$ in Corollary 6.32. Another basis consists of the indicator functions of one-element subsets of G, and the two bases must have the same number of elements.

In (b), again we have two ways of computing a dimension, one from (d) in the previous problem, and the other from indicator functions of single conjugacy classes. The two computations must give the same result.

In (c), representatives of the possible cycle structures are (1234), (123), (12), $(12)(34)$, (1). By (b), the number of $\Phi^{(\alpha)}$'s is 5. Two of these have degree 1. For the other three the sums of the squares of the degrees must equal $24 - 1 - 1 = 22$. The only possibility is $22 = 9 + 9 + 4$, and thus the degrees are $1, 1, 2, 3, 3$.

21. Let $\Omega \subseteq G$ be the set of products ST, and let $K = S \cap T$. The group $S \times T$ acts continuously on Ω by $(s,t)\omega = s\omega t^{-1}$, and the isotropy subgroup at 1 is the closed subgroup diag K. Thus the map $(s,t) \mapsto st^{-1}$ descends to a map of $(S \times T)/\text{diag}\,K$ onto Ω. Since Ω is assumed open in G, it is locally compact Hausdorff in the relative

topology. Then Problem 3 shows that the map of $(S \times T)/\operatorname{diag} K$ onto Ω is open, and it follows by taking compositions that the multiplication map of $S \times T$ to Ω is open.

22. In the two parts, AN and MAN are subgroups closed under limits of sequences, hence are closed subgroups. Consider the decompositions in (a) and (b). For the decomposition in (a), we multiply out the relation $k_\theta a_x n_y = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and solve for θ, x, and y, obtaining

$$e^x = \sqrt{a^2 + c^2}, \qquad \cos\theta = e^{-x}a, \qquad \sin\theta = e^{-x}c, \qquad y = e^{-2x}(ab + cd).$$

Hence we have the required unique decomposition. Since KAN equals all of G, the image under multiplication of $K \times AN$ is open in G. For the decomposition in (b), we multiply out the relation $v_t m_\pm a_x n_y = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and solve for t, $m_\pm$, x, and y, obtaining

$$\pm = \operatorname{sgn} a, \qquad e^x = |a|, \qquad y = b/a, \qquad t = c/a.$$

Hence we have the required unique decomposition if $a \neq 0$, and the decomposition fails if $a = 0$. Since $VMAN$ equals the open subset of G where the upper left entry is nonzero, the image under multiplication of $V \times MAN$ is open in G.

The group $G = SL(2, \mathbb{R})$ is unimodular, being generated by commutators, and hence the formula in Theorem 12.9 simplifies to $\int_G f(x)\,dx = \int_{S\times T} f(st)\,d_ls\,d_rt$. For (a), we apply this formula with $S = K$ and $T = AN$. The group K is unimodular, so that d_ls becomes $d\theta$, and we easily compute that d_rt can be taken to be $e^{2x}\,dy\,dx$. For (b), we apply the formula with $S = V$ and $T = MAN$. The group V is unimodular, and we find that the right Haar measure for MAN can be taken to be $e^{2x}\,dy\,dx$ on the m_+ part and the same thing on the m_- part.

25. If h is in $C([0,\pi])$, the previous two problems produce a unique $f = f_h$ in $C(G)$ such that f_h is constant on conjugacy classes and has $h(\theta) = f_h(t_\theta)$. Define $\ell(h) = \int_G f_h(x)\,dx$. This is a positive linear functional on $C([0,\pi])$ and yields the measure μ, by the Riesz Representation Theorem. If f is any member of $C(G)$ and $f_0(x) = \int_G f(gxg^{-1})\,dg$, then $\int_G f(x)\,dx = \int_G f_0(x)\,dx$ and f_0 is f_h for the function $h(\theta) = f_0(t_\theta)$. The construction of μ makes $\int_{[0,\pi]} f_0(t_\theta)\,d\mu = \int_G f_0(x)\,dx$. Substitution gives $\int_{[0,\pi]} \big[\int_G f(gt_\theta g^{-1})\,dg\big]\,d\mu = \int_G f_0(x)\,dx = \int_G f(x)\,dx$.

26. The crux of the matter is (a). The formula of Problem 25, together with the character formula for χ_n, gives

$$\delta_{n0} = \int_G \chi_n \overline{\chi_0}\,dx = \int_{[0,\pi]} (e^{in\theta} + e^{i(n-2)\theta} + \cdots + e^{-in\theta})\,d\mu(\theta).$$

This says that $\int_{[0,\pi]} d\mu(\theta) = 1$ for $n = 0$, $\int_{[0,\pi]} (e^{i\theta} + e^{-i\theta})\,d\mu(\theta) = 0$ for $n = 1$, and $\int_{[0,\pi]} (e^{2i\theta} + 1 + e^{-2i\theta})\,d\mu(\theta) = 0$ for $n = 2$. The middle term of the integrand for $n = 2$ has already been shown to produce 1, and thus the $n = 2$ result may be rewritten as $\int_{[0,\pi]}(e^{2i\theta}+e^{-2i\theta})\,d\mu(\theta)=-1$. For $n\geq 3$, comparison of the displayed formula for n with what it is for $n-2$ gives $0=\int_{[0,\pi]}(e^{in\theta}+e^{-in\theta})\,d\mu(\theta)+\delta_{n-2,0}$. Since $n-2>0$, we obtain $\int_{[0,\pi]}(e^{in\theta}+e^{-in\theta})\,d\mu(\theta)=0$ for $n>2$.

For the rest we replace θ by $-\theta$ in our integrals and see that the integral $\int_{[-\pi,0]}(e^{in\theta}+e^{-in\theta})\,d\mu(-\theta)$ is 0 for $n=1$ and $n\geq 3$, and is -1 for $n=2$. Therefore $\int_{[-\pi,\pi]}(e^{in\theta}+e^{-in\theta})\,d\mu'(\theta)$ is 0 for $n=1$ and $n\geq 3$, and is -1 for $n=2$. We can regard μ' as a periodic Stieltjes measure whose Fourier series may be written in terms of cosines and sines. Since $\mu'(E)=\mu'(-E)$, only the cosine terms contribute. There are no point masses since only finitely many Fourier coefficients are nonzero. Since $\cos 2\theta$ has a cosine series with no other $\cos k\theta$ contributing, $\int_{[-\pi,\pi]}\cos n\theta\,d\mu'(\theta)=-\frac{1}{2}\delta_{n,2}=-\frac{1}{2\pi}\int_{[-\pi,\pi]}\cos n\theta\cos 2\theta\,d\theta$ for all $n>0$. Taking into account that $\mu'([-\pi,\pi])=1$, we conclude from the Fourier coefficients that $d\mu'(\theta)=\frac{1}{2\pi}(1-\cos 2\theta)\,d\theta=\frac{1}{\pi}\sin^2\theta\,d\theta$. Since $\int_G f(x)\,dx=\int_{[-\pi,\pi]}\int_G f(gt_\theta g^{-1})\,dg\,d\mu'(\theta)$, substitution into the formula of Problem 25 gives the desired result.

27. Problem 19d shows that the irreducible characters give an orthonormal basis for the subspace of L^2 functions on $SU(2)$ invariant under conjugation. In view of Problem 26d, the restrictions of these characters to the diagonal subgroup T therefore form an orthonormal basis of the subspace of all functions χ in $L^2\big([-\pi,\pi],\frac{1}{\pi}\sin^2\theta\,d\theta\big)$ with $\chi(\theta)=\chi(-\theta)$. Since $\sin^2\theta=\frac{1}{4}|e^{i\theta}-e^{-i\theta}|^2$, the conditions to have a new χ are that it be a continuous function with $\chi(\theta)=\chi(-\theta)$ such that

$$\int_{-\pi}^{\pi}(e^{i\theta}-e^{-i\theta})\chi(\theta)(e^{i(n+1)\theta}-e^{-i(n+1)\theta})=0$$

for every integer $n\geq 0$. Using the condition $\chi(\theta)=\chi(-\theta)$, we can write the Fourier series of χ as $\chi(\theta)\sim\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos k\theta$. For $n\geq 1$, the orthogonality condition says that $\int_{-\pi}^{\pi}\chi(\theta)(\cos(n+2)\theta-\cos n\theta)\,d\theta=0$. Hence $a_{n+2}=a_n$ for $n\geq 1$. By the Riemann–Lebesgue Lemma, all a_n are 0 for $n\geq 1$. Thus χ is constant. Since $\chi_0=1$ is already a known character, $\chi=0$.

28. Let F be a compact topological field. If F is discrete, then each one-point set is open, and the compactness forces F to be finite. Otherwise, every point in F is a limit point. Take a net $\{x_\alpha\}$ in $F-\{0\}$ with limit 0, and form the net $\{x_\alpha^{-1}\}$. By compactness this has a convergent subnet $\{x_{\alpha_\mu}^{-1}\}$ with some limit x_0. By continuity of multiplication, $\{x_{\alpha_\mu}^{-1}x_{\alpha_\mu}\}$ converges to $0x_0=0$. On the other hand, every term of the subnet is 1, and we conclude that a net that is constantly 1 is converging to 0. This behavior means that F is not Hausdorff, contradiction.

29. In (a), the argument that $c\mapsto |c|_F$ is continuous and satisfies $|c_1c_2|_F=|c_1|_F|c_2|_F$ is the same as in Problem 11a.

For (b), we have $d(cx)/|cx|_F=(|c|_F\,dx)/(|c|_F|x|_F)=dx/|x|_F$. For (c), $|x|_F=|x|$ if $F=\mathbb{R}$, and $|x|_F=|x|^2$ if $F=\mathbb{C}$. For (d), $|x|_F=|x|_p$ if $F=\mathbb{Q}_p$. For (e), we have $I=p\mathbb{Z}_p$, and therefore the Haar measure of I is the product of $|p|_p=p^{-1}$ times the Haar measure of $\mathbb{Z}_p$. Hence the Haar measure of I is p^{-1}.

30. In (a), the image of a multiplicative character must be a subgroup of S^1, and the only subgroup of S^1 contained within a neighborhood of radius 1 about the identity is $\{1\}$. Thus as soon as n is large enough so that $p^n\mathbb{Z}_p$ is mapped into the unit "ball" about 1, $p_n\mathbb{Z}_p$ is mapped to 1.

In (b), $\mathbb{Q}_p/\mathbb{Z}_p$ is discrete since $\mathbb{Z}_p$ is open. Hence the cosets of the members of $\mathbb{Q}$ exhaust $\mathbb{Q}_p/\mathbb{Z}_p$, and it is enough to define a multiplicative character of the additive group $\mathbb{Q}$ that is 1 on every member of $\mathbb{Q} \cap \mathbb{Z}_p$. Let a/b be in lowest terms with $b > 0$ and with $|a/b|_p = p^k$. If $k \le 0$, then set $\varphi_0(a/b) = 1$. If $k \ge 0$, write $b = b'p^k$. Since b' and p^k are relatively prime, we can choose integers c and d with $cp^k + b'd = a$, and then $\dfrac{a}{b'p^k} = \dfrac{c}{b'} + \dfrac{d}{p^k}$. We set $\varphi_0(a/b) = e^{2\pi i d/p^k}$. The result is well defined because if $c'p^k + b'd' = a$, then $(c - c')p^k + (d - d')b' = 0$ shows that $d - d'$ is divisible by p^k and hence that $e^{2\pi i d/p^k} = e^{2\pi i d'/p^k}$. One has to check that φ_0 has the required properties.

In (c), we may assume that φ is not trivial. The p-adic number k can be formed by an inductive construction. Use (a) to choose the smallest possible (i.e., most negative) integer n such that φ is trivial on $p^n\mathbb{Z}_p$. Then $x \mapsto \varphi(p^n x)$ is trivial on $\mathbb{Z}_p$ and must be a power of $e^{2\pi i/p}$ on p^{-1}. We match this, adjust φ, iterate the construction through powers of p^{-1}, and prove convergence of the series obtained for k.

31. Write r in $\mathbb{Q}$ as $r = \pm m/n$, assume without loss of generality that $r \ne 0$, and factor m and n as products of powers of primes. Only finitely many primes can appear, and $|r|_p = 1$ if p is prime but is not one of those primes. The only other v is ∞, and thus $|r|_v = 1$ except for finitely many v.

32. With $r \ne 0$ and with $r = \pm m/n$ in lowest terms, factor m and n into products of primes as $m = \prod_{i=1}^k p_i^{a_i}$ and $n = \prod_{j=1}^l q_j^{b_j}$. Then $|r|_{p_i} = p_i^{-a_i}$ and $|r|_{q_j} = q_j^{b_j}$. Hence

$$\prod_{p \text{ prime}} |r|_p = \Big(\prod_{i=1}^k p_i^{-a_i}\Big)\Big(\prod_{j=1}^l q_j^{b_j}\Big) = |r|^{-1} \qquad \text{and} \qquad \prod_{v \in P} |r|_p = |p|^{-1}|p|_\infty = 1.$$

33. The product of topological groups is a topological group, and thus each $X(S)$ is a topological group. The defining properties of a group depend only on finitely many elements at a time, and these will all be in some $X(S)$. Thus X acquires a group structure. The operations are continuous because again they can be considered in a suitable neighborhood of each point, and these points can be taken to be in some $X(S) \times X(S)$ in the case of multiplication, or in some $X(S)$ in the case of inversion. Thus X is a topological group. The assertions about the situation with topological rings are handled similarly.

35. By continuity of translations, it is enough to find an open neighborhood U of 0 in $\mathbb{A}_\mathbb{Q}$ with $U \cap \mathbb{Q} = \{0\}$. Since each $\mathbb{A}_\mathbb{Q}(S)$ is open in $\mathbb{A}_\mathbb{Q}$, it is enough to find this U in some $\mathbb{A}_\mathbb{Q}(S)$. We do so for $S = \{\infty\}$. Let $U = (-1/2, 1/2) \times \big(\bigtimes_{p \text{ prime}} \mathbb{Z}_p\big)$. If x is in U, then $|x|_p \le 1$ for all primes p and $|x|_\infty < 1/2$. By Problem 32, x cannot be

in $\mathbb{Q}$ unless $x = 0$. Hence $U \cap \mathbb{Q} = \{0\}$. Proposition 6.3b shows that $\mathbb{Q}$ is therefore discrete.

36. If $x = (x_v)$ is in $\mathbb{A}_\mathbb{Q}$, let $p_1, \dots, p_n$ be the primes p where $|x_p|_p > 1$, and let $|x_p|_{p_j} = p_j^{a_j}$. If $r = \prod_{j=1}^n p_j^{-a_j}$ and if we regard r as embedded diagonally in $\mathbb{A}_\mathbb{Q}$, then $|x_p r^{-1}|_p \le 1$ for every prime p. Hence xr^{-1} is in $\mathbb{A}_\mathbb{Q}(\{\infty\})$. Choose an integer n such that $|x_\infty r^{-1} - n|_\infty \le 1$. If we then regard n as embedded diagonally in $\mathbb{A}_\mathbb{Q}$, then $|n|_p \le 1$ for all primes p, and hence n is in $\mathbb{A}_\mathbb{Q}(\{\infty\})$. Thus $xr^{-1} - n$ is in the compact subset $K = [-1, 1] \times \big(\bigtimes_{p \text{ prime}} \mathbb{Z}_p \big)$ of $\mathbb{A}_\mathbb{Q}$. The continuous image of K in $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ is compact, and we have just seen that this image is all of $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$. Thus $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ is compact.

37. Fix a finite subset S of P containing $\{\infty\}$. Then the projection of $\bigtimes_{w \in S} \mathbb{Q}_w^\times$ to $\mathbb{Q}_v^\times$ is continuous for each $v \in S$. Since also the inclusion $\mathbb{Q}_v^\times \to \mathbb{Q}_v$ is continuous, the composition $\bigtimes_{w \in S} \mathbb{Q}_w^\times \to \mathbb{Q}_v$ is continuous. Thus the corresponding mapping $\bigtimes_{w \in S} \mathbb{Q}_w^\times \to \bigtimes_{w \in S} \mathbb{Q}_w$ is continuous. In similar fashion $\bigtimes_{w \notin S} \mathbb{Z}_w^\times \to \mathbb{Z}_v$ is a continuous function as a composition of continuous functions. Thus $\bigtimes_{w \notin S} \mathbb{Z}_w^\times \to \bigtimes_{w \notin S} \mathbb{Z}_w$ is continuous. Putting these two compositions together shows that $\mathbb{A}_\mathbb{Q}^\times(S) \to \mathbb{A}_\mathbb{Q}(S)$ is continuous, and therefore $\mathbb{A}_\mathbb{Q}^\times(S) \to \mathbb{A}_\mathbb{Q}$ is continuous. Since this is true for each S, it follows that $\mathbb{A}_\mathbb{Q}^\times \to \mathbb{A}_\mathbb{Q}$ is continuous.

The topologies on the adeles $\mathbb{A}_\mathbb{Q}$ and the ideles $\mathbb{A}_\mathbb{Q}^\times$ are regular and Hausdorff, and they are both separable. Hence $\mathbb{A}_\mathbb{Q}$ and $\mathbb{A}_\mathbb{Q}^\times$ are metric spaces, and the distinction between the topologies can be detected by sequences. Let p_n be the n^{th} prime, and let $x_n = (x_{n,v})$ be the adele with $x_{n,v} = p_n$ if $v = p_n$ and $x_{n,v} = 1$ if $v \ne p_n$. The result is a sequence $\{x_n\}$ of ideles, and we show that it converges to the idele (1) in the topology of the adeles but does not converge in the topology of ideles. In fact, each x_n lies in $\mathbb{A}_\mathbb{Q}(\{\infty\})$, which is an open set in $\mathbb{A}_\mathbb{Q}$. For each prime p, $x_{n,p} = 1$ if n is large enough, and also $x_{n,\infty} = 1$ for all n. Since $\mathbb{A}_\mathbb{Q}(\{\infty\})$ has the product topology, $\{x_n\}$ converges to (1). On the other hand, if $\{x_n\}$ were to converge to some limit x in $\mathbb{A}_\mathbb{Q}^\times$, then x would have to lie in some $\mathbb{A}_\mathbb{Q}^\times(S)$, and the ideles x_n would have to be in $\mathbb{A}_\mathbb{Q}^\times(S)$ for large n. But $(x_{n,v})$ is not in $\mathbb{A}_\mathbb{Q}^\times(S)$ as soon as v is outside S.

39. In (a), let f be in $C(K)$. Corollary 6.7 shows that the map $k \mapsto kf$ of K into the left translates of f is continuous into $C(K)$. The continuous image of a compact set is compact, and thus f is left almost periodic. Similarly f is right almost periodic.

In (b), let g be in G. Then $(gf)(x) = f(g^{-1}x) = F(\iota(g^{-1}x)) = F(\iota(g)^{-1}\iota(x)) = ((\iota(g)F)(\iota(x))$ shows that the set of left translates of f can be regarded as a subset of the set of left translates of F. The latter is compact, and hence the closure of the former is compact.

40. We may view the unitary representation Φ as a continuous homomorphism of G into the compact group $K = U(N)$ for some N. If $f(x) = \Phi(x)_{ij}$, then $f(x) = F(\Phi(x))$, where $F : U(N) \to \mathbb{C}$ is the $(i, j)^{\text{th}}$ entry function. Thus Problem 39b applies.

41. In (a), assume the contrary. Then for some $\epsilon > 0$ and for every neighborhood N of the identity, there exists g_N in N with $\|g_N f - f\|_{\sup} \geq \epsilon$. Here $\{g_N f\}$ is a net in the compact metric space K_f, and there must be a convergent subnet $\{g_{N_\alpha} f\}$ with limit some function h in K_f. Since $\|g_{N_\alpha} f - h\|_{\sup}$ tends to 0, h is not f. Thus $g_{N_\alpha} f$ converges uniformly to h while, by continuity, tending pointwise to f. Since $h \neq f$, we have arrived at a contradiction.

Part (b) follows from the formula $\|g_0(g_1 f) - g_0(g_2 f)\|_{\sup} = \|g_1 f - g_2 f\|_{\sup}$, and part (c) follows from (b), uniform continuity, and completeness of the compact set K_f.

42. Part (a) follows from a remark with Ascoli's Theorem when stated as Theorem 2.56 of *Basic*: the remark says that if we have an equicontinuous sequence of functions from a compact metric space into a compact metric space, then there is a uniformly convergent subsequence. Here if we have a sequence $\{\varphi_n\}$ of isometries of X onto itself, then the φ_n are equicontinuous with $\delta = \epsilon$. Since the domain X is compact and the image X is compact, the sequence has a uniformly convergent subsequence, and we readily check that the limit is an isometry. Since every sequence in Γ has a convergent subsequence, Γ is compact.

For (b), let members of Γ have $\varphi_n \to \varphi$ and $\psi_n \to \psi$. Then

$$\rho(\varphi_n \circ \psi_n, \varphi \circ \psi) \leq \rho(\varphi_n \circ \psi_n, \varphi_n \circ \psi) + \rho(\varphi_n \circ \psi, \varphi \circ \psi).$$

The first term on the right side equals $\rho(\psi_n, \psi)$ because φ_n is an isometry, and the second term equals $\rho(\varphi_n, \varphi)$ because $\psi(x)$ describes all members of X as x varies through X. These two terms tend to 0 by assumption and hence $\varphi_n \circ \psi_n \to \varphi \circ \psi$. This proves continuity of multiplication. Similarly inversion is continuous.

For (c), let $\gamma_n \to \gamma$ and $x_n \to x$. Then

$$d(\gamma_n(x_n), \gamma(x)) \leq d(\gamma_n(x_n), \gamma(x_n)) + d(\gamma(x_n), \gamma(x)) \leq \rho(\gamma_n, \gamma) + d(\gamma(x_n), \gamma(x)),$$

and both terms on the right side tend to 0.

43. In (a), let $\{g_n\}$ be a net convergent to g_0 in G, and form $\{\iota(g_n)\}$. Then $\rho(\iota(g_n), \iota(g_0)) = \sup_{h \in K_f} \|\iota(g_n)h - \iota(g_0)h\|_{\sup} = \sup_{h \in K_f, x \in G} |\iota(g_n)h(x) - \iota(g_0)h(x)|$ $= \sup_{h \in K_f, x \in G} |h(g_n^{-1}x) - h(g_0^{-1}x)| = \sup_{y \in G, x \in G} |(yf)(g_n^{-1}x) - (yf)(g_0^{-1}x)| =$ $\sup_{y \in G, x \in G} |f(y^{-1}g_n^{-1}x) - f(y^{-1}g_0^{-1}x)|$. If this does not tend to 0 as g_n tends to g_0, then we can find a subnet of $\{g_n\}$, which we write without any change in notation, and some $\epsilon > 0$ such that this supremum is $\geq \epsilon$ for every n. To each such n, we associate some y_n such that $\sup_{x \in G} |f(y_n^{-1}g_n^{-1}x) - f(y_n^{-1}g_0^{-1}x)| \geq \epsilon/2$. By left almost periodicity we can find a subnet of $\{y_n f\}$ that converges uniformly to some function, say H. This function H has to be left uniformly continuous, and we may suppose that $\|y_n f - H\|_{\sup} \leq \epsilon/8$ for $n \geq N$. Then $n \geq N$ implies

$$\begin{aligned} &|(y_n f)(g_n^{-1}x) - (y_n f)(g_0^{-1}x)| \\ &\leq |(y_n f)(g_n^{-1}x) - H(g_n^{-1}x)| + |H(g_n^{-1}x) - H(g_0^{-1}x)| + |H(g_0^{-1}x) - (y_n f)(g_0^{-1}x)| \\ &\leq \tfrac{\epsilon}{8} + |H(g_n^{-1}x) - H(g_0^{-1}x)| + \tfrac{\epsilon}{8}. \end{aligned}$$

The left uniform continuity of H implies that the right side is eventually $\leq \frac{3\epsilon}{8}$. This contradicts the condition $\sup_{x\in G}|f(y_n^{-1}g_n^{-1}x) - f(y_n^{-1}g_0^{-1}x)| \geq \epsilon/2$, and (a) is proved.

In (b), the action $\Gamma_f \times K_f \to K_f$ is continuous by Problem 42c, and therefore $\gamma \mapsto \gamma^{-1}h$ is continuous. Evaluation of members of K_f at 1 is continuous, and hence $F_f(h)$ is continuous on Γ_f. If $\{g_n\}$ is a net with $g_n f \to h$, then $F_f(h)(\iota_f(g_0)) = ((\iota_f(g_0))^{-1}h)(1) = \lim_n((\iota_f(g_0))^{-1}g_n f)(1) = \lim_n(g_n f)(g_0) = h(g_0)$.

For (c), we apply (b) with $h = f$. Then f arises from the compact group Γ_f via the construction in Problem 39b. Therefore f is right almost periodic.

44. If f is a given almost periodic function, the function F to use takes an element $\prod_{f'}(\gamma_{f'})$ to $F_f(\gamma_f)$. Then the equality $F(\iota(x)) = F_f(\iota_f(x)) = f(x)$ shows that f arises from the compact group Γ.

45. Problem 44 produces an isomorphism of the algebra $LAP(G)$ of almost periodic functions on G onto $C(\Gamma)$, and the Stone Representation Theorem (Theorem 4.15) produces an isomorphism of $LAP(G)$ with $C(S_1)$, where S_1 is the Bohr compactification of G. The result then follows after applying Problem 23 in Chapter IV.

46. Finite-dimensional unitary representations of Γ give rise to finite-dimensional unitary representations of G, and thus Corollary 6.33 for Γ gives the desired result.

47. Any continuous multiplicative character of K yields a continuous multiplicative character of G. Conversely any continuous multiplicative character of G is almost periodic by Problem 40 and therefore yields a continuous function on K. The multiplicative property of this continuous function on the dense set $p(G)$, together with continuity of multiplication on K, implies that the function on K is a multiplicative character.

Chapter VII

1. If x_0 is in Ω, let φ be a compactly supported smooth function on Ω equal to $(x - x_0)^\alpha$ in an open neighborhood V of x_0. Then $0 = (P(x, D)u)(x) = (\alpha!)a_\alpha(x)$ on V, and hence $a_\alpha(x) = 0$ for x in V.

2. Within the Banach space $C(\Omega^{\mathrm{cl}}, \mathbb{R})$, S is the vector subspace of all functions u with $Lu = 0$ on Ω. It contains the constants and hence is not 0. The restriction mapping $R : S \to C(\partial\Omega, \mathbb{R})$ is one-one by the maximum principle (Theorem 7.12), and it has norm 1. Let V be the image of R, and let $R^{-1} : V \to S$ be the inverse of $R : S \to V$. The operator R^{-1} has norm 1 as a consequence of the maximum principle. If e_p denotes evaluation at the point p of Ω, then $e_p \circ R^{-1}$ is a bounded linear functional on V of norm 1. The Hahn–Banach Theorem shows that $e_p \circ R^{-1}$ extends to a linear functional ℓ on $C(\partial\Omega, \mathbb{R})$ of norm 1. We know that $\ell(1) = e_p \circ R^{-1}(1) = e_p(1) = 1$. If $f \geq 0$ is a nonzero element in $C(\partial\Omega, \mathbb{R})$, then $1 - f/\|f\|_{\sup}$ has norm ≤ 1. Therefore $|\ell(1 - f/\|f\|_{\sup})| \leq 1$ and $0 \leq \ell(f/\|f\|_{\sup}) \leq 2$. Thus the

linear functional ℓ is positive. By the Riesz Representation Theorem, ℓ is given by a measure μ_p. Consequently every u is S has $u(p) = \int_{\partial\Omega} u(x)\,d\mu_p(x)$. Taking $u = 1$ shows that $\mu_p(\partial\Omega) = 1$ for every p.

3. In (a), the line integral $\oint_{|(x,y)|=\varepsilon} (P\,dx + Q\,dy)$ is equal to

$$\int_0^{2\pi} \varphi(\varepsilon\cos\theta, \varepsilon\sin\theta)\varepsilon^{-2}\big((\varepsilon\cos\theta)(-\varepsilon\sin\theta) + (\varepsilon\sin\theta)(\varepsilon\cos\theta)\big)\,d\theta,$$

and the integrand is identically 0. Part (b) is just a computation of partial derivatives. If (c), we know from Green's Theorem that for any positive numbers $\varepsilon < R$,

$$\Big(\oint_{|(x,y)|=R} - \oint_{|(x,y)|=\varepsilon}\Big)(P\,dx + Q\,dy) = \iint_{\varepsilon\le|(x,y)|\le R} \big(\tfrac{\partial Q}{\partial x} - \tfrac{\partial P}{\partial y}\big)\,dx\,dy.$$

With our P and Q, for sufficiently large R, the line integral $\oint_{|(x,y)|=R}$ is 0 since P and Q have compact support, and (a) says that the limit of the line integral $\oint_{|(x,y)|=\varepsilon}$ is 0 as ε decreases to 0. The function $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{y\varphi_x - x\varphi_y}{x^2+y^2}$ is integrable near $(0,0)$, and we thus conclude from the complete additivity of the integral that $\iint_{\mathbb{R}^2} \big(\frac{y\varphi_x - x\varphi_y}{x^2+y^2}\big)\,dx\,dy = 0$.

In (d), with a new P and Q, the line integral $\oint_{|(x,y)|=\varepsilon} (P\,dx + Q\,dy)$ is equal to

$$\int_0^{2\pi} \varphi(\varepsilon\cos\theta, \varepsilon\sin\theta)\varepsilon^{-2}\big((-\varepsilon\sin\theta)(-\varepsilon\sin\theta) + (\varepsilon\cos\theta)(\varepsilon\cos\theta)\big)\,d\theta.$$

This simplifies to $\int_0^{2\pi} \varphi(\varepsilon\cos\theta, \varepsilon\sin\theta)\,d\theta$, which tends to $2\pi\varphi(0,0)$ by continuity of φ. Part (e) is just a computation of partial derivatives, and part (f) is proved in the same way as part (c).

For (g), we have $z^{-1}\frac{\partial\varphi}{\partial\bar z} = z^{-1}(\varphi_x + i\varphi_y) = \frac{x-iy}{x^2+y^2}(\varphi_x + i\varphi_y) = \frac{x\varphi_x + y\varphi_y}{x^2+y^2} + \frac{i(x\varphi_y - y\varphi_x)}{x^2+y^2}$. Combining (c) and (f) gives $\iint_{\mathbb{R}^2} z^{-1}\frac{\partial\varphi}{\partial\bar z}\,dx\,dy = -2\pi\varphi(0,0) + i0$, and hence $\frac{1}{2\pi}\iint_{\mathbb{R}^2} z^{-1}\frac{\partial\varphi}{\partial\bar z} = -\varphi(0,0)$.

For (h), we use (g) and obtain $\langle\frac{\partial T}{\partial\bar z}, \varphi\rangle = -\langle T, \frac{\partial\varphi}{\partial\bar z}\rangle = -\iint_{\mathbb{R}^2}(2\pi z)^{-1}\frac{\partial\varphi}{\partial\bar z}\,dx\,dy = \varphi(0,0) = \langle\delta, \varphi\rangle$.

4. In (a), let φ be in $C^\infty_{\text{com}}(\mathbb{R}^1)$. Then $\langle D_x H, \varphi\rangle = -\langle H, \varphi'\rangle = -\int_{-\infty}^{\infty} H(x)\varphi'(x)dx = -\int_0^\infty \varphi'(x)\,dx = -\lim_N [\varphi(x)]_0^N = \varphi(0) = \langle\delta, \varphi\rangle$.

In (b) let φ be in $C^\infty_{\text{com}}((-1,1))$. We are to verify that $\int_{-1}^1 \max\{x,0\}\varphi'(x)\,dx = -\int_{-1}^1 H(x)\varphi(x)\,dx$, i.e., that $\int_0^1 x\varphi'(x)\,dx = -\int_0^1 \varphi(x)\,dx$. This follows from integration by parts because $\int_0^1 x\varphi'(x)\,dx = [x\varphi(x)]_0^1 - \int_0^1 \varphi(x)\,dx = -\int_0^1 \varphi(x)\,dx$.

The answer to (c) is no. If g were a weak derivative, then the left side of the equality $\int_{-1}^1 H(x)\varphi'(x)\,dx = -\int_{-1}^1 g(x)\varphi(x)\,dx$ would be 0 whenever $\varphi \in C^\infty_{\text{com}}((-1,1))$ vanishes in a neighborhood of 0. Then $g(x)$ would have to be 0 almost everywhere for $x \ne 0$, and we would necessarily have $0 = \int_0^1 \varphi'(x)\,dx = [\varphi(x)]_0^1 = -\varphi(0)$ for all φ in $C^\infty_{\text{com}}((-1,1))$.

In (d), $\langle D_x(H \times \delta), \varphi\rangle = -\langle H \times \delta, D_x\varphi\rangle = -\int_0^\infty (D_x\varphi)(x,0)\,dx$, and this $= -\lim_N [\varphi(x,0)]_{x=0}^{x=N} = \varphi(0,0) = \langle \delta, \varphi\rangle$.

In (e), the support of H is $[0,\infty)$ and the singular support is $\{0\}$, while for $H \times \delta$ the support and the singular support are both $\mathbb{R} \times \{0\}$.

5. We apply Lemma 7.8 to $R(x) = P(ix)$. The preliminary step in the proof multiplies the given distribution f by something so that f has support near 0. We form $e^{-i\alpha\cdot x} f$ as a member of $\mathcal{E}'((-2\pi, 2\pi)^N)$ and restrict it to a member of $\mathcal{P}'(T^N)$. Then it has a Fourier series $e^{-i\alpha\cdot x} f \sim \sum_k d_k e^{ik\cdot x}$. Put $c_k = \frac{d_k}{R(k+\alpha)}$, α being the member of $\mathbb{R}^N$ produced by the lemma. Then $|c_k| \le C(1+|k|^2)^p$ for some p, and (b) produces a distribution S in $\mathcal{E}'((-2\pi, 2\pi)^N)$ with $\langle S, e^{-ik\cdot x}\rangle = c_k$ for all k. Define $u = e^{i\alpha\cdot x} S$ as a member of $\mathcal{E}'((-2\pi, 2\pi)^N)$. Let $\psi(x)$ be a smooth function with compact support near 0, and extend ψ to be periodic, i.e., to be in $C^\infty(T^N)$. The multiple Fourier series of ψ is then of the form $\psi(x) = \sum_k \gamma_k e^{ik\cdot x}$ with γ_k decreasing faster than any power of $|k|$. The function $\varphi(x) = \psi(x)e^{-i\alpha\cdot x}$ is in $C^\infty((-2\pi, 2\pi)^N)$ but is not necessarily periodic. Applying $P(D)$ to u and having the result act on φ, we write

$$\langle P(D)u, \varphi\rangle = \langle P(D)u, \textstyle\sum_k \gamma_k e^{i(k-\alpha)\cdot x}\rangle = \langle P(D)u, \textstyle\sum_k \gamma_{-k} e^{-i(k+\alpha)\cdot x}\rangle.$$

Since the γ_k are rapidly decreasing and $P(D)u$ is continuous on $C^\infty((-2\pi, 2\pi)^N)$, we can interchange the summation and the operation of $P(D)u$. Thus the right side of the display is

$$\begin{aligned}
\textstyle\sum_k \gamma_{-k}\langle P(D)u, e^{-i(k+\alpha)\cdot x}\rangle &= \textstyle\sum_k \gamma_{-k}\langle u, P(-D)(e^{-i(k+\alpha)\cdot x})\rangle \\
&= \textstyle\sum_k \gamma_{-k}\langle e^{i\alpha\cdot x} S, P(i(k+\alpha))e^{-i(k+\alpha)\cdot x}\rangle = \sum_k \gamma_{-k}\langle S, P(i(k+\alpha))e^{-ik\cdot x}\rangle \\
&= \textstyle\sum_k \gamma_{-k} c_k P(i(k+\alpha)) = \sum_k \gamma_{-k} \frac{d_k}{R(k+\alpha)} P(i(k+\alpha)) = \sum_k \gamma_{-k} d_k.
\end{aligned}$$

Now $d_k = \langle e^{-i\alpha\cdot x} f, e^{-ik\cdot x}\rangle$. The rapid convergence of the series $\sum_k \gamma_{-k} e^{-ik\cdot x}$ means that $\langle e^{-i\alpha\cdot x} f, \psi\rangle = \sum_k \gamma_{-k}\langle e^{-i\alpha\cdot x} f, e^{-ik\cdot x}\rangle = \sum_k \gamma_{-k} d_k$. Therefore $\langle P(D)u, \varphi\rangle = \sum_k \gamma_{-k} d_k = \langle e^{-i\alpha\cdot x} f, \psi\rangle = \langle e^{-i\alpha\cdot x} f, e^{i\alpha\cdot x}\varphi\rangle = \langle f, \varphi\rangle$. Near 0, the function φ is an arbitrary smooth function, and thus $P(D)u = f$ near 0.

6. The coefficient of x^α in $(x_1 + \cdots + x_N)^{|\alpha|}$ is the multinomial coefficient $\binom{|\alpha|}{\alpha_1,\dots,\alpha_N} = \frac{|\alpha|!}{\alpha!}$. This is a positive integer, and hence $\alpha! \le |\alpha|!$. Fixing $|\alpha| = l$ and putting $x_1 = \cdots = x_N = 1$, we obtain the formula $N^l = \sum_{|\alpha|=l} \frac{l!}{\alpha!}$, and therefore $\sum_{|\alpha|=l} (1/\alpha!) = N^l/l!$. The identity with z can be proved by induction on q, the base case being $q = 0$, where the expansion is a geometric series. If the case q is known, we differentiate both sides and divide by $q+1$ to obtain the case $q+1$. Alternatively, one can derive the identity from the binomial series expansion in Section I.7 of *Basic*.

7. Here is the solution apart from some details. The argument uses induction, the base case being $m = 1$, where the result describes the given system of differential equations. Assuming that D_t^{m-1} is of the asserted form, we differentiate the expression

with respect to t. In the 2^{m-1} terms of the first kind, the derivative acts on some expression $D_x^\alpha u$, giving $D_x^\alpha D_t u$. We substitute for $D_t u$ from the given system and sort out what happens; we get 2^m terms involving an x derivative of u and 2^{m-1} terms involving a derivative of F. In the $2^{m-1}-1$ terms of the second kind, the derivative acts on some iterated partial derivative of F and just raises the order of differentiation. The total number of terms involving F is then $2^{m-1}+2^{m-1}-1=2^m-1$.

8. In (a), just apply D_x^β to the formula for $D_t^m u$ in the previous problem. The operator gets applied to each u or F that appears in the formula, and there is no simplification. Then one evaluates at $(0,0)$. In (b), the asserted finiteness implies that the multiple power series

$$U(x,t)=\sum_\beta \sum_{m\geq 0} \tfrac{D_x^\beta D_t^m u(0,0)}{\beta!\,m!}\, x^\beta t^m$$

converges when $|t|<r$ and $|x_j|<r$ for all j and that $D_x^\beta D_t^m U(0,0)=D_x^\beta D_t^m u(0,0)$ for all β and m. Then it follows that the sum $U(x,t)$ solves the given Cauchy problem for these values of (x,t). Since r is arbitrary, the series converges for all $(x,t)\in\mathbb{C}^{N+1}$ and the sum $U(x,t)$ solves the Cauchy problem globally.

9. In (a), we consider a single term of the expansion of $D_t^m u(0,0)$, namely $T_1\cdots T_m D_x^\alpha u(0,0)=T_1\cdots T_m D_x^\alpha g(0)$. Here each of $T_1,\dots,T_m$ is equal to some A_{j_i} or to B, and D_x^α is the product over i of the D_{j_i} for those T_i with $T_i=A_{j_i}$. The term has $\|T_1\cdots T_m D_x^\alpha g(0)\|_\infty \leq M^m\|D_x^\alpha g(0)\|_\infty$, and the boundedness of the series involving $g(0)$ implies that $(\alpha!)^{-1}\|D_x^\alpha g(0)\|_\infty R^{|\alpha|}\leq C$. Let k be the number of factors of $T_1\cdots T_m$ equal to B. Then $|\alpha|=m-k$, and hence $M^m\|D_x^\alpha g(0)\|_\infty\leq CM^m\alpha!R^{-(m-k)}$. Each T_i equal to A_{j_i} has to be summed over the N values of j_i, and we get a contribution of N^{m-k} from all these sums. Finally the number of such terms involving k factors B is the number of subsets of k elements in a set of m elements and is $\binom{m}{k}$, and $\alpha!\leq(m-k)!$ by Problem 6. The desired estimate results.

In (b), we adjust the above estimate by replacing $\|D_x^\alpha g(0)\|_\infty$ by $\|D_x^{\alpha+\beta}g(0)\|_\infty$. Then $C\alpha!R^{-(m-k)}$ gets replaced by $C(\alpha+\beta)!R^{-(m-k+l)}$, where $l=|\beta|$. Since $(\alpha+\beta)!\leq(m-k+l)!$, the term is $\leq \sum_{k=0}^m CM^mN^{m-k}(m-k+l)!\binom{m}{k}R^{-(m-k+l)}$.

In (c), we are to sum the product of the estimate in (b) by $\dfrac{r^{l+m}}{\beta!m!}$, the sum extending over all $m\geq 0$, all $l\geq 0$, and all β with $|\beta|=l$. Thus we are to bound

$$\sum_{m=0}^\infty\sum_{l=0}^\infty\sum_{|\beta|=l}\sum_{k=0}^m \frac{CM^mN^{k-m}(m-k+l)!\binom{m}{k}R^{-(m-k+l)}r^{l+m}}{\beta!m!}$$
$$=\sum_{m=0}^\infty\sum_{l=0}^\infty\sum_{k=0}^m \frac{CM^mN^{m-k+l}(m-k+l)!\binom{m}{k}R^{-(m-k+l)}r^{l+m}}{l!m!}$$

$$= C\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\Big[\sum_{l=0}^{\infty}\binom{m-k+l}{l}\Big(\frac{Nr}{R}\Big)^{l}\Big]\frac{M^m N^{m-k}R^{-(m-k)}r^m}{k!}$$
$$= C\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\Big(1-\frac{Nr}{R}\Big)^{-(m-k)-1}\frac{M^m N^{m-k}R^{-(m-k)}r^m}{k!},$$

the first and third steps using Problem 6 and the third step requiring the assumption on R that $Nr/R < 1$. If we assume in fact that $Nr/R \le 1/2$, then $\big(1-\frac{Nr}{R}\big)^{-1} \le 2$, and the above expression is

$$\le C\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{2^{m-k+1}M^m N^{m-k}R^{-(m-k)}r^m}{k!} \le 2C\sum_{m=0}^{\infty} e^{R/(2N)}\Big(\frac{2MrN}{R}\Big)^m,$$

the second inequality following from the series expansion of the exponential function. The series on the right is convergent if $2MrN/R < 1$. This proves (c).

In (d), the analog of (a) is to consider a term $T_1\cdots T_s D_x^{\alpha}D_t^{m-1-s}F$, where each T_i is some A_{j_i} or B. Let k be the number of factors B, so that $s-k$ factors are some A_j and $|\alpha| = s-k$. The contributions to D_x^{α} come from the factors A_j; regard the $m-1-s$ contributions to D_t^{m-1-s} as coming from factors of the identity I. In this way the two phenomena can be handled at the same time. Ignore the fact that I commutes with the other matrices; it is easier to treat it as if its occurrences in different positions were different. The effect is the same as expanding the set of n matrices A_j to include I, yielding a set of $N+1$ matrices. The requirement $M \ge 1$ makes it so that the estimate $\|Iv\|_\infty \le M\|v\|_\infty$ is valid for the new member of the set, as well as the old members. The steps for imitating (b) and (c) are then essentially the same as before except that m is replaced by $m-1$ and N is sometimes replaced by $N+1$.

10. The crux of the matter is to show that if $\{u^{i,j}(x,y)\}$ solves the Cauchy problem for the first-order system, then $u^{i,j}(x,y) = D_x^i D_y^j u^{0,0}(x,y)$ for $i+j \le m$ and hence $u^{0,0}(x,y)$ solves the m^{th}-order equation. The proof proceeds by induction on j. The case $j=0$ is okay because the first-order system has $D_x u^{i,0} = u^{i+1,0}$ for $i<m$. Suppose the identity holds for some j. Then $D_x u^{i,j+1} = D_y u^{i+1,j}$ from the system, and this is $= D_y D_x u^{i,j}$ by induction. Hence $D_x(u^{i,j+1} - D_y u^{i,j}) = 0$, and we obtain $u^{i,j+1} - D_y u^{i,j} = c(y)$. Put $x=0$ and get $u^{i,j+1}(0,y) = D_y^{j+1} f^{(i)}(y) = D_y D_y^j f^{(i)}(y) = D_y u^{i,j}(0,y)$. Therefore $c(y)=0$, and $u^{i,j+1} = D_y u^{i,j} = D_x^i D_y^{j+1} u^{0,0}$. This completes the induction.

11. The second index (j in Problem 10) is replaced by an $(N-1)$-tuple $\alpha = (\alpha_1,\dots,\alpha_{N-1})$. If $\beta \ne 0$, the equation for $D_x u^{i,\beta}$ is $D_x u^{i,\beta} = D_{y_j} u^{i,\alpha}$, where j is the first index for which $\alpha_j \ne 0$ and where α is obtained from β by reducing the j^{th} index by 1. If $\beta = 0$, the equations are as in Problem 10. The Cauchy data are $u^{i,\beta}(0,y) = D_y f^{(i)}(y)$ except when $(i,\beta) = (m,0)$, and they are the data of Problem 10 when $(i,\beta) = (m,0)$. The argument now inducts on $\beta_1,\dots,\beta_{N-1}$, and

the functions $c(y)$ that appear are of the form $c(y_1, \dots, y_{N-1})$. The Cauchy data are for $x = 0$, and we get an equation $c(y_1, \dots, y_{N-1}) = 0$ in one step in each case.

12. The equations $D_x u^{i,j+1} = D_y u^{i+1,j}$ involve first partial derivatives in the y direction with coefficient 1, and $D_x u^{i,0} = u^{i+1,0}$ involves an undifferentiated variable with coefficient 1. The equation for $D_x u^{m,0}$ involves a linear combination of variables and first partial derivatives in the y direction of variables, plus the term F_x, which is an entire holomorphic function of (x, y). So the equations of the first-order system are as in Problems 6–9.

Chapter VIII

1. What needs checking is that the two charts are smoothly compatible. The set $M_{\kappa_1} \cap M_{\kappa_2}$ is $S^n - \{(0, \dots, 0, \pm 1)\}$, and the image of this under κ_1 and κ_2 is $\mathbb{R}^n - \{(0, \dots, 0)\}$. Put $y_j = x_j/(1-x_{n+1})$, so that $\kappa_1^{-1}(y_1, \dots, y_n) = (x_1, \dots, x_{n+1})$. Then

$$\begin{aligned}\kappa_2 \circ \kappa_1^{-1}(y_1, \dots, y_n) &= (x_1/(1+x_{n+1}), \dots, x_n/(1+x_{n+1})) \\ &= (y_1(1-x_{n+1})/(1+x_{n+1}), \dots, y_n(1-x_{n+1})/(1+x_{n+1})).\end{aligned}$$

To compute $(1-x_{n+1})/(1+x_{n+1})$, we take $|x| = 1$ into account and write $1 = \sum_{j=1}^{n+1} x_j^2 = x_{n+1}^2 + \sum_{j=1}^n y_j^2(1-x_{n+1})^2$. Then $\sum_{j=1}^n y_j^2 = (1-x_{n+1}^2)/(1-x_{n+1})^2 = (1+x_{n+1})/(1-x_{n+1})$, and

$$\kappa_2 \circ \kappa_1^{-1}(y_1, \dots, y_n) = \big(y_1 \big/ \textstyle\sum_{j=1}^n y_j^2, \dots, y_n \big/ \sum_{j=1}^n y_j^2\big).$$

The entries on the right are smooth functions of y since $y \neq 0$, and the two charts are therefore smoothly compatible.

3. If it is σ-compact, it is Lindelöf. If it is Lindelöf, countably many charts suffice to cover X. If there is a countable dense set, then we can choose one chart for each member of the dense set, and these will have to cover X. This proves (a). For (b), each chart has a countable base, and the union of these countable bases, as the chart varies, is a countable base for X.

4. For (a), multiplication is given by polynomial functions, which are smooth. Inversion, according to Cramer's rule, is given by polynomial functions and division by the determinant, and inversion is therefore smooth.

For (b), we have

$$\begin{aligned}\widetilde{A}_g f &= (dl_g)_1(A)(f) = A(f \circ l_g) = A(f(g\,\cdot)) = \sum_{i,j} A_{ij} \tfrac{\partial(f(g\,\cdot))}{\partial x_{ij}}(1) \\ &= \sum_{i,j} A_{ij} \textstyle\sum_{r,s} \tfrac{\partial f}{\partial x_{rs}}(g) \tfrac{\partial((gx)_{rs})}{\partial x_{ij}}(1) = \displaystyle\sum_{i,j,r,s} A_{ij} \tfrac{\partial f}{\partial x_{rs}}(g) g_{ri} \delta_{sj} \\ &= \sum_{j,r,s} (gA)_{rj} \delta_{sj} \tfrac{\partial f}{\partial x_{rs}}(g) = \sum_{r,s} (gA)_{rs} \tfrac{\partial f}{\partial x_{rs}}(g).\end{aligned}$$

For (c), the condition for smoothness, by Proposition 8.8, is that all $\widetilde{A}x_{ij}$ be smooth functions. Part (b) gives $\widetilde{A}x_{ij}(g) = \widetilde{A}_g(x_{ij}) = \sum_{r,s}(gA)_{rs}\delta_{ir}\delta_{js} = (gA)_{ij} = \sum_k g_{ik}A_{kj}$, and the right side is a smooth function of the entries of g. For the left invariance, let $F = l_h$, and put $g' = F^{-1}(g) = h^{-1}g$. We are to check that $(dF)_{g'}(\widetilde{A}_{g'})(f) = \widetilde{A}_g(f)$ if f is defined near g. The left side is equal to $\widetilde{A}_{g'}(f \circ l_h) = ((dl_{g'})_1(A))(f \circ l_h) = (dl_h)_{g'}(dl_{g'})_1(A)(f)$, and the right side is $\widetilde{A}_g(f) = (dl_g)_1(A)(f)$. These two expressions are equal by Proposition 8.7.

Parts (d) and (e) amount to the same thing. For (d), the question is whether $\widetilde{A}_{g_0 \exp tA} f = (dc)_t\big(\frac{d}{dt}\big)(f)$. The right side is $\frac{d}{dt} f(g_0 \exp tA)$, and that is why (d) and (e) amount to the same thing. The left side is $\sum_{r,s}(g_0(\exp tA)A)_{rs}\frac{\partial f}{\partial x_{rs}}(g_0 \exp tA)$ by (b), and this expression equals $\frac{d}{dt} f(g_0 \exp tA)$ by the chain rule and the formula $\frac{d}{dt}\exp tA = (\exp tA)A$ known from *Basic*.

5. For (a), fix l. Choose, for each p in L_l, a compatible chart about p such that the closure of the domain of the chart is a compact subset of U_l. The domains of these charts form an open cover of L_l, and we extract a finite subcover. Taking the union of such subcovers on l, we obtain the atlas $\{\kappa_\alpha\}$.

For (b) and (d), the solution will be a translation into the language of smooth manifolds of a proof given in introducing Corollary 3.19: In (b), let the domains of the charts constructed at stage l be $M_{\kappa_1}, \dots, M_{\kappa_r}$. Lemma 3.15b of *Basic* constructs an open cover $\{W_1, \dots, W_r\}$ of L_l such that W_j^{cl} is a compact subset of M_{κ_j} for each j. A second application of Lemma 3.15b of *Basic* produces an open cover $\{V_1, \dots, V_r\}$ of L_l such that V_j^{cl} is compact and $V_j^{\mathrm{cl}} \subseteq W_j$ for each j. Proposition 8.2 constructs a smooth function $g_j \ge 0$ that is 1 on V_j^{cl} and is 0 off W_j. Then $\sum_{j=1}^r g_j$ is > 0 on L_l and has compact support in $\bigcup_{j=1}^r M_{\kappa_j}$. If we write $\{\eta_\alpha\}$ for the union of the sets $\{g_1, \dots, g_r\}$ as l varies, then the functions $\varphi_\alpha = \eta_\alpha \big/ \sum_\beta \eta_\beta$ have the required properties.

For (c), we apply (b) to the smooth manifold U. The construction in (b) is arranged so that about each point is an open neighborhood on which only finitely many φ_α's can be nonzero. As this point varies through K, the open neighborhoods cover K, and there is a finite subcover. Therefore only finitely many φ_α's have the property that they are somewhere nonzero on K. The sum of this finite subcollection of all φ_α's is then a smooth function with values in $[0, 1]$ that is 1 everywhere on K and has compact support in U.

For (d), we argue as in (b) with two applications of Lemma 3.15b of *Basic* to produce an open cover $\{V_1, \dots, V_r\}$ of K such that for each j, V_j^{cl} is a compact subset of W_j, whose closure is a compact subset of U_j. Part (c) constructs a smooth function $g_j \ge 0$ that is 1 on V_j^{cl} and is 0 off W_j. Then $g = \sum_{j=1}^r g_j$ is > 0 everywhere on K and has compact support in $\bigcup_{j=1}^r U_j$. A second application of (c) produces a smooth function $h \ge 0$ on M with values in $[0, 1]$ that is 1 on K and is compactly supported within the set where $g > 0$. Then $g + (1 - h)$ is smooth and everywhere positive on M, and the functions $\varphi_j = g_j/(g + (1 - h))$ have the required properties.

6. In the notation of Proposition 8.6, the matrix $\left[\dfrac{\partial F_i}{\partial u_j}\Big|_{(u_1,\dots,u_n)=(x_1(p),\dots,x_n(p))}\right]$, which is of size k-by-n, has rank k. Choose k linearly independent columns. Possibly after a change of notation that will not affect the conclusion, we may assume that they are the first k columns. Call the n functions $y_1 \circ F, \dots, y_k \circ F, x_{k+1}, \dots, x_n$ by the names $f_1, \dots, f_n$. These are in $C^\infty(M_\kappa)$ and have matrix $\left[\dfrac{\partial(f_i \circ \kappa^{-1})}{\partial u_j}\right]$ of the block form

$$\begin{pmatrix} \left[\dfrac{\partial F_i}{\partial u_j}\right] & \left[\dfrac{\partial F_i}{\partial u_j}\right] \\ 0 & 1 \end{pmatrix}$$

at the point where $(u_1, \dots, u_n) = (x_1(p), \dots, x_n(p))$. The upper left corner is invertible by the condition of rank k, and hence the whole matrix is invertible. Then the result follows from Proposition 8.4.

7. In the notation of Proposition 8.6, the matrix $\left[\dfrac{\partial F_i}{\partial u_j}\Big|_{(u_1,\dots,u_n)=(x_1(p),\dots,x_n(p))}\right]$, which is of size k-by-n, has rank n. Choose n linearly independent rows. Since $F_i = (y_i \circ F) \circ \kappa^{-1}$, Proposition 8.4 shows that the corresponding functions $y_i \circ F$ generate a system of local coordinates near p. This proves (a).

8. A little care is needed with the definition of measure 0 for a manifold because the sets of measure 0 that arise are not shown to be Borel sets. However, for points in the intersection of the domains of two charts κ_1 and κ_2, the change-of-variables theorem shows that the two versions of Lebesgue measure near the two images in Euclidean space of a point are of the form dx and $(\kappa_1 \circ \kappa_2^{-1})'(x)\,dx$, and the sets of measure 0 are the same for these.

The solution of the problem as written is a question of localizing matters so that the Euclidean version of Sard's Theorem (Theorem 6.35 of *Basic*) applies. For each point p in M, one can find a chart κ_p with $p \in M_{\kappa_p}$ and a chart λ_p with $F(p) \in N_{\lambda_p}$ such that $F(M_{\kappa_p}) \subseteq N_{\lambda_p}$. The Euclidean theorem applies to $\lambda_p \circ F \circ \kappa_p^{-1}$. The separability implies that countably many of these M_{κ_p}'s cover M. We get measure 0 for the critical values within each $F(M_{\kappa_p})$, and the countable union of sets of measure 0 has measure 0.

9. Here we localize and apply Corollary 6.36 of *Basic*.

10. The reflexive condition follows with $h = 1$, and the transitive condition follows by using the composition of two h's. Strictly equivalent is the condition "equivalent" with $h = 1$.

11. Substitution of the definitions gives

$$\bar{g}_{kj}(x)g_{ji}(x) = \phi'^{-1}_{k,x} \circ h_x \circ \phi_{j,x} \circ \phi_{j,x}^{-1} \circ \phi_{i,x} = \phi'^{-1}_{k,x} \circ h_x \circ \phi_{i,x} = \bar{g}_{ki}(x).$$

This proves the first identity, and the second identity is similar.

12. For (a), if x lies in $M_{\kappa_j} \cap M_{\kappa'_k}$ and y lies in $\mathbb{F}^n$, then the only way that h can have the correct mapping function $x \mapsto \bar{g}_{kj}(x)$ is to have $\bar{g}_{kj}(x)(y) = {\phi'_{k,x}}^{-1} h \phi_{j,x}(y)$. Therefore we must have $h(\phi_{j,x}(y)) = \phi'_{k,x} \bar{g}_{kj}(x)(y)$, and h is unique.

In (b), if h exists, then it is apparent from the formula for it that it is a diffeomorphism. In this case the function h^{-1} exhibits the relation "equivalent" as symmetric.

13. For (a), if x lies also in $M_{\kappa_i} \cap M_{\kappa'_l}$, then we have

$$p_j(b) = \phi_{j,x}^{-1}(b) = \phi_{j,x}^{-1} \phi_{i,x} \phi_{i,x}^{-1}(b) = g_{ji}(x)(p_i(b))$$

and hence

$$\begin{aligned} h_{kj}(b) &= \phi'_{k,x} \bar{g}_{kj}(x)(p_j(b)) = \phi'_{k,x} \bar{g}_{kj}(x) g_{ji}(x)(p_i(b)) = \phi'_{k,x} \bar{g}_{ki}(x)(p_i(b)) \\ &= \phi'_{l,x} g'_{lk}(x) \bar{g}_{ki}(x)(p_i(b)) = \phi'_{l,x} \bar{g}_{li}(x)(p_i(b)) = h_{li}(b). \end{aligned} \tag{$*$}$$

The sets $p^{-1}(M_{\kappa_j} \cap M_{\kappa'_k})$ are open and cover B as j and k vary, and the consistency condition $(*)$ therefore shows that the functions h_{kj} piece together as a single smooth function $h : B \to B'$.

For (b), let y be in $\mathbb{F}^n$. Put $b = \phi_{j,x}(y)$ in the definition of $h_{kj}(b)$, so that $y = \phi_{j,x}^{-1}(b) = p_j(b)$, and then we have

$${\phi'_{k,x}}^{-1} h \phi_{j,x}(y) = {\phi'_{k,x}}^{-1} h(b) = {\phi'_{k,x}}^{-1} \phi'_{k,x} \bar{g}_{kj}(x)(p_j(b)) = \bar{g}_{kj}(x)(y).$$

This shows that the functions $x \mapsto \bar{g}_{kj}(x)$ coincide with the mapping functions of h.

Chapter IX

1. The formula is $\mu_{|x|} = \mu_x + \mu_x^\vee - \frac{1}{2}\mu(\{0\})$, where $\mu_x^\vee$ is the measure on $\mathbb{R}$ defined by $\mu_x^\vee(A) = \mu_x(-A)$.

2. Both sides equal $\int_\Omega \Phi(x_1, \dots, x_n)\, dP$.

3. For (a), we have $\sigma_n^2 = \int_{\mathbb{R}} (t - E)^2\, d\mu_n(t) \geq \int_{|t-E| \geq \delta} (t - E)^2\, d\mu_n(t) \geq \delta^2 P(\{|y_n - E| \geq \delta\})$.

For (b), we calculate

$$\begin{aligned} |E(\Phi(y_n)) - \Phi(E)| &= \left| \textstyle\int_{\mathbb{R}} [\Phi(t) - \Phi(E)]\, d\mu_n(t) \right| \leq \textstyle\int_{\mathbb{R}} |\Phi(t) - \Phi(E)|\, d\mu_n(t) \\ &= \textstyle\int_{|t-E|<\delta} + \int_{|t-E|\geq\delta} \leq \int_{|t-E|<\delta} \epsilon\, d\mu_n(t) + 2MP(\{|y_n - E| \geq \delta\}) \\ &\leq \epsilon + 2M\sigma_n^2 \delta^{-2}. \end{aligned}$$

In (c), let $\epsilon > 0$ be given, and choose the δ of continuity for Φ and ϵ. Then the calculation in (b) applies. Since $\lim \sigma_n^2 = 0$, the right side is $\leq 2\epsilon$ for n large enough. For such n, we have $|E(\Phi(y_n)) - \Phi(E)| \leq 2\epsilon$.

In (d), the argument of (c) depends only on the continuity of Φ at E and the global boundedness of Φ. In the situation of Theorem 9.7 with independent identically distributed random variables x_n, we put $s_n = x_1 + \cdots + x_n$ and take $y_n = \frac{1}{n}s_n$. We saw that if $E(x_k) = E$ and $\operatorname{Var}(x_k) = \sigma^2$, then $E(y_n) = E$ and $\operatorname{Var}(y_n) = \frac{1}{n}\sigma^2$. Thus (c) applies.

4. Part (a) is a direct application of the Kolmogorov Extension Theorem. One starts with the measure on $\mathbb{R}^1$ that assigns mass p to $\{1\}$ and mass $1-p$ to $\{0\}$, forms the n-fold product to model n independent tosses, and obtains the space for a sequence of tosses from the Kolmogorov Theorem.

In (b), the mean is $p \cdot 1 + (1-p)\cdot 0 = p$. The computation for the variance is $p \cdot 1^2 + (1-p)\cdot 0^2 - p^2 = p - p^2 = p(1-p)$.

For (c), the answer is the number of ways of obtaining k heads and $n-k$ tails in n tosses, namely $\binom{n}{k}$, times the probability of getting a specific sequence of k heads and $n-k$ tails, which is $p^k(1-p)^{n-k}$.

In (d), we put $y_n = \frac{1}{n}s_n$. In view of (c), $E(y_n)$ is $\sum_{k=0}^{n}\Phi\big(\frac{k}{n}\big)\binom{n}{k}p^k(1-p)^{n-k}$, and (a) shows that $\Phi(E)$ is $\Phi(p)$. The variance of y_n is $\frac{p(1-p)}{n}$, in view of (b); since this tends to 0, Problem 3c is applicable and establishes the limit formula.

For (e), we go over the solution of Problem 3. The relevant facts for making an estimate that is uniform in p are that Φ is uniformly continuous and that the convergence of the variance to 0 is uniform in p.

6. For the regularity any set in $\mathcal{F}$ is in some $\mathcal{F}_n$. The sets in $\mathcal{F}_n$ are of the form $\widetilde{E} = E \times \big(\bigtimes_{k=n+1}^{\infty} X_k\big)$ with $E \subseteq \Omega^{(n)}$ and $\nu(\widetilde{E}) = \nu_n(E)$. Given $\epsilon > 0$, choose K compact and U open in $\Omega^{(n)}$ with $K \subseteq E \subseteq U$ and $\nu_n(U - K) < \epsilon$. In Ω, $\widetilde{K}$ is compact, $\widetilde{U}$ is open, $\widetilde{K} \subseteq \widetilde{E} \subseteq \widetilde{U}$, and $\nu(\widetilde{U} - \widetilde{K}) < \epsilon$.

7. Let $E = \bigcup_{n=1}^{\infty} E_n$ disjointly in $\mathcal{F}$. Since ν is nonnegative additive, we have $\sum_{n=1}^{\infty}\nu(E_n) \le \nu(E)$. For the reverse inequality let $\epsilon > 0$ be given. Choose K compact and U_n open with $K \subseteq E$, $E_n \subseteq U_n$, $\nu(U_n - E_n) < \epsilon/2^n$, and $\nu(E-K) < \epsilon$. Then $K \subseteq \bigcup_{n=1}^{\infty} U_n$, and the compactness of K forces $K \subseteq \bigcup_{n=1}^{N} U_n$ for some N. Then $\nu(E) \le \nu(K) + \epsilon \le \nu\big(\bigcup_{n=1}^{N} U_n\big) + \epsilon \le \sum_{n=1}^{N}\nu(U_n) + \epsilon \le \sum_{n=1}^{N}\nu(E_n) + 2\epsilon \le \sum_{n=1}^{\infty}\nu(E_n) + 2\epsilon$. Since ϵ is arbitrary, $\nu(E) \le \sum_{n=1}^{\infty}\nu(E_n)$.

8. The key is that Ω is a separable metric space. Every open set is therefore the countable union of basic open sets, which are in the various $\mathcal{F}_n$'s.

10. The collection of subsets of Ω that are of type J for some countable J is a σ-algebra containing $\mathcal{A}'$, and thus it contains $\mathcal{A}$.

11. Continuity cannot be ensured by conditions at only countably many points, as we see by altering the value of the function at a point not in a prospective such countable set of points.

12. A nonempty set of $\mathcal{A}$ that is contained in C must be defined in terms of what happens at countably many points, and no such conditions are possible, just as in the

previous problem. So the set must be empty. Since $\rho_*(C)$ is the supremum of ρ of all such sets, we obtain $\rho_*(C) = 0$.

13. If ω is in C_j but not E, then the uniform continuity of ω means that $\omega\big|_J$ extends to a member of C. In other words, there is a member ω' of Ω that is 0 on J such that $\omega + \omega'$ is in C. Since $C \subseteq E$, $\omega + \omega'$ is in E. The set E is by assumption of type J, and therefore the sum of any member of E with a member of Ω that vanishes on J is again in E. Hence $\omega = (\omega + \omega') - \omega'$ is in E, contradiction.

14. Problem 13 shows that the infimum of $\rho(E)$ for all E in $\mathcal{A}$ containing C equals the infimum over all countable J of $\rho(C_J)$. Under the assumption this infimum is 1. Thus $\rho^*(C) = 1$.

15. Proceeding inductively and using the convergence in probability, we can construct a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $P(|x_{n_k} - x| \le 2^{-k}) \le 2^{-k}$ for $k \ge 1$. The series $\sum_{k=1}^{\infty} P(|x_{n_k} - x| \le 2^{-k})$ converges, and the Borel–Cantelli Lemma (Lemma 9.9) shows that except for ω in a set Z of measure 0, $|x_{n_k}(\omega) - x(\omega)| > 2^{-k}$ only finitely often. Thus except when ω is in Z, $\sum_k |x_{n_k}(\omega) - x(\omega)|$ converges. Since

$$|x_{n_{k+1}}(\omega) - x_{n_k}(\omega)| \le |x_{n_{k+1}}(\omega) - x(\omega)| + |x_{n_k}(\omega) - x(\omega)|$$

for all k, $\sum_k |x_{n_{k+1}}(\omega) - x_{n_k}(\omega)|$ converges. Therefore $\sum_k (x_{n_{k+1}}(\omega) - x_{n_k}(\omega))$ converges. The partial sum of this series through the ℓ^{th} term is $x_{n_{\ell+1}}(\omega) - x_{n_1}(\omega)$, and therefore the series $\sum_k x_{n_k}(\omega)$ converges for all ω not in Z. Since $\{x_{n_k}\}$ converges to some random variable almost surely, Proposition 9.12 shows that the convergence is to x almost surely.

16. Chebyshev's inequality (Section VI.10 of *Basic*) shows that $\int_X |f|^2\,d\mu \ge \xi^2\mu\big(\{x \mid |f(x)| \ge \xi\}\big)$ for all $\xi > 0$ on any measure space. We apply this with $\mu = P$ and with $f = x_n - c$ to obtain $\xi^2 P\big(|x_n - c| \ge \xi\big) \le E((x_n - c)^2) = \mathrm{Var}(x_n)$. The right side tends to 0 as n tends to infinity, and thus $P\big(|x_n - c| \ge \xi\big)$ tends to 0. In other words, $\{x_n\}$ tends to c in probability.

17. Take μ_n to be a unit mass at $\{n\}$, and let $\mu = 0$.

18. According to Problem 4, the mean is $E = p$, and the variance is $\sigma = p(1-p)$. Thus the result follows directly by substituting into Theorem 9.19.

19. In (a), the Binomial Theorem gives $\sum_{k=0}^{n} \binom{n}{k}\big(\frac{\lambda}{n}\big)^k\big(1-\frac{\lambda}{n}\big)^{n-k} = \big(\frac{\lambda}{n}+(1-\frac{\lambda}{n})\big)^n$, and the right side is just 1. Also $\sum_{k=0}^{\infty} \frac{\lambda}{k!}\, e^{-\lambda} = e^{\lambda}e^{-\lambda} = 1$.

In (b), for each $k \ge 0$, we have

$$\begin{aligned} p_{n,\lambda}(k) &= \tbinom{n}{k}\big(\tfrac{\lambda}{n}\big)^k\big(1 - \tfrac{\lambda}{n}\big)^{n-k} \\ &= \Big[\tfrac{n(n-1)\dots(n-k+1)}{k!}\big(\tfrac{1/n}{1-\lambda/n}\big)^k\Big]\lambda^k\big(1 - \tfrac{\lambda}{n}\big)^n. \end{aligned}$$

With k fixed and n tending to infinity, the factor in brackets tends to $\frac{1}{k!}$. Thus $\lim_n p_{n,\lambda}(k) = \frac{1}{k!}\,\lambda^k e^{-\lambda} = p_\lambda(k)$. The cumulative distribution function $F_{x_{n,\lambda}}$ of $x_{n,\lambda}$

at each point is the sum of certain values of $p_{n,\lambda}(k)$, and the cumulative distribution function of F_{x_λ} at that same point is the sum of the corresponding values of $p_\lambda(k)$. Therefore $\lim_n F_{x_{n,\lambda}} = F_{x_\lambda}$ pointwise. By definition $x_{n,\lambda}$ tends to x_λ in distribution.

In (c), the mean is λ, and the variance is λ. In fact, we have $E(x_\lambda) = \sum_k k\frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}\lambda\frac{d}{d\lambda}\big(\sum_k \frac{\lambda^k}{k!}\big) = e^{-\lambda}\lambda e^{\lambda} = \lambda$. Also

$$\begin{aligned}\operatorname{Var}(x_\lambda) &= E(x_\lambda^2) - \lambda^2 = \textstyle\sum k^2\frac{\lambda^k}{k!}e^{-\lambda} - \lambda^2\\ &= e^{-\lambda}\lambda^2 \textstyle\sum \big(k(k-1)+k\big)\frac{\lambda^{k-2}}{k!} - \lambda^2\\ &= e^{-\lambda}\lambda^2\big(\frac{d^2}{d\lambda^2}e^{\lambda}\big) + e^{-\lambda}\lambda\big(\frac{d}{d\lambda}e^{\lambda}\big) - \lambda^2 = \lambda^2+\lambda-\lambda^2 = \lambda.\end{aligned}$$

20. In (a), consider $u(s) = \log(1+c/s)^s = s(\log(s+c) - \log s)$. This has derivative $u'(s) = \log(s+c) - \log s + s/(s+c) - s/s = \log(1+c/s) - c/(s+c)$. Since $u''(s) = 1/(s+c) - 1/s + c/(s+c)^2 = -c^2/(s(s+c)^2)$ is positive for $s > 0$, $u'(s)$ is a strictly decreasing function. By inspection, $\lim_{s\to+\infty} u'(s) = 0$. Thus $u'(s) > 0$ for all $s > 0$. We conclude that $u(s)$ is an increasing function for $s > 0$, and so is its exponential, which is $(1+c/s)^s$.

In (b), we know that $\lim_{s\to+\infty}(1+c/s)^s = e^c$, and (a) says that this is an increasing limit. Taking reciprocals shows that $\lim_{s\to+\infty}(1+c/s)^{-s} = e^{-c}$, a decreasing limit. If we put $c = t^2/2$ and $s = (n-1)/2$, then we obtain

$$\lim_{n\to+\infty}\big(1+\tfrac{t^2}{n-1}\big)^{(n-1)/2} = e^{-t^2/2},$$

a decreasing limit. The second statement follows because $(1+\frac{t^2}{n-1}\big)^{(n-1)/2} = c_n^{-1}f_n(t)$.

In (c), we see from (b) that for $n \ge 2$, $(1+\frac{t^2}{n-1}\big)^{-(n-1)/2}$ is dominated by the case $n = 2$, where the function is $(1+t^2)^{-1/2}$. Multiplying by $(1+t^2)^{-1/2}$, we see that $(1+\frac{t^2}{n-1}\big)^{-n/2} \le (1+t^2)^{-1}$ for $n \ge 2$. The function $(1+t^2)^{-1}$ is integrable, and thus dominated convergence allows us to conclude that $\lim_n \int_{\mathbb{R}}(1+\frac{t^2}{n-1}\big)^{-n/2}\,dt = \int_{\mathbb{R}} \lim_n(1+\frac{t^2}{n-1}\big)^{-n/2}\,dt$. By (b), the right side equals $\int_{\mathbb{R}} e^{-t^2/2}\,dt$, which is $\sqrt{2\pi}$. Since $\int_{\mathbb{R}} f_n(t)\,dt = 1$ for all n, the left side is $\lim_n c_n^{-1}$. Thus $\lim_n c_n^{-1} = \sqrt{2\pi}$.

21. Because of the dominated convergence in the previous problem, $\int_a^b f_n(t)\,dt$ has limit $(2\pi)^{-1/2}e^{-t^2/2}\,dt$, and this is just the statement of the convergence in distribution.

22–23 and 25. The style of argument for these problems is all the same. In the case of Problem 22, we have

$$P(a < x+c < b) = P(a-c < x < b-c) = \int_{a-c}^{b-c} f(t)\,dt = \int_a^b f(s-c)\,ds,$$

and thus the probability distribution of $x+c$ is $f(t-x)\,dt$. Similarly in Problem 23 the probability distribution of cx is $c^{-1}f(c^{-1}t)\,dt$, and in Problem 25 the probability distribution of w_n is $\frac{1}{\sigma^2\sqrt{2\pi}}\,e^{-t^2/(2\sigma^2)}$.

24. We are to consider $\frac{1}{\sigma^2\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)} * \frac{1}{\sigma'^2\sqrt{2\pi}}\, e^{-(x-\mu')^2/(2\sigma'^2)}$. If we write out the convolution of the two functions and complete the square in the exponent, we see that the result is a multiple of some quadratic exponential, hence is normal. The means have to add, and the independence implies that the variances have to add, by a computation in Section 4. Thus the probability distribution of the sum has to be $N(\mu+\mu', \sigma^2+\sigma'^2)$.

26. The probability distribution of w_n is equal to $\frac{1}{\sigma^2\sqrt{2\pi}}\, e^{-t^2/(2\sigma^2)}$ for every n, and we are considering the limit of a constant sequence.

Chapter X

1. Examination of the proof shows that equality can fail only at one step, and that the inequality at that step step holds by the Schwarz inequality. For two nonzero functions, equality holds in the Schwarz inequality if and only if the functions are proportional to one another. Therefore the condition is that $tf(t)$ is proportional to $f'(t)$, i.e., that $f'(t) = ktf(t)$ for some constant k. Solving, we get $f(t) = ce^{\frac{1}{2}kt^2}$. This function is in the Schwartz space if and only if $\operatorname{Re} k < 0$.

2. The function $f_1(t) = t^{-1}\sin \pi t$ vanishes at every integer except 0, and the Fourier transform $\mathcal{F}f_1$ is supported in $[-\frac{1}{2}, \frac{1}{2}]$. If $f_2(t) = f_1(\frac{1}{2}t)$, then f_1 vanishes at every half integer except 0, and the Fourier transform is supported in $[-1, 1]$. Finally the function f_3 with $f_3(t) = f_2(t-\frac{1}{2})$ vanishes at every half integer except $\frac{1}{2}$, and the Fourier transform is supported in $[-1, 1]$. Thus f_3 has the required properties.

3. For (b), write $\nu(y) = \sum_{k=M}^{N} c_k e^{-2\pi iky}$ with $c_M \neq 0$ and $c_N \neq 0$. Then $1 = |\nu(y)|^2$ is a trigonometric polynomial of the form

$$\sum_{k=M}^{N} c_k e^{-2\pi iky} \sum_{j=M}^{N} \overline{c_j} e^{2\pi ijy} = \sum_{k=M}^{N} \sum_{l=-N}^{-M} c_k \overline{c_{-l}} e^{-2\pi i(k+l)y}.$$

The lowest order exponential that appears is $e^{-2\pi i(N-M)}$, and it has coefficient $c_N\overline{c_M} \neq 0$. Since the exponentials are linearly independent, $N - M = 0$, and $\upsilon(y)$ is a multiple of a single exponential.

4. For (a), define $f^\#(x) = f(-x)$ for any function f on $\mathbb{R}$. Let V_j be the set of all $f^\#$ such that f is in V_j. Then $\varphi^\#$ and $\{V_j^\#\}$ form a multiresolution analysis.

Part (b) is routine.

For (c), the idea is that the Daubechies φ is constructed using a function $\mathcal{L}(y)$ built from all the roots within the unit disk of a certain polynomial Q, while $\varphi^\#$, apart from an integer translation, arises from the same construction with the corresponding

$\mathcal{L}(y)$ built from all the roots outside the unit disk. In more detail let $Q(z)$ be as in the construction of φ, and define

$$\mathcal{L}_1(y) = \prod_{|\alpha_j|<1} (e^{-2\pi iy} - \alpha_j) \prod_{\beta_k} (e^{-2\pi iy} - \beta_k)^{m_k}.$$

We set $c_1 = \mathcal{L}_1(0)^{-1}$, and then $\mathcal{L}(y) = c_1\mathcal{L}_1(y)$ is the function $\mathcal{L}$ that appears in Proposition 10.34. For each factor of $\mathcal{L}_1$, we have

$$(e^{2\pi iy} - \alpha) = \alpha e^{2\pi iy}(\alpha^{-1} - e^{-2\pi iy}) = -\alpha e^{2\pi iy}(e^{-2\pi iy} - \alpha^{-1}).$$

Taking the product of all the factors, we obtain

$$\mathcal{L}_1^{\#}(y) = c_2 e^{2\pi ipy} \prod_{|\alpha_j|>1} (e^{-2\pi iy} - \alpha_j) \prod_{\beta_k} (e^{-2\pi iy} - \beta_k)^{m_k} \quad \text{with } c_2 \neq 0 \text{ and } p \in \mathbb{Z}.$$

Let $\mathcal{L}_2(y)$ be the product on the right, so that $\mathcal{L}_1^{\#}(y) = c_2 e^{2\pi ipy}\mathcal{L}_2(y)$. Now

$$m_0(y) = \Big(\frac{1+e^{-2\pi iy}}{2}\Big)^N \mathcal{L}(y)$$

implies

$$\begin{aligned} m_0^{\#}(y) &= \Big(\frac{1+e^{2\pi iy}}{2}\Big)^N \mathcal{L}^{\#}(y) \\ &= e^{2\pi iNy}\Big(\frac{1+e^{-2\pi iy}}{2}\Big)^N \mathcal{L}^{\#}(y) \\ &= c_1 e^{2\pi iNy}\Big(\frac{1+e^{-2\pi iy}}{2}\Big)^N \mathcal{L}_1^{\#}(y) \\ &= c_1 c_2 e^{2\pi i(N+p)y}\Big(\frac{1+e^{-2\pi iy}}{2}\Big)^N \mathcal{L}_2(y). \end{aligned}$$

Form the h function that corresponds to $m_0^{\#}$. The exponential $e^{2\pi i(N+p)y}$ contributes exactly $e^{2\pi i(N+p)y}$ to the infinite product, and $\mathcal{F}^{-1}$ carries the exponential to an integer translation. For the constants we have

$$1 = \mathcal{L}(0) = \mathcal{L}^{\#}(0) = c_1\mathcal{L}_1^{\#}(0) = c_1c_2\mathcal{L}_2(0),$$

and this is the correct normalization to have in the $\mathcal{L}$ part of the function $m_0^{\#}$. Consequently $\varphi^{\#}$ comes from the same construction as φ but with the roots α_j with $|\alpha_j| > 1$ in place of the roots α_j with $|\alpha_j| < 1$, possibly in combination with a translation by the integer $N + p$.

5. Let V_0 be the subspace of all functions in $L^2(\mathbb{R})$ that are a.e. constant on each interval $\left[n-\frac{2}{3}, n+\frac{1}{3}\right)$ for $n \in \mathbb{Z}$. Then the integer translates of φ form an orthonormal basis of V_0. We obtain V_j by dilation from V_0 as usual, and the resulting sequence $\{V_j\}_{j\in\mathbb{Z}}$ of spaces forms a multiresolution analysis with φ.

6. For (a), we have $m_0(y) = \frac{1}{2}(1 + e^{4\pi y}) = e^{-2\pi i y}\cos 2\pi y$ and

$$\begin{aligned}
&2^n \sin(2\pi y/2^n) \prod_{j=1}^{n} m_0(2^{-j}y) \\
&\qquad = 2^n e^{-2\pi i y(\frac{1}{2}+\cdots+\frac{1}{2^n})}\Big(\prod_{j=1}^{n}\cos(2\pi y/2^j)\Big)\sin(2\pi y/2^n) \\
&\qquad = 2^{n-1} e^{-2\pi i y(\frac{1}{2}+\cdots+\frac{1}{2^n})}\Big(\prod_{j=1}^{n-1}\cos(2\pi y/2^j)\Big)\sin(2\pi y/2^{n-1}) \\
&\qquad = \cdots = 2e^{-2\pi i y(\frac{1}{2}+\cdots+\frac{1}{2^n})}\cos(2\pi y/2)\sin(2\pi y/2) \\
&\qquad = e^{-2\pi i y(\frac{1}{2}+\cdots+\frac{1}{2^n})}\sin(2\pi y).
\end{aligned}$$

Therefore

$$\frac{\sin(2\pi y/2^n)}{2\pi y/2^n}\prod_{j=1}^{n} m_0(2^{-j}y) = e^{-2\pi i y(\frac{1}{2}+\cdots+\frac{1}{2^n})}\Big(\frac{\sin 2\pi y}{2\pi y}\Big).$$

Letting n tend to infinity shows that

$$h(y) = e^{-2\pi i y}\Big(\frac{\sin 2\pi y}{2\pi}\Big) = \frac{e^{-2\pi i y}}{2i(2\pi)}(e^{2\pi i y} - e^{-2\pi i y}) = \frac{1 - e^{-4\pi i y}}{4\pi i y} = (\mathcal{F} I_{[0,2]})(y).$$

Thus $\varphi(x) = I_{[0,2]}(x)$.

For (b), $\varphi(x)$ and $\varphi(x-1)$ are not orthogonal, since they are ≥ 0 and their supports overlap. Hypothesis (iii) is not satisfied, since $\frac{1}{2}(1 + e^{4\pi y})$ is not > 0 for $|y| \leq \frac{1}{4}$, merely ≥ 0 .

7. $P_m f$ is meaningful for f in $L^1(\mathbb{R})$ by Proposition 10.5. The formula is

$$P_m f(x) = |I_{m,k}|^{-1}\textstyle\int_{I_{m,k}} f(y)\,dy \qquad \text{for } x \in I_{m,k}.$$

Thus

$$\textstyle\int_{I_{m,k}} |P_m f(x)|\,dx = |I_{m,k}||I_{m,k}|^{-1}\big|\int_{I_{m,k}} f(y)\,dy\big| \leq \int_{I_{m,k}} |f(y))|\,dy.$$

Summing on k, we obtain

$$\textstyle\int_{\mathbb{R}} |P_m f(x)|\,dx \leq \int_{\mathbb{R}} |f(y)|\,dy,$$

i.e., $\|P_m f\|_1 \le \|f\|_1$. To prove the convergence in L^1, we observe from Corollary 10.7 that $P_m g$ tends to g uniformly if g is in $C_{\text{com}}(\mathbb{R})$. Since the convergence all takes place within a compact set, $P_m g$ tends to g in $L^1(\mathbb{R})$. Given $f \in L^1(\mathbb{R})$ and $\epsilon > 0$, choose $g \in C_{\text{com}}(\mathbb{R})$ with $\|g - f\|_1 < \epsilon/3$. If m is taken large enough so that $\|P_m g - g\|_1 < \epsilon/3$, then

$$\begin{aligned}\|P_m f - f\|_1 &\le \|P_m f - P_m g\|_1 + \|P_m g - g\|_1 + \|g - f\|_1 \\ &\le 2\|f - g\|_1 + \|P_m g - g\|_1 \\ &\le 2\epsilon/3 + \|P_m g - g\|_1 < \epsilon.\end{aligned}$$

8. From the first formula in the solution of Problem 7, we have $\int_{I_{m,k}} P_m f(x)\,dx = \int_{I_{m,k}} f(x)\,dx$. Summing on k then gives $\int_{\mathbb{R}} P_m f(x)\,dx = \int_{\mathbb{R}} f(x)\,dx$. Thus the continuous linear functional $\ell(f) = \int_{\mathbb{R}} f(x)\,dx$ on $L^1(\mathbb{R})$ has the property that $\ell(P_m f) = \ell(f)$. If f is a function in $L^1(\mathbb{R})$ for which $P_m f$ tends to 0 in $L^1(\mathbb{R})$, then the continuity of ℓ says we must have $\lim_m \ell(P_m f) = 0$ and hence $\ell(f) = 0$. Since $\ell(f) \ne 0$ for $f = I_{[0,1]}$, $\lim_{m\to-\infty} P_m f$ cannot be 0 for $f = I_{[0,1]}$.

9. The value of $P_m f(\frac{1}{3})$ for this f is $2^m\big|I_{m,k} \cap [0, \frac{1}{3})\big|$, where

$$I_{m,k} = \{y \in \mathbb{R} \mid k \le 2^m y < k + 1\},$$

and $\frac{1}{3}$ is to be in $I_{m,k}$. The binary expansion of $\frac{1}{3}$ comes from the geometric series $\frac{1}{3} = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots$, and the sets $I_{m,k}$ containing $\frac{1}{3}$ are determined as follows. If $m = 2r$ is even, then $k \le 4^r/3 < k + 1$ for $k = 4^r(\frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^r})$. The y's that are in $I_{m,k}$ are the ones with $k \le 4^r y < k + 1$, the smallest of which is $4^{-r}k$, i.e., $y = \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^r}$. The interval of such y's is to be intersected with $[0, \frac{1}{3})$, and the measure of the result is $\frac{1}{3} - (\frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^r}) = \frac{1}{3} - \frac{1-4^{-r}}{3} = 4^{-r}/3$. The normalization by the factor 2^m in the formula for $P_m f(\frac{1}{3})$ then yields $1/3$. So the value of $P_m f(\frac{1}{3})$ is $\frac{1}{3}$ for every even m. A similar computation for odd m gives $\frac{2}{3}$. Therefore

$$\liminf P_m(\tfrac{1}{3}) = \tfrac{1}{3} < \tfrac{2}{3} = \limsup P_m(\tfrac{1}{3}).$$

10. Let f be in $L^2(\mathbb{R})$ with support in $[0, 1]$. We can write out the one-sided Haar series expansion of f as

$$f(x) = \sum_{k\in\mathbb{Z}} \big(\textstyle\int_{\mathbb{R}} f(y)\overline{\varphi_{0,k}(y)}\,dy\big)\varphi_{0,k}(x) + \displaystyle\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}} \big(\textstyle\int_{\mathbb{R}} f(y)\overline{\psi_{j,k}(y)}\,dy\big)\psi_{j,k}(x).$$

In the first term, $\int_{\mathbb{R}} f(y)\overline{\varphi_{0,k}(y)}\,dy = 0$ for $k \ne 0$, since f is supported in $[0, 1]$. In the second term, $\int_{\mathbb{R}} f(y)\overline{\psi_{j,k}(y)}\,dy = 0$ unless $\psi_{j,k}$ is supported in $[0, 1]$. The result is that

$$f(x) = \big(\textstyle\int_{\mathbb{R}} \overline{\varphi(y)}\,dy\big)\varphi(x) + \displaystyle\sum_{\substack{j\ge0,\ k\in\mathbb{Z},\\ \psi_{j,k}\text{ supported in }[0,1]}} \big(\textstyle\int_{\mathbb{R}} \overline{\psi_{j,k}(y)}\,dy\big)\psi_{j,k}(x).$$

This proves that every member of $L^2([0,1])$ is a limit of linear combinations of the stated restrictions of functions. The restrictions are orthonormal as well, and hence they form an orthonormal basis.

11. Let $f(x) = f(2x) + f(2x-1)$, and let $\widehat{f} = \mathcal{F}f$. Taking Fourier transforms, we get

$$\begin{aligned}
\widehat{f}(y) &= \textstyle\int_{\mathbb{R}} f(2x)e^{-2\pi ixy}\,dx + \int_{\mathbb{R}} f(2x-1)e^{-2\pi ixy}dx \\
&= \tfrac{1}{2}\textstyle\int_{\mathbb{R}} f(x)e^{-2\pi ix\frac{1}{2}y}\,dx + \tfrac{1}{2}\int_{\mathbb{R}} f(x)e^{-2\pi i(x+1)\frac{1}{2}y}\,dx \\
&= \tfrac{1}{2}\widehat{f}(\tfrac{1}{2}y) + \tfrac{1}{2}e^{-\pi iy}\widehat{f}(\tfrac{1}{2}y) \\
&= \tfrac{1}{2}(1+e^{-\pi iy})\widehat{f}(\tfrac{1}{2}y) \\
&= \tfrac{1}{4}(1+e^{-\pi iy})(1+e^{-\pi iy/2})\widehat{f}(\tfrac{1}{4}y) \\
&= \dots \\
&= e^{-\pi iy/2}\cos(\pi y/2)e^{-\pi iy/4}\cos(\pi y/4)\cdots e^{-\pi iy/2^n}\cos(\pi y/2^n)\widehat{f}(2^{-n}y).
\end{aligned}$$

Thus

$$\begin{aligned}
&\widehat{f}(y)\sin(\pi y/2^n) \\
&\quad = e^{-\pi iy/2}\cdots e^{-\pi iy/2^n}\cos(\pi y/2)\cos(\pi y/4)\cdots\cos(\pi y/2^{n-1})\sin(\pi y/2^{n-1})\tfrac{1}{2} \\
&\qquad \times \widehat{f}(2^{-n}y) \\
&\quad = e^{-\pi iy(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n})}\cos(\pi y/2)\sin(\pi y/2)\tfrac{1}{2^{n-1}}\widehat{f}(2^{-n}y),
\end{aligned}$$

and we obtain

$$\widehat{f}(y)\frac{\sin(\pi y2^{-n})}{2^{-n}} = e^{-\pi iy(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n})}\sin(\pi y)\widehat{f}(2^{-n}y).$$

Dividing both sides by πy, we let n tend to infinity with y fixed, and the result is

$$\widehat{f}(y) = \frac{e^{-\pi iy}\sin(\pi y)}{\pi y}\widehat{f}(0),$$

since $\widehat{f}$ is continuous at 0. We can rewrite this equality as

$$\widehat{f}(y) = \frac{1-e^{-2\pi iy}}{2\pi iy}\widehat{f}(0) = (\mathcal{F}I_{[0,1]}(y)\widehat{f}(0).$$

Therefore we can conclude that f is a multiple of $I_{[0,1]}$.

12. Taking the inner product of both sides with $\Phi(2x-n)$ and using the orthogonality of the functions $\Phi(2x-k)$ for $k\in\mathbb{Z}$, we obtain

$$\int \Phi(x)\Phi(2x-n)\,dx = a_n\int_{\mathbb{R}}\Phi(2x-n)^2\,dx = \tfrac{1}{2}a_n\int_{\mathbb{R}}\Phi(x)^2\,dx = \tfrac{1}{2}a_n.$$

So

$$a_n = 2\int_{\mathbb{R}}\Phi(x)\Phi(2x-n)\,dx = 2\int_{-1/2}^{1/2}\Phi(2x-n)\,dx = \int_{-1}^{1}\Phi(x-n)\,dx.$$

We conclude that $a_0=1$ and $a_{-1}=a_1=\frac{1}{2}$. The given identity thus has to be

$$\Phi(x)=\Phi(2x)+\tfrac{1}{2}\Phi(2x-1)+\tfrac{1}{2}\Phi(2x+1).$$

For $\frac{1}{2}<x<\frac{3}{4}$, the left side is 0 while the right side is $0+\frac{1}{2}+0=\frac{1}{2}$, contradiction.

13. Since ν is of class C^m, the derivatives of P must match those of ν through order m at $x=0$ and $x=1$. Then $P^{(k}(0)=0$ for $0\le k\le m$, $P(1)=1$, and $P^{(k)}(1)=0$ for $1\le k\le m$. The first of these conditions says that $P(x)$ is divisible by x^{m+1}. Thus P is admissible.

14. The difference of two admissible polynomials is divisible by x^{m+1} and $(x-1)^m$, hence by $x^m(x-1)^m$. Thus it has degree $\ge 2m+2$. This proves uniqueness of admissible polynomials of degree $\le 2m+1$. If P is admissible, then the Euclidean algorithm allows us to write $P(x)=A(x)x^{m+1}(x-1)^{m+1}+B(x)$ with $B=0$ or $\deg B\le 2m+1$, and B will be admissible. This proves existence of admissible polynomials of degree $\le 2m+1$ under the assumption that an admissible polynomial of some degree exists.

15. If $P(x)$ is admissible, then so is $1-P(1-x)$. The uniqueness in Problem 14 forces $P(x)+P(1-x)=1$.

16. For (a), the Binomial Theorem gives

$$\begin{aligned}(1-z)^{m+1} &= \sum_{k=0}^{m+1}(-1)^k\binom{m+1}{k}z^k\\ &= \sum_{k=0}^{p}(-1)^k\binom{m+1}{k}z^k+[z^{p+1}]\\ &= \sum_{q=0}^{p}(-1)^{p-q}\binom{m+1}{p-q}z^{p-q}+[z^{p+1}],\end{aligned}$$

the last equality following after the change of indices $k=p-q$.

For (b), let $D=\frac{d}{dz}$. The binomial series, convergent for $|z|<1$, is

$$\begin{aligned}(1-z)^{-(m+1)} &= \sum_{k=0}^{\infty}\tfrac{1}{k!}\big(D^k(1-z)^{-(m+1)}\big)(0)z^k\\ &= 1+\sum_{k=1}^{\infty}\tfrac{1}{k!}(m+1)(m+2)\cdots(m+k)z^k\\ &= \sum_{k=0}^{\infty}\binom{m+k}{k}z^k,\end{aligned}$$

and the conclusion follows.

For (c), we multiply the results of (a) and (b) and obtain

$$\begin{aligned}1 &= (1-z)^{m+1}(1-z)^{-(m+1)}\\ &= \Big(\sum_{l=0}^{p}(-1)^{p-l}\binom{m+1}{p-l}z^{p-l}\Big)\Big(\sum_{k=0}^{p}\binom{m+k}{k}z^k\Big)+[z^{p+1}].\end{aligned}$$

Equating the coefficients of z^p on the two sides of the equation gives the desired result, since the z^p term in the product arises exactly when $k=l$.

17. For (a), we set $f(x)=x^{m+1}$ and $g(x)=\sum_{k=0}^{m}\binom{m+k}{k}(1-x)^k$. Then we have

$$D^pP(x)=\sum_{q=0}^{p}D^{p-q}(x^{m+1})D^qg(x)$$

with

$$\begin{aligned}D^{p-q}(x^{m+1}) &= (m+1)(m)(m-1)\cdots(m-p+q+2)x^{m+1-p+q}\\ D^{p-q}(x^{m+1})(1) &= \frac{(m+1)!}{(m-p+q+1)!}\\ D^qg(x) &= \sum_{k=0}^{m}(-1)^qk(k-1)\cdots(k-q+1)(1-x)^{k-q}\\ D^qg(1) &= \binom{m+q}{q}q!(-1)^q.\end{aligned}$$

So

$$\begin{aligned}D^pP(1) &= \sum_{q=0}^{p}\binom{p}{q}\frac{(m+1)!}{m-p+q+1)!}\binom{m+q}{q}(-1)^qq!\\ &= \sum_{q=0}^{p}\frac{p!}{q!(p-q)!}\binom{m+1}{p-q}(p-q)!\binom{m+q}{q}q!(-1)^q\\ &= p!\sum_{q=0}^{p}(-1)^q\binom{m+1}{p-q}\binom{m+q}{q}.\end{aligned}$$

For (b), we compute $P(1)$ from (a) with $p=0$ and obtain $P(1)=0!\binom{m+1}{0}\binom{m+0}{0}=1$. We compute $P^{(p)}(1)$ for $1\le p\le m$ from (a) with $p>0$ and obtain

$$\begin{aligned}P^{(p)}(1) &= p!\sum_{q=0}^{p}(-1)^q\binom{m+1}{p-q}\binom{m+q}{q}\\ &= (-1)^pp!\Big(\sum_{q=0}^{p}(-1)^{p-q}\binom{m+1}{p-q}\binom{m+q}{q}\Big).\end{aligned}$$

The sum in parentheses on the right side is 0 by Problem 16c, and thus $P^{(p)}(1) = 0$ for $p > 0$. The polynomial P is manifestly divisible by x^{m+1}, and thus it is admissible. Since P has degree $2m+1$, Problems 14 and 15 show that P is usable as the polynomial in the definition of the Meyer wavelet of index m.

18. It is enough to treat the scaling function φ, since the wavelet equation shows that ψ is a linear combination of the functions $\varphi_{1,k}$. We have

$$\|\varphi\|_{H^s}^2 = \int_{|y|\le 1} |(\mathcal{F}\varphi)(y)|^2 (1+y^2)^s\,dy + \sum_{j=1}^{\infty} \int_{2^{j-1}\le|y|\le 2^j} |(\mathcal{F}\varphi)(y)|^2(1+y^2)^s\,dy.$$

Since $(1+y^2)^s \le 2^s$ for $|y| \le 1$, the first term on the right side is $\le 2^s \int_{\mathbb{R}} |\mathcal{F}\varphi|^2\,dy$ and is harmless. The sum on the right side is

$$\le \sum_{j=1}^{\infty} 2^j \Big(\tfrac{2^{N-1}}{\sqrt{\pi N}}\Big)(\sqrt{4\pi N})^{-j}(1+2^{2j})^s,$$

and this is finite by the ratio test if the equal quantities

$$\sum_{j=1}^{\infty} 2^j(\sqrt{4\pi N})^{-j}\,2^{2sj} = \sum_{j=1}^{\infty} (\sqrt{\pi N})^{-j}(4^s)^j$$

are finite, and in turn these quantities are finite if $4^s < \sqrt{\pi N}$. Taking $\log_2$ of both sides shows that a sufficient condition for finiteness is that $s < \frac{1}{4}\log_2(\pi N)$, as required.

19. Problem 12c from Chapter III says that the members of $H^s(\mathbb{R})$ are of class C^m if $s > \frac{1}{2} + m$. Thus we want $\frac{1}{2} + m < \frac{1}{4}\log_2(\pi N)$ or

$$m < \tfrac{1}{4}\log_2(\pi N) - \tfrac{1}{2} = \tfrac{1}{4}\big(\log_2(\pi N) - \log_2 4\big) = \tfrac{1}{4}\log_2(\pi N/4).$$