I. Introduction to Boundary-Value Problems, 1-33

DOI: 10.3792/euclid/9781429799911-1

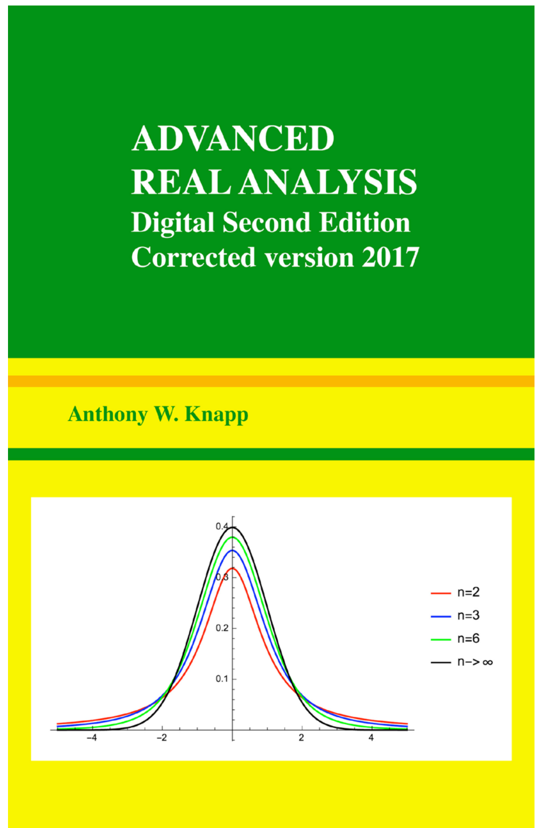
from

Advanced Real Analysis

Digital Second Edition
Corrected version 2017

Anthony W. Knapp

Full Book DOI: 10.3792/euclid/9781429799911
ISBN: 978-1-4297-9991-1



Anthony W. Knapp
81 Upper Sheep Pasture Road
East Setauket, N.Y. 11733–1729, U.S.A.
Email to: aknapp@math.stonybrook.edu
Homepage: www.math.stonybrook.edu/∼aknapp

Title: Advanced Real Analysis
Cover: Normal distribution as a limit of Gosset's t distribution; see page 421.

Mathematics Subject Classification (2010): 46–01, 42–01, 43–01, 35–01, 34–01, 47–01, 58–01, 60A99, 60F05, 28C10, 42C40, 65T60.

First Edition, ISBN-13 978-0-8176-4382-9

Published by Birkhäuser Boston

Digital Second Edition, not to be sold, no ISBN

Published by the Author

CHAPTER I

Introduction to Boundary-Value Problems

Abstract. This chapter applies the theory of linear ordinary differential equations to certain boundary-value problems for partial differential equations.

Section 1 briefly introduces some notation and defines the three partial differential equations of principal interest—the heat equation, Laplace's equation, and the wave equation.

Section 2 is a first exposure to solving partial differential equations, working with boundary-value problems for the three equations introduced in Section 1. The settings are ones where the method of "separation of variables" is successful. In each case the equation reduces to an ordinary differential equation in each independent variable, and some analysis is needed to see when the method actually solves a particular boundary-value problem. In simple cases Fourier series can be used. In more complicated cases Sturm's Theorem, which is stated but not proved in this section, can be helpful.

Section 3 returns to Sturm's Theorem, giving a proof contingent on the Hilbert–Schmidt Theorem, which itself is proved in Chapter II. The construction within this section finds a Green's function for the second-order ordinary differential operator under study; the Green's function defines an integral operator that is essentially an inverse to the second-order differential operator.

1. Partial Differential Operators

This chapter contains a first discussion of linear partial differential equations. The word "equation" almost always indicates that there is a single unknown function, and the word "partial" indicates that this function probably depends on more than one variable. In every case the equation will be **homogeneous** in the sense that it is an equality of terms, each of which is the product of the unknown function or one of its iterated partial derivatives to the first power, times a known coefficient function. Consequently the space of solutions on the domain set is a vector space, a fact that is sometimes called the **superposition principle**. The emphasis will be on a naive-sounding method of solution called "separation of variables" that works for some equations in some situations but not for all equations in all situations. This method, which will be described in Section 2, looks initially for solutions that are products of functions of one variable and hopes that all solutions can be constructed from these by taking linear combinations and passing to the limit.

For the basic existence-uniqueness results with ordinary differential equations, one studies single ordinary differential equations in the presence of initial data of the form $y(t_0) = y_0, \ldots, y^{(n-1)}(t_0) = y_0^{(n-1)}$. Implicitly the independent variable is regarded as time. For the partial differential equations in the settings that we study in this section, the solutions are to be defined in a region of space for all time $t \geq 0$, and the corresponding additional data give information to be imposed on the solution function at the boundary of the resulting domain in space-time. Behavior at $t = 0$ will not be sufficient to determine solutions uniquely; we shall need further conditions that are to be satisfied for all $t \geq 0$ when the space variables are at the edge of the region of definition. We refer to these two types of conditions as **initial data** and **space-boundary data**. Together they are simply **boundary data** or **boundary values**.

For the most part the partial differential equations will be limited to three—the heat equation, the Laplace equation, and the wave equation. Each of these involves space variables in some $\mathbb{R}^n$, and the heat and wave equations involve also a time variable t. To simplify the notation, we shall indicate partial differentiations by subscripts; thus u_{xt} is shorthand for $\partial^2 u \big/ \partial x \partial t$. The space variables are usually $x_1, \ldots, x_n$, but we often write x, y, z for them if $n \leq 3$. The linear differential operator Δ given by

$$\Delta u = u_{x_1 x_1} + \cdots + u_{x_n x_n}$$

is involved in the definition of all three equations and is known as the **Laplacian** in n space variables.

The first partial differential equation that we consider is the **heat equation**, which takes the form

$$u_t = \Delta u,$$

the unknown function $u(x_1, \ldots, x_n, t)$ being real-valued in any physically meaningful situation. Heat flows by conduction, as a function of time, in the region of the space variables, and this equation governs the temperature on any open set where there are no external influences. It is usually assumed that external influences come into play on the boundary of the space region, rather than the interior. They do so through a given set of space-boundary data. Since time and distance squared have distinct physical units, some particular choice of units has been incorporated into the equation in order to make a certain constant reduce to 1.

The second partial differential equation that we consider is the **Laplace equation**, which takes the form

$$\Delta u = 0,$$

the unknown function $u(x_1, \ldots, x_n)$ again being real-valued in any physically meaningful situation. A C^2 function that satisfies the Laplace equation on an open set is said to be **harmonic**. The potential due to an electrostatic charge is harmonic on any open set where the charge is 0, and so are steady-state solutions of the heat equation, i.e., those solutions with time derivative 0.

The third and final partial differential equation that we consider is the **wave equation**, which takes the form

$$u_{tt} = \Delta u,$$

the unknown function $u(x_1, \dots, x_n)$ once again being real-valued in any physically meaningful situation. Waves of light or sound spread in some medium in space as a function of time. In our applications we consider only cases in which the number of space variables is 1 or 2, and the function u is interpreted as the displacement as a function of the space and time variables.

2. Separation of Variables

We shall describe the method of separation of variables largely through what happens in examples. As we shall see, the rigorous verification that separation of variables is successful in a particular example makes serious analytic demands that bring together a great deal of real-variable theory as discussed in Chapters I–IV of *Basic*.[1] The general method of separation of variables allows use of a definite integral of multiples of the basic product solutions, but we shall limit ourselves to situations in which a sum or an infinite series of multiples of basic product solutions is sufficient. Roughly speaking, there are four steps:

(i) Search for basic solutions that are the products of one-variable functions, and form sums or infinite series of multiples of them (or integrals in a more general setting).

(ii) Use the boundary data to determine what specific multiples of the basic product solutions are to be used.

(iii) Address completeness of the expansions as far as dealing with all sets of boundary data is concerned.

(iv) Justify that the obtained solution has the required properties.

Steps (i) and (ii) are just a matter of formal computation, but steps (iii) and (iv) often require serious analysis. In step (iii) the expression "all sets of boundary data" needs some explanation, as far as smoothness conditions are concerned. The normal assumption for the three partial differential equations of interest is that the data have two continuous derivatives, just as the solutions of the equations are to have. Often one can verify (iii) and carry out (iv) for somewhat rougher

[1] Throughout this book the word *"Basic"* indicates the companion volume *Basic Real Analysis*.

data, but the verification of (iv) in this case may be regarded as an analysis problem separate from solving the partial differential equation.

The condition that the basic product solutions in (i) form a discrete set, so that the hoped-for solutions are given by infinite series and not integrals, normally results from assuming that the space variables are restricted to a bounded set and that sufficiently many boundary conditions are specified. In really simple situations the benefit that we obtain is that an analytic problem potentially involving Fourier integrals is replaced by a more elementary analytic problem with Fourier series; in more complicated situations we obtain a comparable benefit. Step (iii) is crucial since it partially addresses the question whether the solution we seek is at all related to basic product solutions. Let us come back to what step (iii) entails in a moment. Step (iv) is a matter of interchanges of limits. One step consists in showing that the expected solution satisfies the partial differential equation, and this amounts to interchanging infinite sums with derivatives. It often comes down to the standard theorem in real-variable theory for that kind of interchange, which is proved in the real-valued case as Theorem 1.23 of *Basic* and extended to the vector-valued case later. We restate it here in the vector-valued case for handy reference.

Theorem 1.1. Suppose that $\{f_n\}$ is a sequence of functions on an interval with values in a finite-dimensional real or complex vector space V. Suppose further that the functions are continuous for $a \le t \le b$ and differentiable for $a < t < b$, that $\{f_n'\}$ converges uniformly for $a < t < b$, and that $\{f_n(x_0)\}$ converges in V for some x_0 with $a \le x_0 \le b$. Then $\{f_n\}$ converges uniformly for $a \le t \le b$ to a function f, and $f'(x) = \lim_n f_n'(x)$ for $a < x < b$, with the derivative and the limit existing.

Another step in handling (iv) consists in showing that the expected solution has the asserted boundary values. This amounts to interchanging infinite sums with passages to the limit as certain variables tend to the boundary, and the following result can often handle that.

Proposition 1.2. Let X be a set, let Y be a metric space, let $A_n(x)$ be a sequence of complex-valued functions on X such that $\sum_{n=1}^{\infty} |A_n(x)|$ converges uniformly, and let $B_n(y)$ be a sequence of complex-valued functions on Y such that $|B_n(y)| \le 1$ for all n and y and such that $\lim_{y\to y_0} B_n(y) = B_n(y_0)$ for all n. Then

$$\lim_{y\to y_0} \sum_{n=1}^{\infty} A_n(x)B_n(y) = \sum_{n=1}^{\infty} A_n(x)B_n(y_0),$$

and the convergence is uniform in x if, in addition to the above hypotheses, each $A_n(x)$ is bounded.

PROOF. Let $\epsilon > 0$ be given, and choose N large enough so that $\sum_{n=N+1}^{\infty} |A_n(x)|$ is $< \epsilon$. Then

$$\begin{aligned}\Big|\sum_{n=1}^{\infty} A_n(x)B_n(y) - \sum_{n=1}^{\infty} A_n(x)B_n(y_0)\Big| &= \Big|\sum_{n=1}^{\infty} A_n(x)\big(B_n(y) - B_n(y_0)\big)\Big| \\ &\le \sum_{n=1}^{N} |A_n(x)|\,|B_n(y) - B_n(y_0)| + 2\sum_{n=N+1}^{\infty} |A_n(x)| \\ &< 2\epsilon + \sum_{n=1}^{N} |A_n(x)|\,|B_n(y) - B_n(y_0)|.\end{aligned}$$

For y close enough to y_0, the second term on the right side is $< \epsilon$, and the pointwise limit relation is proved. The above argument shows that the convergence is uniform in x if $\max_{1\le n\le N} |A_n(x)| \le M$ independently of x. □

In combination with a problem[2] in *Basic*, Proposition 1.2 shows, under the hypotheses as stated, that if X is a metric space and if $\sum_{n=1}^{\infty} A_n(x)B_n(y)$ is continuous on $X \times (Y - \{y_0\})$, then it is continuous on $X \times Y$. This conclusion can be regarded, for our purposes, as tying the solution of the partial differential equation well enough to one of its boundary conditions. It is in this sense that Proposition 1.2 contributes to handling part of step (iv).

Let us return to step (iii). Sometimes this step is handled by the completeness of Fourier series as expressed through a uniqueness theorem[3] or Parseval's Theorem.[4] But these methods work in only a few examples. The tools necessary to deal completely with step (iii) in all discrete cases generate a sizable area of analysis known in part as "Sturm–Liouville theory," of which Fourier series is only the beginning. We do not propose developing all these tools, but we shall give in Theorem 1.3 one such tool that goes beyond ordinary Fourier series, deferring any discussion of its proof to the next section.

For functions defined on intervals, the behavior of the functions at the endpoints will be relevant to us: we say that a continuous function $f : [a, b] \to \mathbb{C}$ with a derivative on (a, b) has a continuous derivative at one or both endpoints if f' has a finite limit at the endpoint in question; it is equivalent to say that f extends to a larger set so as to be differentiable in an open interval about the endpoint and to have its derivative be continuous at the endpoint.

Theorem 1.3 (Sturm's Theorem). Let p, q, and r be continuous real-valued functions on $[a, b]$ such that p' and r'' exist and are continuous and such that p

[2]Problem 6 at the end of Chapter II.
[3]Corollaries 1.60 and 1.66 in *Basic*.
[4]Theorem 1.61 in *Basic*.

and r are everywhere positive for $a \le t \le b$. Let c_1, c_2, d_1, d_2 be real numbers such that c_1 and c_2 are not both 0 and d_1 and d_2 are not both 0. Finally for each complex number λ, let (SL) be the following set of conditions on a function $u : [a, b] \to \mathbb{C}$ with two continuous derivatives:

$$(p(t)u')' - q(t)u + \lambda r(t)u = 0, \qquad \text{(SL1)}$$

$$c_1u(a) + c_2u'(a) = 0 \quad \text{and} \quad d_1u(b) + d_2u'(b) = 0. \qquad \text{(SL2)}$$

Then the system (SL) has a nonzero solution for a countably infinite set of values of λ. If E denotes this set of values, then the members λ of E are all real, they have no limit point in $\mathbb{R}$, and the vector space of solutions of (SL) is 1-dimensional for each such λ. The set E is bounded below if $c_1c_2 \le 0$ and $d_1d_2 \ge 0$, and E is bounded below by 0 if these conditions and the condition $q \ge 0$ are all satisfied. In any case, enumerate E as $\lambda_1, \lambda_2, \ldots$, let $u = \varphi_n$ be a nonzero solution of (SL) when $\lambda = \lambda_n$, define $(f, g)_r = \int_a^b f(t)\overline{g(t)}\, r(t)\, dt$ and $\|f\|_r = \big(\int_a^b |f(t)|^2 r(t)\, dt\big)^{1/2}$ for continuous f and g, and normalize φ_n so that $\|\varphi_n\|_r = 1$. Then $(\varphi_n, \varphi_m)_r = 0$ for $m \ne n$, and the functions φ_n satisfy the following completeness conditions:

(a) any u having two continuous derivatives on $[a, b]$ and satisfying (SL2) has the property that the series $\sum_{n=1}^\infty (u, \varphi_n)_r \varphi_n(t)$ converges absolutely uniformly to $u(t)$ on $[a, b]$,

(b) the only continuous φ on $[a, b]$ with $(\varphi, \varphi_n)_r = 0$ for all n is $\varphi = 0$,

(c) any continuous φ on $[a, b]$ satisfies $\|\varphi\|_r^2 = \sum_{n=1}^\infty |(\varphi, \varphi_n)_r|^2$.

REMARK. The expression **converges absolutely uniformly** in (a) means that $\sum_{n=1}^\infty |(u, \varphi_n)_r \varphi_n(t)|$ converges uniformly.

EXAMPLE. The prototype for Theorem 1.3 is the constant-coefficient case $p = r = 1$ and $q = 0$. The equation (SL1) is just $u'' + \lambda u = 0$. If λ happens to be > 0, then the solutions are $u(t) = C_1 \cos pt + C_2 \sin pt$, where $\lambda = p^2$. Suppose $[a, b] = [0, \pi]$. The condition $c_1u(0) + c_2u'(0) = 0$ says that $c_1C_1 + pc_2C_2 = 0$ and forces a linear relationship between C_1 and C_2 that depends on p. The condition $d_1u(\pi) + d_2u'(\pi) = 0$ gives a further such relationship. These two conditions may or may not be compatible. An especially simple special case is that $c_2 = d_2 = 0$, so that (SL2) requires $u(0) = u(\pi) = 0$. From $u(0) = 0$, we get $C_1 = 0$, and then $u(\pi) = 0$ forces $\sin p\pi = 0$ if u is to be a nonzero solution. Thus p must be an integer. It may be checked that $\lambda \le 0$ leads to no nonzero solutions if $c_2 = d_2 = 0$. Part (a) of the theorem therefore says that any twice continuously differentiable function $u(t)$ on $[0, \pi]$ vanishing at 0 and π has an expansion $u(t) = \sum_{p=1}^\infty b_p \sin pt$, the series being absolutely uniformly convergent.

The first partial differential equation that we consider is the **heat equation** $u_t = \Delta u$, and we are interested in real-valued solutions.

EXAMPLES WITH THE HEAT EQUATION.

(1) We suppose that there is a single space variable x and that the set in 1-dimensional space is a rod $0 \le x \le l$. The unknown function is $u(x, t)$, and the boundary data are

$u(x, 0) = f(x)$	(initial temperature equal to $f(x)$),
$u(0, t) = u(l, t) = 0$	(ends of rod at absolute 0 temperature for all $t \ge 0$).

Heat flows in the rod for $t \ge 0$, and we want to know what happens. The equation for the heat flow is $u_t = u_{xx}$, and we search for solutions of the form $u(x, t) = X(x)T(t)$. Unless $T(t)$ is identically 0, the boundary data force $X(x)T(0) = f(x)$ and $X(0) = X(l) = 0$. Substitution into the heat equation gives

$$X(x)T'(t) = X''(x)T(t).$$

We divide by $X(x)T(t)$ and obtain

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

A function of t alone can equal a function of x alone only if it is constant, and thus

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = c$$

for some real constant c. The bound variable is x, and we hope that the possible values of c lie in a discrete set. Suppose that c is > 0, so that $c = p^2$ with $p > 0$. The equation $X''(x)/X(x) = p^2$ would say that $X(x) = c_1e^{px} + c_2e^{-px}$. From $X(0) = 0$, we get $c_2 = -c_1$, so that $X(x) = c_1(e^{px} - e^{-px})$. Since $e^{px} - e^{-px}$ is strictly increasing, $c_1(e^{px} - e^{-px}) = 0$ is impossible unless $c_1 = 0$. Thus we must have $c \le 0$. Similarly $c = 0$ is impossible, and the conclusion is that $c < 0$. We write $c = -p^2$ with $p > 0$. The equation is $X''(x) = -p^2X(x)$, and then $X(x) = c_1 \cos px + c_2 \sin px$. The condition $X(0) = 0$ says $c_1 = 0$, and the condition $X(l) = 0$ then says that $p = n\pi/l$ for some integer n. Thus

$$X(x) = \sin(n\pi x/l),$$

up to a multiplicative constant. The t equation becomes $T'(t) = -p^2T = -(n\pi/l)^2T(t)$, and hence

$$T(t) = e^{-(n\pi/l)^2 t},$$

up to a multiplicative constant. Our product solution is then a multiple of $e^{-(n\pi/l)^2 t}\sin(n\pi x/l)$, and the form of solution we expect for the boundary-value problem is therefore

$$u(x,t)=\sum_{n=1}^{\infty} c_n e^{-(n\pi/l)^2 t}\sin(n\pi x/l).$$

The constants c_n are determined by the condition at $t=0$. We extend $f(x)$, which is initially defined for $0\le x\le l$, to be defined for $-l\le x\le l$ and to be an odd function. The constants c_n are then the Fourier coefficients of f except that the period is $2l$ rather than 2π:

$$f(x)\sim\sum_{n=1}^{\infty} c_n\sin\tfrac{n\pi x}{l}\quad\text{with } c_n=\tfrac{1}{l}\int_{-l}^{l} f(y)\sin\tfrac{n\pi y}{l}\,dy=\tfrac{2}{l}\int_0^l f(y)\sin\tfrac{n\pi y}{l}\,dy.$$

Normally the Fourier series would have cosine terms as well as sine terms, but the cosine terms all have coefficient 0 since f is odd. In any event, we now have an explicit infinite series that we hope gives the desired solution $u(x,t)$. Checking that the function $u(x,t)$ defined above is indeed the desired solution amounts to handling steps (iii) and (iv) in the method of separation of variables. For (iii), we want to know whether $f(x)$ really can be represented in the indicated form. This example is simple enough that (iii) can be handled by the theory of Fourier series as in Chapter I of *Basic*: since f is assumed to have two continuous derivatives on $[0,l]$, the Fourier series converges uniformly by the Weierstrass M test, and the sum must be f by the uniqueness theorem. Another way of handling (iii) is to apply Theorem 1.3 to the equation $y''+\lambda y=0$ subject to the conditions $y(0)=0$ and $y(l)=0$: The theorem gives us a certain unique abstract expansion without giving us formulas for the explicit functions that are involved. It says also that we have completeness and absolute uniform convergence. Since our explicit expansion with sines satisfies the requirements of the unique abstract expansion, it must agree with the abstract expansion and it must converge absolutely uniformly. Whichever approach we use, the result is that we have now handled (iii). Step (iv) in the method is the justification that $u(x,t)$ has all the required properties: we have to check that the function in question solves the heat equation and takes on the asserted boundary values. The function in question satisfies the heat equation because of Theorem 1.1 and the rapid convergence of the series $\sum_{n=1}^{\infty} e^{-(n\pi/l)^2 t}$ and its first and second derivatives. The question about boundary values is completely settled by Proposition 1.2. For the condition $u(x,0)=f(x)$, we take $X=[0,l]$, $Y=[0,+\infty)$, $y=t$, $A_n(x)=c_n\sin(n\pi x/l)$, $B_n(t)=e^{-(n\pi/l)^2 t}$, and $y_0=0$ in the proposition; uniform convergence of $\sum|A_n(x)|$ follows either from Theorem 1.3 or from the

Fourier-series estimate $|c_n| \le C/n^2$, which in turn follows from the assumption that f has two continuous derivatives. The conditions $u(0,t) = u(l,t) = 0$ may be verified in the same way by reversing the roles of the space variable and the time variable. To check that $u(0,t) = 0$, for example, we use Proposition 1.2 with $X = (\delta, +\infty)$, $Y = [0,l]$, and $y_0 = 0$. Our boundary-value problem is therefore now completely solved.

(2) We continue to assume that space is 1-dimensional and that the object of interest is a rod $0 \le x \le l$. The unknown function for heat flow in the rod is still $u(x,t)$, but this time the boundary data are

$$\begin{aligned} &u(x,0) = f(x) && \text{(initial temperature equal to } f(x)\text{)},\\ &u_x(0,t) = u_x(l,t) = 0 && \text{(ends of rod perfectly insulated for all } t \ge 0\text{)}. \end{aligned}$$

In the same way as in Example 1, a product solution $X(x)T(t)$ leads to a separated equation $T'(t)/T(t) = X''(x)/X(x)$, and both sides must be some constant $-\lambda$. The equation for $X(x)$ is then

$$X'' + \lambda X = 0 \qquad \text{with } X'(0) = X'(l) = 0.$$

We find that λ has to be of the form p^2 with $p = n\pi/l$ for some integer $n \ge 0$, and $X(x)$ has to be a multiple of $\cos(n\pi x/l)$. Taking into account the formula $\lambda = p^2$, we see that the equation for $T(t)$ is

$$T'(t) = -p^2 T(t).$$

Then $T(t)$ has to be a multiple of $e^{-(n\pi/l)^2 t}$, and our product solution is a multiple of $e^{-(n\pi/l)^2 t}\cos(n\pi x/l)$. The form of solution we expect for the boundary-value problem is therefore

$$u(x,t) = \sum_{n=0}^{\infty} c_n e^{-(n\pi/l)^2 t}\cos(n\pi x/l).$$

We determine the coefficients c_n by using the initial condition $u(x,0) = f(x)$, and thus we want to represent $f(x)$ by a series of cosines:

$$f(x) \sim \sum_{n=0}^{\infty} c_n \cos\tfrac{n\pi x}{l}.$$

We can do so by extending $f(x)$ from $[0,l]$ to $[-l,l]$ so as to be even and using ordinary Fourier coefficients. The formula is therefore $c_n = \frac{2}{l}\int_0^l f(y)\cos\frac{n\pi y}{l}\,dy$ for $n > 0$, with $c_0 = \frac{1}{l}\int_0^l f(y)\,dy$. Again as in Example 1, we can carry out step (iii) of the method either by using the theory of Fourier series or by appealing to Theorem 1.3. In step (iv), we can again use Theorem 1.1 to see that the prospective function $u(x,t)$ satisfies the heat equation, and the boundary-value conditions can be checked with the aid of Proposition 1.2.

(3) We still assume that space is 1-dimensional and that the object of interest is a rod $0 \le x \le l$. The unknown function for heat flow in the rod is still $u(x, t)$, but this time the boundary data are

$$\begin{aligned} &u(x,0) = f(x) && \text{(initial temperature equal to } f(x)\text{)},\\ &u(0,t) = 0 && \text{(one end of rod held at temperature 0)},\\ &u_x(l,t) = -hu(l,t) && \text{(other end radiating into a medium of temperature 0)}, \end{aligned}$$

and h is assumed positive. In the same way as in Example 1, a product solution $X(x)T(t)$ leads to a separated equation $T'(t)/T(t) = X''(x)/X(x)$, and both sides must be some constant $-\lambda$. The equation for $X(x)$ is then

$$X'' + \lambda X = 0 \qquad \text{with} \quad \begin{cases} X(0) = 0, \\ hX(l) + X'(l) = 0. \end{cases}$$

From the equation $X'' + \lambda X = 0$ and the condition $X(0) = 0$, $X(x)$ has to be a multiple of $\sinh px$ with $\lambda = -p^2 < 0$, or of x with $\lambda = 0$, or of $\sin px$ with $\lambda = p^2 > 0$. In the first two cases, $hX(l) + X'(l)$ equals $h \sinh pl + p \cosh pl$ or $hl + 1$ and cannot be 0. Thus we must have $\lambda = p^2 > 0$, and $X(x)$ is a multiple of $\sin px$. The condition $hX(l) + X'(l) = 0$ then holds if and only if $h \sin pl + p \cos pl = 0$. This equation has infinitely many positive solutions p, and we write them as $p_1, p_2, \ldots$. See Figure 1.1 for what happens when $l = \pi$.

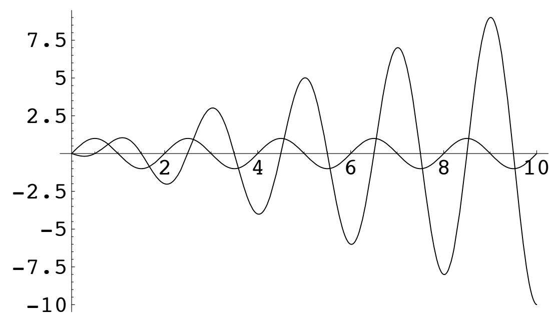


FIGURE 1.1. Graphs of $\sin \pi p$ and $-p \cos \pi p$. The graphs intersect for infinitely many values of $\pm p$.

If $\lambda = p_n^2$, then the equation for $T(t)$ is $T'(t) = -p_n^2 T(t)$, and $T(t)$ has to be a multiple of $e^{-p_n^2 t}$. Thus our product solution is a multiple of $e^{-p_n^2 t} \sin p_n x$, and the form of solution we expect for the boundary-value problem is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-p_n^2 t} \sin p_n x.$$

Putting $t = 0$, we see that we want to choose constants c_n such that

$$f(x) \sim \sum_{n=1}^{\infty} c_n \sin p_n x.$$

There is no reason why the numbers p_n should form an arithmetic progression, and such an expansion is not a result in the subject of Fourier series. To handle step (iii), this time we appeal to Theorem 1.3. That theorem points out the remarkable fact that the functions $\sin p_n x$ satisfy the orthogonality property $\int_0^l \sin p_n x \sin p_m x \, dx = 0$ if $n \neq m$ and therefore that

$$c_n = \int_0^l f(y) \sin p_n y \, dy \bigg/ \int_0^l \sin^2 p_n y \, dy \ .$$

Even more remarkably, the theorem gives us a completeness result and a convergence result. Thus (iii) is completely finished. In step (iv), we use Theorem 1.1 to check that $u(x,t)$ satisfies the partial differential equation, just as in Examples 1 and 2. The same technique as in Examples 1 and 2 with Proposition 1.2 works to recover the boundary value $u(x,0)$ as a limit; this time we use Theorem 1.3 for the absolute uniform convergence in the x variable. For $u(0,t)$, one new comment is appropriate: we take $X = (\delta, +\infty)$, $Y = [0,l]$, $y_0 = 0$, $A_n(x) = e^{-p_n^2 t}$, and $B_n(y) = c_n \sin p_n x$; although the estimate $|B_n(y)| \leq 1$ may not be valid for all n, it is valid for n sufficiently large because of the uniform convergence of $\sum c_n \sin p_n x$.

4) This time we assume that space is 2-dimensional and that the object of interest is a circular plate. The unknown function for heat flow in the plate is $u(x,y,t)$, the differential equation is $u_t = u_{xx} + u_{yy}$, and the assumptions about boundary data are that the temperature distribution is known on the plate at $t = 0$ and that the edge of the plate is held at temperature 0 for all $t \geq 0$. Let us use polar coordinates (r,θ) in the (x,y) plane, let us assume that the plate is described by $r \leq 1$, and let us write the unknown function as $v(r,\theta,t) = u(r\cos\theta, r\sin\theta, t)$. The heat equation becomes

$$v_t = v_{rr} + r^{-1} v_r + r^{-2} v_{\theta\theta},$$

and the boundary data are given by

$$\begin{aligned} &v(r,\theta,0) = f(r,\theta) && \text{(initial temperature equal to } f(r,\theta)\text{)}, \\ &v(1,\theta,t) = 0 && \text{(edge of plate held at temperature 0)}. \end{aligned}$$

We first look for solutions of the heat equation of the form $R(r)\Theta(\theta)T(t)$. Substitution and division by $R(r)\Theta(\theta)T(t)$ gives

$$\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} = \frac{T'(t)}{T(t)} = -c,$$

so that $T(t)$ is a multiple of e^{-ct}. The equation relating R, Θ, and c becomes

$$\frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = -cr^2.$$

Therefore

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = -\lambda = -\frac{r^2R''(r)}{R(r)} - \frac{rR'(r)}{R(r)} - cr^2.$$

Since $\Theta(\theta)$ has to be periodic of period 2π, we must have $\lambda = n^2$ with n an integer ≥ 0; then $\Theta(\theta) = c_1 \cos n\theta + c_2 \sin n\theta$. The equation for $R(r)$ becomes

$$r^2R'' + rR' + (cr^2 - n^2)R = 0.$$

This has a regular singular point at $r = 0$, and the indicial equation is $s^2 = n^2$. Thus $s = \pm n$. In fact, we can recognize this equation as Bessel's equation of order n by a change of variables: A little argument excludes $c \leq 0$. Putting $k = \sqrt{c}$, $\rho = kr$, and $y(\rho) = R(r)$ leads to $y'' + \rho^{-1}y' + (1 - n^2\rho^{-2})y = 0$, which is exactly Bessel's equation of order n. Transforming the solution $y(\rho) = J_n(\rho)$ back with $r = k^{-1}\rho$, we see that $R(r) = y(\rho) = J_n(\rho) = J_n(kr)$ is a solution of the equation for R. A basic product solution is therefore $\frac{1}{2}a_{0,k}J_0(kr)$ if $n = 0$ or

$$J_n(kr)(a_{n,k}\cos n\theta + b_{n,k}\sin n\theta)e^{-k^2t}$$

if $n > 0$. The index n has to be an integer in order for v to be well behaved at the center, or origin, of the plate, but we have not thus far restricted k to a discrete set. However, the condition of temperature 0 at $r = 1$ means that $J_n(k)$ has to be 0, and the zeros of J_n form a discrete set. The given condition at $t = 0$ means that we want

$$f(r,\theta) \sim \frac{1}{2}\sum_{\substack{k>0 \text{ with} \\ J_0(kr)=0}} a_{0,k}J_0(kr) + \sum_{n=1}^{\infty}\Bigg(\sum_{\substack{k>0 \text{ with} \\ J_n(kr)=0}} (a_{n,k}\cos n\theta + b_{n,k}\sin n\theta)J_n(kr)\Bigg).$$

We do not have the tools to establish this kind of relation, but we can see a hint of what to do. The orthogonality conditions that allow us to write candidates for the coefficients are the usual orthogonality for trigonometric functions and the relation

$$\int_0^1 J_n(kr)J_n(k'r)r\,dr = 0 \qquad \text{if } J_n(k) = J_n(k') = 0 \text{ and } k \neq k'.$$

The latter is not quite a consequence of Theorem 1.3, but it is close since the equation satisfied by $y_k(r) = J_n(kr)$, namely

$$(ry_k')' + (k^2r - n^2r^{-1})y_k = ry_k'' + y_k' + (k^2r - n^2r^{-1})y_k = 0,$$

fails to be of the form in Theorem 1.3 only because of trouble at the endpoint $r = 0$ of the domain interval. In fact, the argument in the next section for the orthogonality in Theorem 1.3 will work also in this case; see Problem 2 at the end of the chapter. Thus put

$$a_n(r) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(r,\theta)\cos n\theta\,d\theta \quad \text{and} \quad b_n(r) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(r,\theta)\sin n\theta\,d\theta,$$

so that

$$f(r,\theta) \sim \tfrac{1}{2}a_0(r) + \sum_{n=1}^{\infty}(a_n(r)\cos n\theta + b_n(r)\sin n\theta) \qquad \text{for each } r.$$

Then put $$a_{n,k} = \int_0^1 a_n(r)y_k(r)r\,dr \Big/ \int_0^1 y_k(r)^2 r\,dr$$

and $$b_{n,k} = \int_0^1 b_n(r)y_k(r)r\,dr \Big/ \int_0^1 y_k(r)^2 r\,dr\ .$$

With these values in place, handling step (iii) amounts to showing that

$$f(r,\theta) = \tfrac{1}{2}\sum_{\substack{k>0 \text{ with} \\ J_0(kr)=0}} a_{0,k}J_0(kr) + \sum_{n=1}^{\infty}\Big(\sum_{\substack{k>0 \text{ with} \\ J_n(kr)=0}}(a_{n,k}\cos n\theta + b_{n,k}\sin n\theta)J_n(kr)\Big)$$

for functions f of class C^2. This formula is valid, but we would need a result from Sturm–Liouville theory that is different from Theorem 1.3 in order to prove it. Step (iv) is to use the convergence from Sturm–Liouville theory, together with application of Proposition 1.2 and Theorem 1.1, to see that the function $u(r,\theta,t)$ given by

$$\frac{1}{2}\sum_{\substack{k>0 \text{ with} \\ J_0(kr)=0}} a_{0,k}J_0(kr)e^{-k^2t} + \sum_{n=1}^{\infty}\Big(\sum_{\substack{k>0 \text{ with} \\ J_n(kr)=0}}(a_{n,k}\cos n\theta + b_{n,k}\sin n\theta)J_n(kr)e^{-k^2t}\Big)$$

has all the required properties.

The second partial differential equation that we consider is the **Laplace equation** $\Delta u = 0$. Various sets of boundary data can be given, but we deal only with the values of u on the edge of its bounded domain of definition. In this case the problem of finding u is known as the **Dirichlet problem**.

EXAMPLES WITH LAPLACE EQUATION.

(1) We suppose that the space domain is the unit disk in $\mathbb{R}^2$. The Laplace equation in polar coordinates (r, θ) is $u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0$. The unknown function is $u(r, \theta)$, and the given boundary values of u for the Dirichlet problem are

$$u(1, \theta) = f(\theta) \qquad \text{(value on unit circle).}$$

It is implicit that $u(r, \theta)$ is to be periodic of period 2π in θ and is to be well behaved at $r = 0$. A product solution is of the form $R(r)\Theta(\theta)$. We substitute into the equation, divide by $r^{-2}R(r)\Theta(\theta)$, and and find that the variables separate as

$$\frac{r^2 R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = c.$$

The equation for Θ is $\Theta'' + c\Theta = 0$, and the solution is required to be periodic. We might be tempted to try to apply Theorem 1.3 at this stage, but the boundary condition of periodicity, $\Theta(-\pi) = \Theta(\pi)$, is not exactly of the right kind for Theorem 1.3. Fortunately we can handle matters directly, using Fourier series in the analysis. The periodicity forces $c = n^2$ with n an integer ≥ 0. Then $\Theta(\theta) = c_1 \cos n\theta + c_2 \sin n\theta$, except that the sine term is not needed when $n = 0$. The equation for R becomes

$$r^2 R'' + rR' - n^2 R = 0.$$

This is an Euler equation with indicial equation $s^2 = n^2$, and hence $s = \pm n$. We discard $-n$ with $n \geq 1$ because the solution r^{-n} is not well behaved at $r = 0$, and we discard also the second solution $\log r$ that goes with $n = 0$. Consequently $R(r)$ is a multiple of r^n, and the product solution is $r^n(a_n \cos n\theta + b_n \sin n\theta)$ when $n > 0$. The expected solution of the Laplace equation is then

$$u(r, \theta) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta).$$

We determine a_n and b_n by formally putting $r = 1$, and we see that a_n and b_n are to be the ordinary Fourier coefficients of $f(x)$. The normal assumption for a boundary-value problem is that f is as nice a function as u and hence has two continuous derivatives. In this case we know that the Fourier series converges to $f(x)$ uniformly. It is immediate from Theorem 1.1 that $u(r, \theta)$ satisfies Laplace's equation for $r < 1$, and Proposition 1.2 shows that $u(r, \theta)$ has the desired boundary values. This completes the solution of the boundary-value problem. In this example the solution $u(r, \theta)$ is given by a nice integral formula: The same easy computation that expresses the partial sums of a Fourier series in terms of the Dirichlet kernel allows us to write $u(r,\theta)$ in terms of the **Poisson kernel**

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2} = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta},$$

namely

$$\begin{aligned} u(r,\theta) &= \sum_{n=-\infty}^{\infty} r^{|n|}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)e^{-in\varphi}\,d\varphi\Big)e^{in\theta} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\Big(\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(\theta-\varphi)}\Big)\,d\varphi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)P_r(\theta-\varphi)\,d\varphi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta-\varphi)P_r(\varphi)\,d\varphi. \end{aligned}$$

The interchange of integral and sum for the second equality is valid because of the uniform convergence of the series $\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(\theta-\varphi)}$ for fixed r. The resulting formula for $u(r,\theta)$ is known as the **Poisson integral formula** for the unit disk.

(2) We suppose that the space domain is the unit ball in $\mathbb{R}^3$. The Laplace equation in spherical coordinates (r,φ,θ), with φ measuring latitude from the point $(x,y,z) = (0,0,1)$, is

$$(r^2u_r)_r + \frac{1}{\sin\varphi}\,((\sin\varphi)u_\varphi)_\varphi + \frac{1}{\sin^2\varphi}\,u_{\theta\theta} = 0.$$

The unknown function is $u(r,\varphi,\theta)$, and the given boundary values of u for the Dirichlet problem are

$$u(1,\varphi,\theta) = f(\varphi,\theta) \qquad \text{(value on unit sphere).}$$

The function u is to be periodic in θ and is to be well behaved at $r=0$, $\varphi=0$, and $\varphi=\pi$. Searching for a solution $R(r)\Phi(\varphi)\Theta(\theta)$ leads to the separated equation

$$\frac{r^2R'' + 2rR'}{R} = -\frac{\Phi'' + (\cot\varphi)\Phi'}{\Phi} - \frac{1}{\sin^2\varphi}\frac{\Theta''}{\Theta} = c.$$

The resulting equation for R is $r^2R''+2rR'-cR=0$, which is an Euler equation whose indicial equation has roots s satisfying $s(s+1)=c$. The condition that a solution of the Laplace equation be well behaved at $r=0$ means that the solution

r^s must have s equal to an integer $m \ge 0$. Then $R(r)$ is a multiple of r^m with m an integer ≥ 0 and with $c = m(m+1)$. The equation involving Φ and Θ is then

$$(\sin^2\varphi)\,\frac{\Phi'' + (\cot\varphi)\Phi'}{\Phi} + \frac{\Theta''}{\Theta} + m(m+1)\sin^2\varphi = 0.$$

This equation shows that $\Theta''/\Theta = c'$, and as usual we obtain $c' = -n^2$ with n an integer ≥ 0. Then $\Theta(\theta) = c_1\cos n\theta + c_2\sin n\theta$. Substituting into the equation for Φ yields

$$(\sin^2\varphi)\,\frac{\Phi'' + (\cot\varphi)\Phi'}{\Phi} - n^2 + m(m+1)\sin^2\varphi = 0.$$

We make the change of variables $t = \cos\varphi$, which has

$$\frac{d}{d\varphi} = -\sin\varphi\,\frac{d}{dt} \qquad \text{and} \qquad \frac{d^2}{d\varphi^2} = -(\cos\varphi)\,\frac{d}{dt} + (\sin^2\varphi)\,\frac{d^2}{dt^2}.$$

Putting $P(t) = P(\cos\varphi) = \Phi(\varphi)$ for $0 \le \varphi \le \pi$ leads to

$$(1-t^2)\Big[\frac{(1-t^2)P'' - tP' + (\cot\varphi)(-\sin\varphi)P'}{P}\Big] - n^2 + m(m+1)(1-t^2) = 0$$

and then to

$$(1-t^2)P'' - 2tP' + \Big[m(m+1) - \frac{n^2}{1-t^2}\Big]P = 0.$$

This is known as an **associated Legendre equation**. For $n = 0$, which is the case of a solution independent of longitude θ, the equation reduces to the ordinary Legendre equation.[5] Suppose for simplicity that f is independent of longitude θ and that we can take $n = 0$ in this equation. One solution of the equation for P is $P(t) = P_m(t)$, the m^{th} Legendre polynomial. This is well behaved at $t = \pm 1$, the values of t that correspond to $\varphi = 0$ and $\varphi = \pi$. Making a change of variables, we can see that the Legendre equation has regular singular points at $t = 1$ and $t = -1$. By examining the indicial equations at these points, we can see that there is only a 1-parameter family of solutions of the equation for P that are well behaved at $t = \pm 1$. Thus $\Phi(\varphi)$ has to be a multiple of $P_m(\cos\varphi)$, and we are led to expect

$$u(r,\varphi,\theta) = \sum_{m=0}^{\infty} c_m r^m P_m(\cos\varphi)$$

[5]The ordinary Legendre equation is $(1-t^2)P'' - 2tP' + m(m+1)P = 0$, as in Section IV.8 of *Basic*.

for solutions that are independent of θ. If $f(\varphi, \theta)$ is independent of θ, we determine c_m by the formula

$$f(\varphi, \theta) \sim \sum_{m=0}^{\infty} c_m P_m(\cos \varphi).$$

The coefficients can be determined because the polynomials P_m are orthogonal under integration over $[-1, 1]$. To see this fact, we first rewrite the equation for P as $((1-t^2)P')' + m(m+1)P = 0$. This is almost of the form in Theorem 1.3, but the coefficient $1 - t^2$ vanishes at the endpoints $t = \pm 1$. Although the orthogonality does not then follow from Theorem 1.3, it may be proved in the same way as the orthogonality that is part of Theorem 1.3; see Problem 2 at the end of the chapter. A part of the completeness question is easily settled by observing that P_m is of degree m and that therefore the linear span of $\{P_0, P_1, \ldots, P_N\}$ is the same as the linear span of $\{1, t, \ldots, t^N\}$. This much does not establish, however, that the series $\sum c_m P_m(t)$ converges uniformly. For that, we would need yet another result from Sturm–Liouville theory or elsewhere. Once the uniform convergence has been established, step (iv) can be handled in the usual way.

The third and final partial differential equation that we consider is the **wave equation** $u_{tt} = \Delta u$. We consider examples of boundary-value problems in one and two space variables.

EXAMPLES WITH WAVE EQUATION.

(1) A string on the x axis under tension is such that each point can be displaced only in the y direction. Let $y = u(x, t)$ be the displacement. The equation for the unknown function $u(x, t)$ in suitable physical units is $u_{tt} = u_{xx}$, and the boundary data are

$$\begin{aligned} u(x, 0) &= f(x) && \text{(initial displacement)}, \\ u_t(x, 0) &= g(x) && \text{(initial velocity)}, \\ u(0, t) &= u(l, t) = 0 && \text{(ends of string fixed for all } t \geq 0). \end{aligned}$$

The string vibrates for $t \geq 0$, and we want to know what happens. Searching for basic product solutions $X(x)T(t)$, we are led to $T''/T = X''/X =$ constant. As usual the conditions at $x = 0$ and $x = l$ force the constant to be nonpositive, necessarily $-\omega^2$ with $\omega \geq 0$. Then $X(x) = c_1 \cos \omega x + c_2 \sin \omega x$. We obtain $c_1 = 0$ from $X(0) = 0$, and we obtain $\omega = n\pi/l$, with n an integer, from $X(l) = 0$. Thus $X(x)$ has to be a multiple of $\sin(n\pi x/l)$, and we may take $n > 0$. Examining the T equation, we are readily led to expect

$$u(x, t) = \sum_{n=1}^{\infty} \sin(n\pi x/l)[a_n \cos(n\pi t/l) + b_n \sin(n\pi t/l)].$$

The conditions $u(x,0) = f(x)$ and $u_t(x,0)$ say that

$$f(x) \sim \sum_{n=1}^{\infty} a_n \sin\left(\tfrac{n\pi x}{l}\right) \qquad \text{and} \qquad g(x) \sim \sum_{n=1}^{\infty} \left(\tfrac{n\pi}{l}\right) b_n \sin\left(\tfrac{n\pi x}{l}\right),$$

so that a_n and $n\pi b_n/l$ are coefficients in the Fourier sine series for f and g. Steps (iii) and (iv) in the method follow in the same way as in earlier examples.

(2) We visualize a vibrating circular drum. A membrane in the (x, y) plane covers the unit disk and is under uniform tension. Each point can be displaced only in the z direction. Let $u(x, y, t) = U(r, \theta, t)$ be the displacement. The wave equation $u_{tt} = u_{xx} + u_{yy}$ becomes $U_{tt} = U_{rr} + r^{-1}U_r + r^{-2}U_{\theta\theta}$ in polar coordinates. Assume for simplicity that the boundary data are

$$\begin{aligned} U(r, \theta, 0) &= f(r) && \text{(initial displacement independent of } \theta\text{)},\\ U_t(r, \theta, 0) &= 0 && \text{(initial velocity 0)},\\ U(1, \theta, t) &= 0 && \text{(edge of drum fixed for all } t \geq 0\text{)}. \end{aligned}$$

Because of the radial symmetry, let us look for basic product solutions of the form $R(r)T(t)$. Substituting and separating variables, we are led to $T''/T = (R'' + r^{-1}R')/R = c$. The equation for R is $r^2R'' + rR' - cr^2R = 0$, and the usual considerations do not determine the sign of c. The equation for R has a regular singular point at $r = 0$, but it is not an Euler equation. The indicial equation is $s^2 = 0$, with $s = 0$ as a root of multiplicity 2, independently of c. One solution is given by a power series in r, while another involves $\log r$. We discard the solution with the logarithm because it would represent a singularity at the middle of the drum. To get at the sign of c, we use the condition $R(1) = 0$ and argue as follows: Without loss of generality, $R(0)$ is positive. Suppose $c > 0$, and let $r_1 \leq 1$ be the first value of $r > 0$ where $R(r_1) = 0$. From the equation $r^{-1}(rR')' = cR$ and the inequality $R(r) > 0$ for $0 < r < r_1$, we see that rR' is strictly increasing for $0 < r < r_1$. Examining the power series expansion for $R(r)$, we see that $R'(0) = 0$. Thus $R'(r) > 0$ for $0 < r < r_1$. But $R(0) > 0$ and $R(r_1) = 0$ imply, by the Mean Value Theorem, that $R'(r)$ is < 0 somewhere in between, and we have a contradiction. Similarly we rule out $c = 0$. We conclude that c is negative, i.e., $c = -k^2$ with $k > 0$. The equation for R is then

$$r^2R'' + rR' + k^2r^2R = 0.$$

The change of variables $\rho = kr$ reduces this equation to Bessel's equation of order 0, and the upshot is that $R(r)$ is a multiple of $J_0(kr)$. The condition $R(1) = 0$ means that $J_0(k) = 0$. If k_n is the n^{th} positive zero of J_0, then the T equation is

$T'' + k_n^2 T = 0$, so that $T(t) = c_1 \cos k_n t + c_2 \sin k_n t$. From $U_t(r, \theta, 0) = 0$, we obtain $c_2 = 0$. Thus $T(t)$ is a multiple of $\cos k_n t$, and we expect that

$$U(r, \theta, t) = \sum_{n=1}^{\infty} c_n J_0(k_n r) \cos k_n t.$$

In step (iii), the determination of the c_n's and the necessary analysis are similar to those in Example 4 for the heat equation, and it is not necessary to repeat them. Step (iv) is handled in much the same way as in the vibrating-string problem.

3. Sturm–Liouville Theory

The name "Sturm–Liouville theory" refers to the analysis of certain kinds of "eigenvalue" problems for linear ordinary differential equations, particularly equations of the second order. In this section we shall concentrate on one theorem of this kind, which was stated explicitly in Section 2 and was used as a tool for verifying that the method of separation of variables succeeded, for some examples, in solving a boundary-value problem for one of the standard partial differential equations. Before taking up this one theorem, however, let us make some general remarks about the setting, about "eigenvalues" and "eigenfunctions," and about "self-adjointness."

Fix attention on an interval $[a, b]$ and on second-order differential operators on this interval of the form $L = P(t)D^2 + Q(t)D + R(t)1$ with $D = d/dt$, so that

$$L(u) = P(t)u'' + Q(t)u' + R(t)u.$$

We shall assume that the coefficient functions P, Q, and R are real-valued; then $L(\bar{u}) = \overline{L(u)}$. As was mentioned in Section 2, the behavior of all functions in question at the endpoints will be relevant to us: we say that a continuous function $f : [a, b] \to \mathbb{C}$ with a derivative on (a, b) has a continuous derivative at one or both endpoints if f' has a finite limit at the endpoint in question; it is equivalent to say that f extends to a larger set so as to be differentiable in an open interval about the endpoint and to have its derivative be continuous at the endpoint.

An **eigenvalue** of the differential operator L is a complex number c such that $L(u) = cu$ for some nonzero function u. Such a function u is called an **eigenfunction**. In practice we often have a particular nonvanishing function r and look for c such $L(u) = cru$ for a nonzero u. In this case, c is an eigenvalue of $r^{-1}L$.

We introduce the inner-product space of complex-valued functions with two continuous derivatives on $[a, b]$ and with $(u, v) = \int_a^b u(t)\overline{v(t)}\,dt$. Computation using integration by parts and assuming suitable differentiability of the coefficients gives

$$\begin{aligned}
(L(u), v) &= \int_a^b (Pu'' + Qu' + Ru)\bar{v}\, dt \\
&= \int_a^b ((u'')(P\bar{v}) + (u')(Q\bar{v}) + (u)(R\bar{v}))\, dt \\
&= \Big[(u')(P\bar{v}) + (u)(Q\bar{v})\Big]_a^b - \int_a^b (u'(P\bar{v})' + (u)(Q\bar{v})' - (u)(R\bar{v}))\, dt \\
&= \Big[(u')(P\bar{v}) + (u)(Q\bar{v}) - (u)(P\bar{v})'\Big]_a^b \\
&\quad + \int_a^b ((u)(P\bar{v})'' - (u)(Q\bar{v})' + (u)(R\bar{v}))\, dt \\
&= (u, L^*(v)) + \Big[(u')(P\bar{v}) + (u)(Q\bar{v}) - u(P\bar{v})'\Big]_a^b,
\end{aligned}$$

where $L^*(v) = Pv'' + (2P' - Q)v' + (P'' - Q' + R)v$. The above computation shows that $(L(u), v) = (u, L^*(v))$ if the integrated terms are ignored; this property is the abstract defining property of L^*. The differential operator L^* is called the **formal adjoint** of L. We shall be interested only in the situation in which $L^* = L$, which we readily see happens if and only if $P' = Q$; when $L^* = L$, we say that L is **formally self adjoint**. If L is formally self adjoint, then substitution of $Q = P'$ shows that the above identity reduces to

$$(L(u), v) - (u, L(v)) = \Big[(P)(u'\bar{v} - u\bar{v}')\Big]_a^b,$$

which is known as **Green's formula**.

Even when L as above is not formally self adjoint, it can be multiplied by a nonvanishing function, specifically $\int^t \exp[(Q(s) - P'(s))/P(s)]\, ds$, to become formally self adjoint. Thus formal self-adjointness by itself is no restriction on our second-order differential operator.

In the formally self-adjoint case, one often rewrites $P(t)D^2 + P'(t)D$ as $D(P(t)D)$. With this understanding, let us rewrite our operator as

$$L(u) = (p(t)u')' - q(t)u$$

and assume that p, p', and q are continuous on $[a, b]$ and that $p(t) > 0$ for $a \le t \le b$. We associate a **Sturm–Liouville eigenvalue problem** called (SL) to the set of data consisting of L, an everywhere-positive function r with two continuous derivatives on $[a, b]$, and real numbers c_1, c_2, d_1, d_2 such that c_1 and c_2 are not both 0 and d_1 and d_2 are not both 0. This is the problem of analyzing simultaneous solutions of

$$L(u) + \lambda r(t)u = 0, \tag{SL1}$$

$$c_1 u(a) + c_2 u'(a) = 0 \quad \text{and} \quad d_1 u(b) + d_2 u'(b) = 0, \tag{SL2}$$

for all values of λ.

Each condition (SL1) and (SL2) depends linearly on u and u' if λ is fixed, and thus the space of solutions of (SL) for fixed λ is a vector space. We know[6] that the vector space of solutions of (SL1) alone is 2-dimensional; let u_1 and u_2 form a basis of this vector space. The Wronskian matrix is $\begin{pmatrix} u_1(t) & u_2(t) \\ u_1'(t) & u_2'(t) \end{pmatrix}$, and the determinant of this matrix, namely

$$u_1(t)u_2'(t) - u_1'(t)u_2(t),$$

is nowhere 0. If u_1 and u_2 were both to satisfy the condition $c_1 u(a) + c_2 u'(a) = 0$ with c_1 and c_2 not both 0, then $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ would be a nontrivial solution of the matrix equation

$$\begin{pmatrix} u_1(a) & u_1'(a) \\ u_2(a) & u_2'(a) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and we would obtain the contradictory conclusion that the Wronskian matrix at a is singular. We conclude that the space of solutions of (SL) for fixed λ is at most 1-dimensional.

Let $(\varphi_1, \varphi_2)_r = \int_a^b \varphi_1(t)\overline{\varphi_2(t)}\, r(t)\, dt$ for any continuous functions φ_1 and φ_2 on $[a, b]$, and let $\|\varphi_1\|_r = ((\varphi_1, \varphi_1)_r)^{1/2}$. The unsubscripted expressions (φ_1, φ_2) and $\|\varphi_1\|$ will refer to $(\varphi_1, \varphi_2)_r$ and $\|\varphi_1\|_r$ with $r = 1$. Then we can restate Theorem 1.3 as follows.

Theorem 1.3$'$ (Sturm's Theorem). The system (SL) has a nonzero solution for a countably infinite set of values of λ. If E denotes this set of values, then the members λ of E are all real, they have no limit point in $\mathbb{R}$, and the space of solutions of (SL) is 1-dimensional for each such λ. The set E is bounded below if $c_1 c_2 \le 0$ and $d_1 d_2 \ge 0$, and E is bounded below by 0 if these conditions and the condition $q \ge 0$ are all satisfied. In any case, enumerate E in any fashion as $\lambda_1, \lambda_2, \dots$, let $u = \varphi_n$ be a nonzero solution of (SL) when $\lambda = \lambda_n$, and normalize φ_n so that $\|\varphi_n\|_r = 1$. Then $(\varphi_n, \varphi_m)_r = 0$ for $m \ne n$, and the functions φ_n satisfy the following completeness conditions:

(a) any u having two continuous derivatives on $[a, b]$ and satisfying (SL2) has the property that the series $\sum_{n=1}^{\infty} (u, \varphi_n)_r \varphi_n(t)$ converges absolutely uniformly to $u(t)$ on $[a, b]$,

(b) the only continuous φ on $[a, b]$ with $(\varphi, \varphi_n)_r = 0$ for all n is $\varphi = 0$,

(c) any continuous φ on $[a, b]$ satisfies $\|\varphi\|_r^2 = \sum_{n=1}^{\infty} |(\varphi, \varphi_n)_r|^2$.

[6]From Theorem 4.6 of *Basic*, for example.

REMARKS. In this section we shall reduce the proof of everything but (b) and (c) to the Hilbert–Schmidt Theorem, which will be proved in Chapter II. Conclusions (b) and (c) follow from (a) and some elementary facts about Hilbert spaces, and we shall return to prove these two conclusions at the time of the Hilbert–Schmidt Theorem in Chapter II.

PROOF EXCEPT FOR STEPS TO BE COMPLETED IN CHAPTER II. By way of preliminaries, let u and v be nonzero functions on $[a, b]$ satisfying (SL2) and having two continuous derivatives. Green's formula gives

$$\begin{aligned}(L(u),v) - (u, L(v)) &= \big[(p)(u'\bar{v} - u\bar{v}')\big]_a^b \\ &= p(b)\big(u'(b)\overline{v(b)} - u(b)\overline{v'(b)}\big) - p(a)\big(u'(a)\overline{v(a)} - u(a)\overline{v'(a)}\big).\end{aligned}$$

Condition (SL2) says that

$$c_1u(a) + c_2u'(a) = 0 \quad \text{and} \quad c_1v(a) + c_2v'(a) = 0.$$

Since c_1 and c_2 are real, these equations yield

$$c_1u(a)\overline{v(a)} + c_2u'(a)\overline{v(a)} = 0 \;\text{ and }\; c_1u(a)\overline{v(a)} + c_2u(a)\overline{v'(a)} = 0,$$

as well as

$$c_1u(a)\overline{v'(a)} + c_2u'(a)\overline{v'(a)} = 0 \;\text{ and }\; c_1u'(a)\overline{v(a)} + c_2u'(a)\overline{v'(a)} = 0.$$

Subtracting, for each of the above two displays, each second equation of a display from the first equation of the display, we obtain

$$c_2\big(u'(a)\overline{v(a)} - u(a)\overline{v'(a)}\big) = 0$$

and

$$c_1\big(u(a)\overline{v'(a)} - u'(a)\overline{v(a)}\big) = 0.$$

Since c_1 and c_2 are not both 0, we conclude that $p(a)(u'(a)\overline{v(a)} - u(a)\overline{v'(a)}) = 0$. A similar computation starting from

$$d_1u(b) + d_2u'(b) = 0 \quad \text{and} \quad d_1v(b) + d_2v'(b) = 0$$

shows that $p(b)(u'(b)\overline{v(b)} - u(b)\overline{v'(b)}) = 0$. Consequently

$$(L(u), v) - (u, L(v)) = 0$$

whenever u and v are functions on $[a, b]$ satisfying (SL2) and having two continuous derivatives.

Now we can begin to establish the properties of the set E of numbers λ for which (SL) has a nonzero solution. Suppose that φ_α and φ_β satisfy $L(\varphi_\alpha) + \lambda_\alpha r\varphi_\alpha = 0$ and $L(\varphi_\beta) + \lambda_\beta r\varphi_\beta = 0$. By what we have just seen,

$$\begin{aligned} 0 &= (L(\varphi_\alpha), \varphi_\beta) - (\varphi_\alpha, L(\varphi_\beta)) \\ &= \int_a^b L(\varphi_\alpha)\bar{\varphi}_\beta\,dt - \int_a^b \varphi_\alpha\overline{L(\varphi_\beta)}\,dt \\ &= (-\lambda_\alpha + \bar{\lambda}_\beta)\int_a^b \varphi_\alpha\bar{\varphi}_\beta\, r\,dt = (-\lambda_\alpha + \bar{\lambda}_\beta)(\varphi_\alpha, \varphi_\beta)_r. \end{aligned}$$

Taking $\varphi_\alpha = \varphi_\beta$ in this computation shows that $\lambda_\alpha = \bar{\lambda}_\alpha$; hence λ_α is real. With λ_α and λ_β real and unequal, this computation shows that $(\varphi_\alpha, \varphi_\beta)_r = 0$. Thus the members of E are real, and the corresponding φ's are orthogonal. We have seen that the dimension of the space of solutions of (SL) corresponding to any member of E is 1-dimensional.

We shall prove that E is at most countably infinite. Let $c = \big(\int_a^b r(t)\,dt\big)^{1/2}$. Any continuous φ on $[a, b]$ satisfies

$$\|\varphi\|_r = \Big(\int_a^b |\varphi(t)|^2 r(t)\,dt\Big)^{1/2} \le (\sup_{a\le t\le b} |\varphi(t)|)\Big(\int_a^b r(t)\,dt\Big)^{1/2} = c\sup|\varphi|.$$

Consider the open ball $B(k; \varphi)$ of radius k and center φ in the space $C([a, b])$ of continuous functions on $[a, b]$; the metric is given by the supremum of the absolute value of the difference of the functions. If ψ is in this ball, then $\sup|\psi - \varphi| < k$, $c\sup|\psi - \varphi| < ck$, and $\|\psi - \varphi\|_r < ck$. Choose k with $ck = \frac{1}{2}$. Suppose that φ_α and φ_β correspond as above to unequal λ_α and λ_β and that φ_α and φ_β have been normalized so that $\|\varphi_\alpha\|_r = \|\varphi_\beta\|_r = 1$. If ψ is in $B(k; \varphi_\alpha) \cap B(k; \varphi_\beta)$, then $\|\psi - \varphi_\alpha\|_r < \frac{1}{2}$ and $\|\psi - \varphi_\beta\|_r < \frac{1}{2}$. The triangle inequality gives $\|\varphi_\alpha - \varphi_\beta\|_r < 1$, whereas the orthogonality implies that

$$\begin{aligned} \|\varphi_\alpha - \varphi_\beta\|_r^2 &= (\varphi_\alpha - \varphi_\beta,\ \varphi_\alpha - \varphi_\beta)_r \\ &= (\varphi_\alpha, \varphi_\alpha)_r - (\varphi_\alpha, \varphi_\beta)_r - (\varphi_\beta, \varphi_\alpha)_r + (\varphi_\beta, \varphi_\beta)_r \\ &= 1 - 0 - 0 + 1 = 2. \end{aligned}$$

The existence of ψ thus leads us to a contradiction, and we conclude that $B(k; \varphi_\alpha)$ and $B(k; \varphi_\beta)$ are disjoint. Since $[a, b]$ is a compact metric space, $C([a, b])$ is separable as a metric space,[7] and hence so is the metric subspace $S = \bigcup_\alpha B(k; \varphi_\alpha)$. The collection of all $B(k; \varphi_\alpha)$ is an open cover of S, and the separability gives us

[7] By Corollary 2.59 of *Basic*.

a countable subcover. Since the sets $B(k; \varphi_\alpha)$ are disjoint, we conclude that the set of all φ_α is countable. Hence E is at most countably infinite.

The next step is to bound E below under additional hypotheses as in the statement of the theorem. Let λ be in E, and let φ be a nonzero solution of (SL) corresponding to λ and normalized so that $\|\varphi\|_r = 1$. Multiplying (SL1) by $\bar\varphi$ and integrating, we have

$$\begin{aligned}
\lambda &= \int_a^b \lambda|\varphi|^2 r\,dt = -\int_a^b (p\varphi')'\bar\varphi\,dt + \int_a^b q|\varphi|^2\,dt \\
&= -\big[p\varphi'\bar\varphi\big]_a^b + \int_a^b p|\varphi'|^2\,dt + \int_a^b q|\varphi|^2\,dt \\
&\geq -p(b)\varphi'(b)\overline{\varphi(b)} + p(a)\varphi'(a)\overline{\varphi(a)} + \int_a^b (|\varphi|^2 r)(r^{-1}q)\,dt \\
&\geq -p(b)\varphi'(b)\overline{\varphi(b)} + p(a)\varphi'(a)\overline{\varphi(a)} + \inf_{a\leq t\leq b}\{r(t)^{-1}q(t)\}.
\end{aligned}$$

Let us show under the hypotheses $c_1c_2 \leq 0$ and $d_1d_2 \geq 0$ that $\varphi'(a)\overline{\varphi(a)} \geq 0$ and $\varphi'(b)\overline{\varphi(b)} \leq 0$, and then the asserted lower bounds will follow. Condition (SL2) gives us $c_1\varphi(a) + c_2\varphi'(a) = 0$. If $c_1 = 0$ or $c_2 = 0$, then $\varphi'(a) = 0$ or $\varphi(a) = 0$, and hence $\varphi'(a)\overline{\varphi(a)} \geq 0$. If $c_1c_2 \neq 0$, then $c_1c_2 < 0$. The identity $c_1\varphi(a) + c_2\varphi'(a) = 0$ implies that $c_1^2|\varphi(a)|^2 + c_1c_2\varphi'(a)\overline{\varphi(a)} = 0$ and hence $-c_1c_2\varphi'(a)\overline{\varphi(a)} = c_1^2|\varphi(a)|^2 \geq 0$. Because of the condition $c_1c_2 < 0$, we conclude that $\varphi'(a)\overline{\varphi(a)} \geq 0$. A similar argument using $d_1d_2 \geq 0$ and $d_1\varphi(b) + d_2\varphi'(b) = 0$ shows that $\varphi'(b)\overline{\varphi(b)} \leq 0$. This completes the verification of the lower bounds for λ.

We have therefore established all the results in the theorem that are to be proved at this time except for

(i) the existence of a countably infinite set of λ for which (SL) has a nonzero solution,
(ii) the fact that E has no limit point in $\mathbb{R}$,
(iii) the assertion (a) about completeness.

Before carrying out these steps, we may need to adjust L slightly. We are studying functions u satisfying $L(u) + \lambda ru = 0$ and (SL2), and we have established that the set E of λ for which there is a nonzero solution is at most countably infinite. Choose a member λ_0 of the complementary set E^c and rewrite the differential equation as $M(u) + \nu ru = 0$, where $M(u) = L(u) + \lambda_0 ru$ and $\nu = (\lambda - \lambda_0)$. Then M has properties similar to those of L, and it has the further property that 0 is not a value of ν for which $M(u) + \nu ru = 0$ and (SL2) together have a nonzero solution. It would be enough to prove (i), (ii), and (iii) for $M(u) + \nu ru = 0$ and (SL2). Adjusting notation, we may assume from the outset that 0 is not in E.

The next step is to prove the existence of a continuous real-valued function $G_1(t,s)$ on $[a,b]\times[a,b]$ such that $G_1(t,s) = G_1(s,t)$, such that the operator T_1 given by

$$T_1 f(t) = \int_a^b G_1(t,s) f(s)\,ds$$

carries the space $C[a,b]$ of continuous functions f on $[a,b]$ one-one onto the space $\mathcal{D}[a,b]$ of functions u on $[a,b]$ satisfying (SL2) and having two continuous derivatives on $[a,b]$, and such that $L : \mathcal{D}[a,b] \to C[a,b]$ is a two-sided inverse function to T_1. The existence will be proved by an explicit construction that will be carried out as a lemma at the end of this section. The function $G_1(t,s)$ is called a **Green's function** for the operator L subject to the conditions (SL2). Assuming that a Green's function indeed exists, we next apply the Hilbert–Schmidt Theorem of Chapter II in the following form:

SPECIAL CASE OF HILBERT–SCHMIDT THEOREM. Let $G(t,s)$ be a continuous complex-valued function on $[a,b]\times[a,b]$ such that $G(t,s) = \overline{G(s,t)}$, and define

$$Tf(t) = \int_a^b G(t,s) f(s)\,ds$$

from the space $C[a,b]$ of continuous functions on $[a,b]$ to itself. Define an inner product $(f,g) = \int_a^b f(t)\overline{g(t)}\,dt$ and its corresponding norm $\|\cdot\|$ on $C[a,b]$. For each complex $\mu \neq 0$, define

$$V_\mu = \big\{ f : [a,b] \to \mathbb{C} \;\big|\; f \text{ is continuous and } T(f) = \mu f \big\}.$$

Then each V_μ is finite dimensional, the space $V_\mu \neq 0$ is nonzero for only countably many μ, the μ's with $V_\mu \neq 0$ are all real, and for any $\epsilon > 0$, there are only finitely many μ with $V_\mu \neq 0$ and $|\mu| \geq \epsilon$. The spaces V_μ are mutually orthogonal with respect to the inner product (f,g), and the continuous functions orthogonal to all V_μ are the continuous functions h with $T(h) = 0$. Let $v_1, v_2, \ldots$ be an enumeration of the union of orthogonal bases of the spaces V_μ with $\|v_j\| = 1$ for all j. Then for any continuous f on $[a,b]$,

$$T(f)(t) = \sum_{n=1}^\infty (T(f), v_n) v_n(t),$$

the series on the right side being absolutely uniformly convergent.

The theorem is applied not to our Green's function G_1 and the operator T_1 as above but to

$$G(t,s) = r(t)^{1/2} G_1(t,s) r(s)^{1/2}$$

and
$$Tf(t) = \int_a^b G(t,s) f(s)\,ds = r(t)^{1/2} T_1(r^{1/2} f)(t).$$

If $T(f) = \mu f$ for a real number $\mu \neq 0$, then we have $T_1(r^{1/2}f) = \mu r^{-1/2} f$. Application of L gives $r^{1/2} f = \mu L(r^{-1/2} f)$. If we put $u = r^{-1/2} f$, then we obtain $\mu L(u) = r^{1/2} f = r(r^{-1/2} f) = ru$. Hence $L(u) + \lambda r u = 0$ for $\lambda = -\mu^{-1}$. Also, the equation $u = r^{-1/2} f = \mu^{-1} T_1(r^{1/2} f)$ exhibits u as in the image of T_1 and shows that u satisfies (SL2). Conversely if $L(u) + \lambda r u = 0$ and u satisfies (SL2), recall that we arranged that 0 is not in E, so that λ has a reciprocal. Define $f = r^{1/2} u$. Application of T_1 to $L(u) + \lambda r u = 0$ gives $0 = u + \lambda T_1(ru) = r^{-1/2} f + \lambda T_1(r^{1/2} f)$. Then $T(f) = r^{1/2} T_1(r^{1/2} f) = -\lambda^{-1} f$. We conclude that the correspondence $f = r^{1/2} u$ exactly identifies the vector subspace of functions u in $\mathcal{D}[a,b]$ satisfying $L(u) + \lambda r u = 0$ with the vector subspace of functions f in $C[a,b]$ satisfying $T(f) = -\lambda^{-1} f$.

The statement of Sturm's Theorem gives us an enumeration $\lambda_1, \lambda_2, \dots$ of E. We know for each $\lambda = \lambda_n$ that the space of functions u solving (SL) for $\lambda = \lambda_n$ in E is 1-dimensional, and the statement of Sturm's Theorem has selected for us a function $u = \varphi_n$ solving (SL) such that $\|\varphi_n\|_r = 1$. Define $v_n = r^{1/2} \varphi_n$ and $\mu_n = -\lambda_n^{-1}$, so that $T(v_n) = \mu_n v_n$ and $\|v_n\| = \|\varphi_n\|_r = 1$. Because of the correspondence of μ's and λ's, the v_n may be taken as the complete list of vectors specified in the Hilbert–Schmidt Theorem. Since the φ_n's are orthogonal for $(\,\cdot\,,\,\cdot\,)_r$, the v_n's are orthogonal for $(\,\cdot\,,\,\cdot\,)$.

The operator T_1 has 0 kernel on $C[a,b]$, being invertible, and the formula for T in terms of T_1 shows therefore that T has 0 kernel. Thus the sequence $\mu_1, \mu_2, \dots$ is infinite, and the Hilbert–Schmidt Theorem shows that it tends to 0. The corresponding sequence $\lambda_1, \lambda_2, \dots$ of negative reciprocals is then infinite and has no finite limit point. This proves results (i) and (ii) announced above.

Let u have two continuous derivatives on $[a,b]$ and satisfy (SL2). Then u is in the image of T_1. Write $u = T_1(f)$ with f continuous, and put $g = r^{-1/2} f$. Then $u = T_1(f) = r^{-1/2} T(r^{-1/2} f) = r^{-1/2} T(g)$ and $(u, \varphi_n)_r = (T(g), v_n)$. Hence

$$r(t)^{1/2} u(t) = T(g)(t)$$

and
$$r(t)^{1/2} (u, \varphi_n)_r \varphi_n(t) = (T(g), v_n) v_n(t).$$

The Hilbert–Schmidt Theorem tells us that the series $\sum_{n=1}^{\infty} (T(g), v_n) v_n(t)$ converges absolutely uniformly to $T(g)(t)$. Because $r(t)^{1/2}$ is bounded above and below by positive constants, it follows that the series $\sum_{n=1}^{\infty} (u, \varphi_n)_r \varphi_n(t)$ converges absolutely uniformly to $u(t)$. This proves result (iii), i.e., the completeness assertion (a) in the statement of Sturm's Theorem, and we are done for now except for the proof of the existence of the Green's function G_1. □

Lemma 1.4. Under the assumption that there is no nonzero solution of (SL) for $\lambda = 0$, there exists a continuous real-valued function $G_1(t, s)$ on $[a, b] \times [a, b]$ such that $G_1(t, s) = G_1(s, t)$, such that the operator T_1 given by

$$T_1 f(t) = \int_a^b G(t, s) f(s)\, ds$$

carries the space $C[a, b]$ of continuous functions f on $[a, b]$ one-one onto the space $\mathcal{D}[a, b]$ of functions u on $[a, b]$ satisfying (SL2) and having two continuous derivatives on $[a, b]$, and such that $L : \mathcal{D}[a, b] \to C[a, b]$ is a two-sided inverse function to T_1.

PROOF. Since $L(u) = pu'' + p'u' - qu$, a solution of $L(u) = 0$ has $u'' = -p^{-1}p'u' + p^{-1}qu$. Fix a point c in $[a, b]$. Let $\varphi_1(t)$ and $\varphi_2(t)$ be the unique solutions of $L(u) = 0$ on $[a, b]$ satisfying

$$\varphi_1(c) = 1 \text{ and } \varphi_1'(c) = 0, \qquad \varphi_2(c) = 0 \text{ and } \varphi_2'(c) = 1.$$

Since the complex conjugate of φ_1 or φ_2 satisfies the same conditions, we must have $\bar{\varphi}_1 = \varphi_1$ and $\bar{\varphi}_2 = \varphi_2$. Hence φ_1 and φ_2 are real-valued. The associated Wronskian matrix is

$$W(\varphi_1, \varphi_2)(t) = \begin{pmatrix} \varphi_1(t) & \varphi_2(t) \\ \varphi_1'(t) & \varphi_2'(t) \end{pmatrix},$$

and its determinant is

$$\det W(\varphi_1, \varphi_2)(t) = \varphi_1(t)\varphi_2'(t) - \varphi_1'(t)\varphi_2(t).$$

Then $\det W(\varphi_1, \varphi_2)(c) = 1$ and $\det W(\varphi_1, \varphi_2)(t)$ satisfies the first-order linear homogeneous differential equation

$$\begin{aligned}
(\det W(\varphi_1, \varphi_2))' &= \varphi_1\varphi_2'' - \varphi_1''\varphi_2 \\
&= \varphi_1(-p^{-1}p'\varphi_2' + p^{-1}q\varphi_2) - \varphi_2(-p^{-1}p'\varphi_1' + p^{-1}q\varphi_1) \\
&= -p^{-1}p'(\varphi_1\varphi_2' - \varphi_1'\varphi_2) \\
&= -p^{-1}p' \det W(\varphi_1, \varphi_2).
\end{aligned}$$

Therefore

$$\det W(\varphi_1,\varphi_2)(t) = \exp\big(-\textstyle\int_c^t p'(s)/p(s)\,ds\big) = \exp\big(-\log p(t) + \log p(c)\big)$$
$$= \exp(\log(p(c)/p(t))) = p(c)/p(t).$$

For f continuous, consider the solutions of the equation $L(u) = f$. A specific solution is given by variation of parameters, as stated in Theorem 4.9 of *Basic*. To use the formula in that theorem, we need L to have leading coefficient 1. For that purpose, we rewrite $L(u) = f$ as $u'' + p^{-1}p'u' - p^{-1}qu = p^{-1}f$. The theorem shows that one solution $u^*(t)$ is given by the first entry of

$$W(\varphi_1,\varphi_2)(t)\int_a^t W(\varphi_1,\varphi_2)(s)^{-1}\begin{pmatrix}0\\ p^{-1}(s)f(s)\end{pmatrix}ds.$$

Since $W(\varphi_1,\varphi_2)(s)^{-1} = (\det W(\varphi_1,\varphi_2)(s))^{-1}\left(\begin{smallmatrix}\varphi_2'(s) & -\varphi_2(s)\\ -\varphi_1'(s) & \varphi_1(s)\end{smallmatrix}\right)$, the result is

$$u^*(t) = \int_a^t \frac{-\varphi_1(t)\varphi_2(s)p^{-1}(s)f(s) + \varphi_2(t)\varphi_1(s)p^{-1}(s)f(s)}{p(c)/p(s)}\,ds$$
$$= p(c)^{-1}\int_a^t \big(-\varphi_1(t)\varphi_2(s) + \varphi_2(t)\varphi_1(s)\big)f(s)\,ds.$$

Define

$$G_0(t,s) = \begin{cases} p(c)^{-1}\big(-\varphi_1(t)\varphi_2(s) + \varphi_2(t)\varphi_1(s)\big) & \text{if } s \le t,\\ 0 & \text{if } s > t.\end{cases}$$

This function is continuous everywhere on $[a,b]\times[a,b]$, including where $s = t$, and it has been constructed so that

$$u^*(t) = \int_a^t G_0(t,s)f(s)\,ds = \int_a^b G_0(t,s)f(s)\,ds$$

is a solution of $u'' + p^{-1}p'u' - p^{-1}qu = p^{-1}f$, i.e., of $L(u) = f$. In particular, the form of the equation shows that u^* has two continuous derivatives on $[a,b]$. Therefore the operator

$$T_0(f)(t) = \int_a^b G_0(t,s)f(s)\,ds$$

carries $C[a,b]$ into the space of twice continuously differentiable functions on $[a,b]$.

The final step is to adjust G_0 and T_0 so that the operator produces twice continuously differentiable functions satisfying (SL2). Fix f continuous, and let $u^*(t) = \int_a^b G_0(t,s) f(s)\,ds$. By assumption the equation $L(u) = 0$ has no nonzero solution that satisfies (SL2). Thus the function $\varphi(t) = x_1\varphi_1(t) + x_2\varphi_2(t)$ does not have both

$$c_1\varphi(a) + c_2\varphi'(a) = 0 \quad \text{and} \quad d_1\varphi(b) + d_2\varphi'(b) = 0$$

unless x_1 and x_2 are both 0. In other words the homogeneous system of equations

$$\begin{pmatrix} c_1\varphi_1(a) + c_2\varphi_1'(a) & c_1\varphi_2(a) + c_2\varphi_2'(a) \\ d_1\varphi_1(b) + d_2\varphi_1'(b) & d_1\varphi_2(b) + d_2\varphi_2'(b) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has only the trivial solution. Consequently the system given by

$$\begin{aligned} &\begin{pmatrix} c_1\varphi_1(a) + c_2\varphi_1'(a) & c_1\varphi_2(a) + c_2\varphi_2'(a) \\ d_1\varphi_1(b) + d_2\varphi_1'(b) & d_1\varphi_2(b) + d_2\varphi_2'(b) \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \\ &\qquad\qquad = -\begin{pmatrix} c_1 u^*(a) + c_2 u^{*\prime}(a) \\ d_1 u^*(b) + d_2 u^{*\prime}(b) \end{pmatrix} \end{aligned} \tag{$*$}$$

has a unique solution $\binom{k_1}{k_2}$ for fixed f. We need to know how k_1 and k_2 depend on f. From the form of G_0, we have

$$u^*(t) = p(c)^{-1}\Big(-\varphi_1(t) \int_a^t \varphi_2(s) f(s)\,ds + \varphi_2(t) \int_a^t \varphi_1(s) f(s)\,ds \Big).$$

By inspection, two terms in the differentiation drop out and the derivative is

$$u^{*\prime}(t) = p(c)^{-1}\Big(-\varphi_1'(t) \int_a^t \varphi_2(s) f(s)\,ds + \varphi_2'(t) \int_a^t \varphi_1(s) f(s)\,ds \Big).$$

Evaluation of these formulas at a and b gives

$$\begin{aligned} u^*(a) &= u^{*\prime}(a) = 0, \\ u^*(b) &= p(c)^{-1}\big(-\varphi_1(b) \int_a^b \varphi_2(s) f(s)\,ds + \varphi_2(b) \int_a^b \varphi_1(s) f(s)\,ds \big), \\ u^{*\prime}(b) &= p(c)^{-1}\big(-\varphi_1'(b) \int_a^b \varphi_2(s) f(s)\,ds + \varphi_2'(b) \int_a^b \varphi_1(s) f(s)\,ds \big). \end{aligned}$$

Thus the right side of the equation $(*)$ that defines k_1 and k_2 is of the form

$$-\begin{pmatrix} c_1 u^*(a) + c_2 u^{*\prime}(a) \\ d_1 u^*(b) + d_2 u^{*\prime}(b) \end{pmatrix} = \begin{pmatrix} 0 \\ \int_a^b (e_1\varphi_1(s) + e_2\varphi_2(s)) f(s)\,ds \end{pmatrix},$$

where e_1 and e_2 are real constants independent of f. Hence k_1 and k_2 are of the form

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} \int_a^b (\alpha\varphi_1(s) + \beta\varphi_2(s)) f(s)\, ds \\ \int_a^b (\gamma\varphi_1(s) + \delta\varphi_2(s)) f(s)\, ds \end{pmatrix},$$

where $\alpha, \beta, \gamma, \delta$ are real constants independent of f. The fact that $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ solves the system $(*)$ means that the function $v(t)$ given by

$$u^*(t)+\varphi_1(t)\int_a^b (\alpha\varphi_1(s)+\beta\varphi_2(s)) f(s)\, ds+\varphi_2(t)\int_a^b (\gamma\varphi_1(s)+\delta\varphi_2(s)) f(s)\, ds$$

satisfies $c_1 v(a) + c_2 v'(a) = 0$ and $d_1 v(b) + d_2 v'(b) = 0$. Put

$$\begin{pmatrix} K_1(s) \\ K_2(s) \end{pmatrix} = \begin{pmatrix} \alpha\varphi_1(s) + \beta\varphi_2(s) \\ \gamma\varphi_1(s) + \delta\varphi_2(s) \end{pmatrix}.$$

We can summarize the above computation by saying that the real-valued continuous function

$$G_1(t, s) = G_0(t, s) + K_1(s)\varphi_1(t) + K_2(s)\varphi_2(t)$$

has, for every continuous f, the property that $v(t) = \int_a^b G_1(t, s) f(s)\, ds$ satisfies $L(v) = f$ and the condition (SL2).

Define $T_1(f)(t) = \int_a^b G_1(t, s) f(s)\, ds$. We have seen that T_1 carries $C[a, b]$ into $\mathcal{D}[a, b]$ and that $L(T_1(f)) = f$. Now suppose that u is in $\mathcal{D}[a, b]$. Since $L(u)$ is continuous, $T_1(L(u))$ is in $\mathcal{D}[a, b]$ and has $L(T_1(L(u))) = L(u)$. Therefore $T_1(L(u)) - u$ is in $\mathcal{D}[a, b]$ and has $L\big(T_1(L(u)) - u\big) = 0$. We have assumed that there is no nonzero solution of (SL) for $\lambda = 0$, and therefore $T_1(L(u)) = u$. Thus T_1 and L are two-sided inverses of one another.

Finally we are to prove that $G_1(t, s) = G_1(s, t)$. Let f and g be arbitrary real-valued continuous functions on $[a, b]$, and put $u = T_1(f)$ and $v = T_1(g)$. We know from Green's formula and (SL2) that $(L(u), v) = (u, L(v))$. Substituting the formulas $f = L(u)$ and $g = L(v)$ into this equality gives

$$\begin{aligned}\int_a^b \int_a^b G_1(t, s) f(t) g(s)\, ds\, dt &= \int_a^b f(t) v(t)\, dt = (L(u), v) \\ &= (u, L(v)) = \int_a^b u(s) g(s)\, ds = \int_a^b \int_a^b G_1(s, t) f(t) g(s)\, dt\, ds.\end{aligned}$$

By Fubini's Theorem the identity

$$\int_a^b \int_a^b (G_1(t, s) - G_1(s, t)) F(s, t)\, dt\, ds = 0$$

holds when F is one of the linear combinations of continuous functions $f(s)g(t)$. We can extend this conclusion to general continuous F by passing to the limit and using uniform convergence because the Stone–Weierstrass Theorem shows that real linear combinations of products $f(t)g(s)$ are uniformly dense in the space of continuous real-valued functions on $[a, b] \times [a, b]$. Taking $F(s, t) = G_1(t, s) - G_1(s, t)$, we see that $\int_a^b \int_a^b (G_1(t, s) - G_1(s, t))^2 \, dt \, ds = 0$. Therefore $G_1(t, s) - G_1(s, t) = 0$ and $G_1(t, s) = G_1(s, t)$. This completes the proof of the lemma. □

HISTORICAL REMARKS. Sturm's groundbreaking paper appeared in 1836. In that paper he proved that the set E in Theorem 1.3′ is infinite by comparing the zeros of solutions of various equations, but he did not address the question of completeness. Liouville introduced integral equations in 1837.

4. Problems

1. Let p_n be the n^{th}-smallest positive real number p such that $h \sin pl + p \cos pl = 0$, as in Example 3 for the heat equation in Section 2. Here h and l are positive constants. Prove directly that $\int_0^l \sin p_n x \sin p_m x \, dx = 0$ for $n \neq m$ by substituting from the trigonometric identity $\sin a \sin b = -\frac{1}{2}\big(\cos(a+b) - \cos(a-b)\big)$.

2. Multiplying the relevant differential operators by functions to make them formally self adjoint, and applying Green's formula from Section 3, prove the following orthogonality relations:
   (a) $\int_{-1}^1 P_n(t) P_m(t) \, dt = 0$ if P_n and P_m are Legendre polynomials and $n \neq m$. The m^{th} Legendre polynomial P_m is a certain nonzero polynomial solution of the Legendre equation $(1 - t^2)P'' - 2tP' + m(m+1)P = 0$. It is unique up to a scalar factor. These polynomials are applied in the second example with the Laplace equation in Section 2.
   (b) $\int_0^1 J_0(k_n r) J_0(k_m r) r \, dr = 0$ if k_n and k_m are distinct zeros of the Bessel function J_0. The function J_0 is the power series solution $J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(n!)^2}$ of the Bessel equation of order 0, namely $t^2 y'' + t y' + t^2 y = 0$. It is applied in the last example of Section 2.

3. In the proof of Lemma 1.4:
   (a) Show directly by expanding out $u^*(t) = \int_a^t G_0(t, s) f(s) \, ds$ that u^* satisfies $L(u^*) = f$.
   (b) Calculate $G_0(t, s)$ and $G_1(t, s)$ explicitly for the case that $L(u) = u'' + u$ when the conditions (SL2) are that $u(0) = 0$ and $u(\pi/2) = 0$.

4. This problem discusses the starting point for Sturm's original theory. Suppose that $p(t)$, $p'(t)$, $g_1(t)$, and $g_2(t)$ are real-valued and continuous on $[a, b]$ and that $p(t) > 0$ and $g_2(t) > g_1(t)$ everywhere on $[a, b]$. Let $y_1(t)$ and $y_2(t)$ be real-valued solutions of the respective equations

$$(p(t)y')' + g_1(t)y = 0 \qquad \text{and} \qquad (p(t)y')' + g_2(t)y = 0.$$

Follow the steps below to show that if t_1 and t_2 are consecutive zeros of $y_1(t)$, then $y_2(t)$ vanishes somewhere on (t_1, t_2).

(a) Arguing by contradiction and assuming that $y_2(t)$ is nonvanishing on (t_1, t_2), normalize matters so that $y_1(t) > 0$ and $y_2(t) > 0$ on (t_1, t_2). Multiply the first equation by y_2, the second equation by y_1, subtract, and integrate over $[t_1, t_2]$. Conclude from this computation that $\big[py_1'y_2 - py_1y_2'\big]_{t_1}^{t_2} > 0$.

(b) Taking the signs of p, y_1, y_2 and the behavior of the derivatives into account, prove that $p(t)y_1'(t)y_2(t) - p(t)y_1(t)y_2'(t)$ is ≤ 0 at $t = t_2$ and is ≥ 0 at t_1, in contradiction to the conclusion of (a). Conclude that $y_2(t)$ must have equaled 0 somewhere on (t_1, t_2).

(c) Suppose in addition that $q(t)$ and $r(t)$ are continuous on $[a, b]$ and that $r(t) > 0$ everywhere. Let $y_1(t)$ and $y_2(t)$ be real-valued solutions of the respective equations

$$(p(t)y')' - q(t)y + \lambda_1 r(t)y = 0 \quad \text{and} \quad (p(t)y')' - q(t)y + \lambda_2 r(t)y = 0,$$

where λ_1 and λ_2 are real with $\lambda_1 < \lambda_2$. Obtain as a corollary of (b) that $y_2(t)$ vanishes somewhere on the interval between two consecutive zeros of $y_1(t)$.

Problems 5–8 concern Schrödinger's equation in one space dimension with a time-independent potential $V(x)$. In suitable units the equation is

$$-\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t) = i\frac{\partial\Psi(x,t)}{\partial t}.$$

5. (a) Show that any solution of the form $\Psi(x,t) = \psi(x)\varphi(t)$ is such that $\psi'' + (E - V(x))\psi = 0$ for some constant E.
   (b) Compute what the function $\varphi(t)$ must be in (a).

6. Suppose that $V(x) = x^2$, so that $\psi'' + (E - x^2)\psi = 0$. Put $\psi(x) = e^{-x^2/2}H(x)$, and show that

$$H'' - 2xH' + (E-1)H = 0.$$

This ordinary differential equation is called **Hermite's equation**.

7. Solve the equation $H'' - 2xH' + 2nH = 0$ by power series. Show that there is a nonzero polynomial solution if and only if n is an integer ≥ 0, and in this case the polynomial is unique up to scalar multiplication and has degree n. For a suitable normalization the polynomial is denoted by $H_n(x)$ and is called a **Hermite polynomial**.

8. Guided by Problem 6, let L be the formally self-adjoint operator

$$L(\psi) = \psi'' - x^2\psi.$$

Using Green's formula from Section 3 for this L on the interval $[-N, N]$ and letting N tend to infinity, prove that

$$\lim_{N\to\infty} \int_{-N}^{N} H_n(x)H_m(x)e^{-x^2}\,dx = 0 \qquad \text{if } n \neq m.$$