# Hints for Solutions of Problems, 126-144

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***Stokes's Theorem and Whitney Manifolds:***
***A Sequel to***
**Basic Real Analysis**

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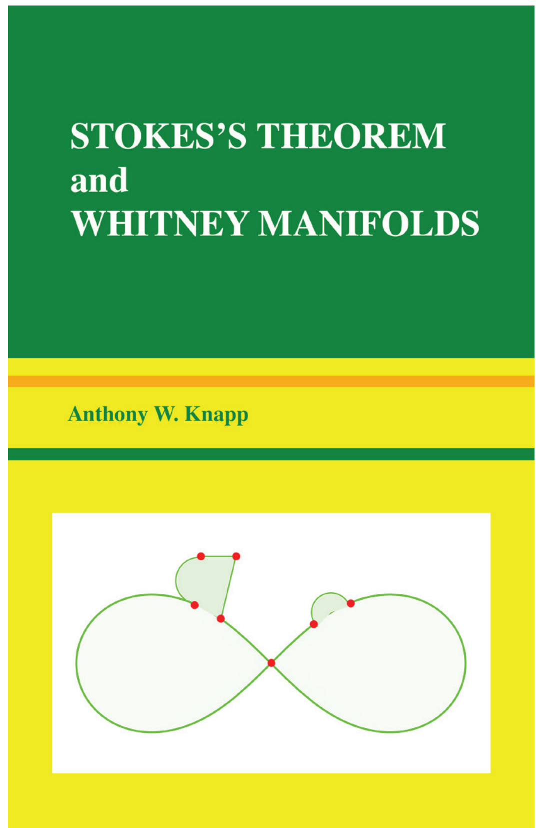

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Cover: An example of a Whitney domain in two-dimensional space. The green portion is a manifold-with-boundary for which Stokes's Theorem applies routinely. The red dots indicate exceptional points of the boundary where a Whitney condition applies that says Stokes's Theorem extends to the whole domain.

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# HINTS FOR SOLUTIONS OF PROBLEMS

## Chapter I

1. The interior of $K_{j+1}$ contains $K_j$ for all $j$, and the union of the $K_j$ equals $M$. The interiors of the sets $K_{j+1}$ therefore form an open cover of $C$. A finite subcover suffices by compactness of $C$, and a single $K_{j+1}$ suffices because the sets are nested.

2. The smooth manifolds will be the same if it is shown that their maximal atlases coincide, and this will happen if it is shown that the charts $C_1$ and $C_2$ are smoothly compatible with the atlas $\{M_1, M_2\}$ and that the charts $M_1$ and $M_2$ are smoothly compatible with the atlas $\{C_1, C_2\}$. One step in the verification is to check that $\varphi_1 \circ \psi_1^{-1}$ is smooth from $\psi_1(M_1 \cap C_1)$ to $\varphi_1(M_1 \cap C_1)$. The function $\varphi_1 \circ \psi_1^{-1}$ carries $t$ to $(\cos t, \sin t)$ and then to $(\cos t)/(1 - \sin t))$ for $-\pi < t < -\pi$ and $t \neq \pi/2$, and the result is a smooth function.

3. For (a), the triangle inequality needs to be checked. Thus we are to show that

$$\min\{|x-y|, |x+y|\} \leq \min\{|x-z|, |x+z|\} + \min\{|z-y|, |z+y|\}.$$

Since

$$|x-y| \leq |x-z| + |z-y| \quad \text{and} \quad |x+y| \leq |x-z| + |z+y|,$$

we have

$$\min\{|x-y|, |x+y|\} \leq |x-z| + \min\{|z-y|, |z+y|\}.$$

Replacing $z$ by $-z$ yields

$$\min\{|x-y|, |x+y|\} \leq |x+z| + \min\{|z-y|, |z+y|\}.$$

Then it follows that

$$\min\{|x-y|, |x+y|\} \leq \min\{|x-z|, |x+z|\} + \min\{|z-y|, |z+y|\},$$

as required. The continuity of $x \mapsto [x]$ is immediate from the inequality $d([x],[y]) \leq |x-y|$. If $x$ is given, then the image of the set of $y$ such that $|x-y| < \varepsilon$ is the set of $[y]$ with $d([x],[y]) < \varepsilon$, and thus open sets map to open sets.

For (b), the checking of the compatibility of the charts is similar to that in Section 1 for the sphere. The continuity of $x \mapsto [x]$ was proved in (a), and the smoothness is straightforward.

4. Let the manifold be $M$. Fix a point $p_0$ in $M$ and consider the set of all points $p$ in $M$ for which there is a diffeomorphism of $M$ carrying $p_0$ to $p$. This set is nonempty since it contains $p_0$, and we prove it is open and closed. Matters come down to considering an open neighborhood of a single point $p$, which may assume in local coordinates is a cube centered at the origin. It is then enough to produce a diffeomorphism of the open unit cube that is the identity near the boundary and carries the origin to any other point. We give the construction in $\mathbb{R}^1$, and then the general case follows by using a product of the functions of one variable. Thus we are to produce a smooth monotone function carrying $(-1, 1)$ onto itself, fixing all points near $-1$ and 1, and carrying 0 to some specified point $p_0$ in $(-1, 1)$. Subtracting the function $g(x) = x$, we see that it is enough to produce a smooth function $f$ of compact support in $(-1, 1)$ such that $-1 < f'(x) < 1$ everywhere and such that $f(0) = p_0$. The assumption about $p_0$ is that $p_0$ is in the interval $(-1, 1)$. Constructing such a function out of standard smooth functions of compact support is easy.

5. This is elementary.

6. These are special cases of the formula $d^2 = 0$ of Proposition 1.23b. See Example 2 in Section 4.

7. This problem was addressed in *Basic Real Analysis* in another guise. Let $\omega = \sum_j P_j\, dx_j$. The condition that $d\omega = 0$ is the condition that $\partial P_j/\partial x_i = \partial P_i/\partial x_j$ for all $i$ and $j$. In the language of Section III.12 of *Basic Real Analysis*, the function $F = (P_1, \dots, P_m)$ is a conservative vector field, and Proposition 3.48 of that book shows that $F$ is the gradient of a function $f$, proceeding by induction on the dimension. This $f$ is the required function.

8. Part (a) comes down to observing that $\frac{\partial}{\partial x}\big(x/(x^2 + y^2)\big) = -\frac{\partial}{\partial y}\big(y/(x^2 + y^2)\big)$ away from $(0, 0)$. Part (b) is a routine computation with several cases. The domain of $\theta$ is the complement in $\mathbb{R}^2$ of the nonnegative $x$-axis. For (c), it has been shown that $f$ and $\theta$ have matching first partial derivatives on the complement of the nonegative real axis. This set is connected, and therefore $f$ and $\theta$ differ by a constant there. Since this set is dense in $\mathbb{R}^2 - \{(0, 0)\}$, the existence of a smooth $f$ on $\mathbb{R}^2 - \{(0, 0)\}$ of this type would imply that $\theta$ has a continuous extension to $\mathbb{R}^2 - \{(0, 0)\}$. There is no continuous extension, and therefore no smooth solution $f$ to $df = \omega$ exists.

9. Choose disjoint open sets $A$ and $B$ such that $E \subseteq A$ and $F \subseteq B$. Next choose by Theorem 1.25 a smooth partition of unity $\{f, g\}$ subordinate to the open cover $\{A, B\}$ of $E \cup F$. Then $f$ and $g$ take values in $[0, 1]$, $f$ equals 0 off a compact subset of $A$, $g$ equals 0 off a compact subset of $B$, and $f + g = 1$ on $E \cup F$. Hence $f$ and $g$ have the required properties.

10. For (a), take $\eta = \varphi_1\, d\varphi_2 \wedge \cdots \wedge d\varphi_k$, for example. In (b), for each $j$ with $1 \le j \le k$, the function $f_j$ is a smooth function of one variable defined on the subset of $x_j \in \mathbb{R}^1$ such that $(x_1, \dots, x_m)$ is in $U$ for some value of the variables other than $x_j$. This subset is a union of open sets in $\mathbb{R}^1$ and is therefore open. For such an open set in $\mathbb{R}^1$, we define a function $F_j$ component by component so that $F_j' = f_j$ on each

component. Then the expansion $\omega = dF_1 \wedge \cdots \wedge dF_k$ exhibits $\omega$ as elementary.

11. We refer to Examples 2 and 3 in Section 3 and find that $\varphi^*(dx) = d\Phi_1 = d(r+s+t) = dr+ds+dt$ and $\varphi^*(dy) = d\Phi_2 = d(rs+st+rt) = r\,ds + s\,dr + s\,dt + t\,ds + r\,dt + t\,dr$. Thus $\varphi^*(dx \wedge dy)$ equals $\varphi^*(dx) \wedge \varphi^*(dy)$, which is

$$\begin{aligned}
&= (dr + ds + dt) \wedge (r\,ds + s\,dr + s\,dt + t\,ds + r\,dt + t\,dr) \\
&= (r+t-s-t)(dr \wedge ds) + (r+s-r-t)(ds \wedge dt) + (s+r-s-t)(dr \wedge dt) \\
&= (r-s)(dr \wedge ds) + (s-t)(ds \wedge dt) + (r-t)(dr \wedge dt).
\end{aligned}$$

12. This is straightforward.

13. For (a), the left side on $(X_2, \ldots, X_k)$ equals $k(\omega_1 \wedge \cdots \wedge \omega_k)(X, X_2, \ldots, X_k)$, which by Corollary 1.16 equals

$$\frac{k}{k!} \det \begin{pmatrix} \omega_1(X) & \omega_2(X) & \cdots & \omega_k(X) \\ \omega_1(X_2) & \omega_2(X_2) & \cdots & \omega_k(X_2) \\ \vdots & & & \\ \omega_1(X_k) & \omega_2(X_k) & \cdots & \omega_k(X_k) \end{pmatrix}.$$

When this determinant is expanded in cofactors about the first row and account is taken of the coefficient, the $i^{\text{th}}$ term of the expansion is exactly

$$(-1)^{i-1}\omega_i(X)(\omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_k)(X_2, \ldots, X_k).$$

The result follows.

For (b), we may assume without loss of generality that $\omega = \omega_1 \wedge \cdots \wedge \omega_k$ and that $\eta = \omega_{k+1} \wedge \cdots \wedge \omega_{k+l}$. Applying (a) to each yields

$$c_X(\omega) = \sum_{i=1}^{k} (-1)^{i-1}\omega_i(X)(\omega_1 \wedge \cdots \widehat{\omega_i} \wedge \cdots \wedge \omega_k)$$

and

$$\begin{aligned}
c_X(\eta) &= \sum_{j=1}^{l} (-1)^{j-1}\omega_{k+j}(X)(\omega_{k+1} \wedge \cdots \widehat{\omega_{k+j}} \wedge \cdots \wedge \omega_{k+l}) \\
&= \sum_{m=k+1}^{k+l} (-1)^{m-k-1}\omega_m(X)(\omega_{k+1} \wedge \cdots \widehat{\omega_m} \wedge \cdots \wedge \omega_{k+l}).
\end{aligned}$$

Therefore

$$\begin{aligned}
c_X(\omega) \wedge \eta + (-1)^k(\omega \wedge c_X(\eta)) &= \sum_{i=1}^{k+l} (-1)^{i-1}\omega_i(X)(\omega_1 \wedge \cdots \widehat{\omega_i} \wedge \cdots \wedge \omega_{k+l}) \\
&= c_X(\omega \wedge \eta).
\end{aligned}$$

14. The expanded formula for $i^*(\omega)$ is

$$i^*(\omega)_p((X_1)_p, \dots, (X_k)_p) = \omega_{i(p)}\big((Di)_p X_1)_{i(p)}, \dots, (Di)_p X_k)_{i(p)}\big),$$

where $(X_1)_p, \dots, (X_k)_p$ are in $T_p(S)$, $(Di)_p$ is the derivative of $i$ at $p$, $\omega$ is an alternating $k$ multilinear form on $M$, and $i^*(\omega)$ is the pullback alternating $k$ multilinear form on $S$. The derivative $(Di)_p$ may be regarded as an inclusion of $T_p(S)$ into $T_{i(p)}(M)$, and the arguments of $\omega_{i(p)}$ within $T_{i(p)}(M)$ are obtained by taking the arguments of $i^*(\omega)_p$ and regarding them as included in $T_{i(p)}(M)$. Inclusions and restrictions are the same thing from a different point of view.

15. We go back to the definition of "orientable" near the beginning of Section 6. Let the two charts be $(M_1, \varphi_1)$ and $M_2, \varphi_2)$. The condition of orientability is that $\det(\varphi_2 \circ \varphi_1^{-1})$ and $\det(\varphi_1 \circ \varphi_2^{-1})$ are both positive. The second determinant is the reciprocal of the first. If they are positive, we are done. If they are negative, then we redefine $\varphi_1$ by following $\varphi_1$ with the map $(x_1, x_2, \dots, x_m) \mapsto (-x_1, x_2, \dots, x_m)$; use of the composition changes the determinants from negative to positive.

16. For (a), a point in $S^n$ may be identified with a vector in $\mathbb{R}^{n+1}$. As a vector, $p = (x_1, \dots, x_{n+1})$ is orthogonal to the tangent space at the point of tangency $p$ on the sphere. Thus the tangent space consists of all $p + x \in \mathbb{R}^{n+1}$ with $x \cdot p = 0$. Viewed as through the origin, the tangent space is simply $\{x \in \mathbb{R}^{n+1} \mid x \cdot p = 0\}$, i.e., the orthogonal complement $(\mathbb{R}p)^\perp$ of the 1 dimensional space $\mathbb{R}p$. Any subspace of a finite dimensional inner product vector space is the direct sum of itself and its orthogonal complement. With these identifications, $\mathbb{R}^{n+1} = (\mathbb{R}p)^\perp \oplus T_p(S^n)$.

In (b), the derivation property of $f \mapsto \frac{d}{dt} f(\gamma_r(t))\big|_{t=0}$ is immediate from the one-variable rule for differentiating products. Write $\gamma_r(t)$ in coordinates as $\gamma_r(t) = \big((x_1(t), \dots, x_{n+1}(t)\big)$, and expand the derivative in question as

$$\frac{d}{dt}(\gamma_r(t))\Big|_{t=0} = \frac{\partial f}{\partial x_1}(p)\,\frac{dx_1}{dt}(0) + \cdots + \frac{\partial f}{\partial x_{n+1}}(p)\,\frac{dx_{n+1}}{dt}(0).$$

To compute this, we write

$$\gamma_r(t) = \frac{p + tr}{|p + tr|} = \frac{p + tr}{\sqrt{(p + tr) \cdot (p + tr)}}.$$

Since $p \cdot p = 1$ and $p \cdot r = 0$, $\gamma_r(t)$ simplifies to $\frac{p+tr}{\sqrt{1+t^2|r|^2}}$, whose derivative at $t = 0$ is $r$ since there are no first-order terms in $t$ in the denominator. The result follows.

17. For $(X_1)_p, \dots, (X_n)_p$ in $T_p(S^n)$, we have

$$i^*(c_X(\omega))_p((X_1)_p, \dots, (X_n)_p) = \omega_{i(p)}(X, i(X_1)_p, \dots, i(X_n)_p),$$

where $i(X_j)_p$ means the effect of the derivative $Di$ on $(X_j)_p$, namely $(Di)_p(X_j)_p$. Take $\{(X_1)_p, \dots, (X_n)_p\}$ at $p$ to be a basis of $T_p(S^n)$. Then $\{i(X_1)_p, \dots, i(X_n)_p\}$ is a basis of the vector space $iT_p(S^n)$, which we know from Problem 16 to equal $(\mathbb{R}p)^\perp$. Since $X_p = p$, $\{X_p, i(X_1)_p, \dots, i(X_n)_p\}$ at $p$ is a basis of $iT_p(S^n) \oplus \mathbb{R}p = \mathbb{R}^{n+1}$. Since the given $\omega$ is nonzero at $p$, its value at $p$ does not vanish on any basis of $\mathbb{R}^{n+1}$. Therefore $i^*(c_X(\omega))_p((X_1)_p, \dots, (X_n)_p) \neq 0$.

18. Corollary 1.16 yields

$$\begin{aligned} c_X(\omega) &= (n+1)(dx_1 \wedge \cdots \wedge dx_{n+1})(X, X_1, \dots, X_n) \\ &= \frac{n+1}{(n+1)!} \det \begin{pmatrix} dx_1(X) & dx_1(X_1) & \cdots & dx_1(X_n) \\ dx_2(X) & dx_2(X_1) & \cdots & dx_2(X_n) \\ \cdots & & & \\ dx_{n+1}(X) & dx_{n+1}(X_1) & \cdots & dx_{n+1}(X_n) \end{pmatrix}. \end{aligned}$$

We can evaluate the entries in the first column as follows. For the $i^{\text{th}}$ entry we have $dx_i(X) = \sum_k x_k\, dx_i\big(\frac{\partial}{\partial x_k}\big) = x_i$. Then we expand the whole determinant by cofactors about the first column. With the coefficient $(n!)^{-1}$ in place, the expansion gives a sum over $i$ of an alternating sign $(-1)^{i-1}$ times the coefficient $x_i$, times the complementary determinant, which is

$$(dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{n+1})(X_1, \dots, X_n).$$

Thus $c_X(\omega) = \sum_i (-1)^{i-1}\,(dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{n+1})$ as required.

19. In (a), symmetry of $\sim$ follows from the fact that $h^2 = 1$. For the transitive property, we observe that if $y = h(x)$ and $z = h(y)$, then $z = h^2(x) = x$ and hence $z \sim x$. In (b), the argument is similar to that for Problem 3, which deals with a special case. In (c) to define a chart about $x$ in $M$, use the open ball about $x$ of each radius less than half the distance from $x$ to $h(x)$,

20. With the proof of Proposition 1.33 as a guide, this is easy.

21. With the proof of Proposition 1.33 as a guide, this is easy.

22. For (b), a nowhere vanishing $n$ form for $S^n$ can be taken to be a restriction of

$$\sum_{j=1}^{n+1} x_j\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{n+1}.$$

The anitpodal map has the effect of sending each $x_i$ into its negative and each $dx_j$ into its negative. Thus it has the effect of introducing $n+1$ minus signs in each term, thus of multiplying the whole expression by $(-1)^{n+1}$. Consequently the $n$ form is preserved by the antipodal map if $n$ is odd and is reversed if $n$ is even. The $n$ form gives the orientation, up to an everywhere positive factor, and so the orientation is preserved if $n$ is odd and is reversed if $n$ is even.

23. This is immediate.

24. In (a), the mapping $\alpha$ and its inverse are continuous because $f$ is continuous. For smoothness of $\sigma$ and its inverse, we are to compose before and after with the chart mappings, and we end up with the identity, which is smooth. In (b), the mappings $\alpha$ and $p$ are smooth, and so is the composition $\alpha \circ I \circ p$; thus the inclusion map $I$ must not be smooth.

25. In (a), $\varphi$ is smooth, and its inverse is $\varphi^{-1}(u, v) = (u, v + f(u))$, which is smooth. Then (b) is an observation.

26. The derivative is $(2\cos 2t, -\sin t)$. For this to be $(0, 0)$, $\sin t$ must be 0, which means that $t$ is a multiple of $\pi$. Then $2t$ is a multiple of $2\pi$, and $\cos 2t = 1$. Thus both entries cannot be 0 for the same $t$, and $\gamma$ is an immersion. It is easy to check that $\gamma$ is one-one over an interval of length $2\pi$. Finally its image is compact, being closed and bounded. Specifically it contains all its limit points, since the only point that needs checking is $(0, 0)$, which is $\gamma(\pi/2) = (0, 0)$ and is therefore already in the image. The topology of the domain of $\gamma$ is that of an open interval, which is not compact, and the topology of the image is compact. Thus the two topologies do not coincide, and the immersion is not an embedding.

27. In (a), $\gamma'(t) = (2\pi i e^{2\pi i t}, 2\pi i c e^{2\pi i c t})$, and neither coordinate is ever 0. So $\gamma'(t)$ nowhere vanishing, and $\gamma$ is an immersion. If $\gamma(t_1) = \gamma(t_2)$, then $e^{2\pi i t_1} = e^{2\pi i t_2}$ and $e^{2\pi i c t_1} = e^{2\pi i c t_2}$. Hence $t_1 - t_2$ is an integer, and so is $c(t_1 - t_2)$. Since $c$ is irrational, this is possible only if $t_1 - t_2 = 0$. Hence $\gamma$ is one-one.

For (b), it follows from (a) that $\{\gamma(k) \mid k \in \mathbb{Z}\}$ is an infinite set. Thus it has a limit point in $\mathbb{C}$, say $z$. Choose a sequence $\{k_n\}$ such that $\lim_n \gamma(k_n) = z$. Given $\varepsilon > 0$, choose two distinct integers $r$ and $s$ in the sequence such that $|\gamma(r) - \gamma(s)| < \varepsilon$. Then $k = r - s$ is a nonzero integer with $|\gamma(k) - \gamma(0)| < \varepsilon$.

For (c), repeating this construction for a sequence of values of $\varepsilon$ tending to 0 shows that there is a sequence of points in $\gamma(\mathbb{Z})$ tending to 1 but not equal to 1. Hence $\gamma(Z)$ does not have the discrete topology, and $\gamma$ is not an embedding.

28. In (a), since the function $x(t)$ is smooth near $t_0$ and its derivative is nonzero there, the one-variable Inverse Function Theorem says that near the point $t_0$, $x(t)$ can in principle be inverted to give a unique smooth inverse function $t = t(x)$. This result can be substituted into the expression $y(t)$ to yield $y(t) = y(t(x))$ as a function of $x$ near $x(t_0)$. More specifically put $x(t_0) = x_0$. Then the set of points $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ in a suitably small rectangle in $\mathbb{R}^2$ about $\begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix}$ is the embedded graph of the smooth function $g(f(t))$.

In (b), fix $x_0$, and suppose that $n$ of the columns of $J(x_0)$ are linearly independent. Possibly by permuting the variables, we may assume that the first $n$ columns are linearly independent. Write $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$, so that the $n$-by-$n$ square matrix $\left\{\left(\frac{\partial (F_1)_i}{\partial x_j}\right)\right\}$ is invertible at $x = x_0$. By the Inverse Function Theorem, we can in principle solve uniquely in a neighborhood of $(x_0, F(x_0))$ to write $x$ as a smooth function $x = x(F_1)$ there. Then the set of points in a suitably small rectangular neighborhood of $\begin{pmatrix} x_0 \\ F(x_0) \end{pmatrix}$

in $\mathbb{R}^{n+k}$ is the embedded graph of the smooth function $F_2(x(F_1))$.

29. For (a), the Jacobian matrix of $(x, y, z)$ with respect to $(s, t)$ is

$$\begin{pmatrix} -\sin s - \frac{1}{2}t\cos(s/2)\sin s - \frac{1}{4}t\sin(s/2)\cos s & \frac{1}{2}\cos(s/2)\cos s \\ \cos s + \frac{1}{2}t\cos(s/2)\cos s - \frac{1}{4}t\sin(s/2)\sin s & \frac{1}{2}\cos(s/2)\sin s \\ \frac{1}{4}t\cos(s/2) & \frac{1}{2}\sin(s/2) \end{pmatrix}.$$

The 2-by-2 determinant from the first two rows is

$$\begin{aligned} &= -\tfrac{1}{2}\cos(s/2)\sin^2 s - \tfrac{1}{4}t\cos^2(s/2)\sin^2 s - \tfrac{1}{8}t\sin(s/2)\cos(s/2)\sin s\cos s \\ &\quad - \tfrac{1}{2}\cos(s/2)\sin^2 s - \tfrac{1}{4}t\cos^2(s/2)\cos^2 s + \tfrac{1}{8}t\sin(s/2) \\ &= -\tfrac{1}{2}\cos(s/2) - \tfrac{1}{4}t\cos^2(s/2) \\ &= -\tfrac{1}{2}\cos(s/2)\big(1 - \tfrac{1}{2}t\cos(s/2)\big), \end{aligned}$$

and this has the same sign as $-\frac{1}{2}\cos(s/2)$. When $\cos(s/2) = 0$, the Jacobian matrix simplifies to

$$\begin{pmatrix} -\sin s - \frac{1}{4}t\sin(s/2)\cos s & 0 \\ \cos s - \frac{1}{4}t\sin(s/2)\sin s & 0 \\ 0 & \frac{1}{2}\sin(s/2) \end{pmatrix}.$$

When $\cos(s/2) = 0$, we see that $\sin(s/2)$ is $\pm 1$, $\sin s$ is 0, and $\cos s$ is $\pm 1$. Thus the determinant from the first and third rows equals $(\pm\frac{1}{2})(\pm\frac{1}{4}t)$, which is nonzero unless $t = 0$. When $\cos(s/2) = 0$ and $t = 0$, then the determinant from the second and third rows equals $(\pm\frac{1}{2})\cos s$, which is not zero. Thus the Jacobian matrix has rank two for every pair $(s, t)$ under consideration.

Part (b) is clear. For (c), the image of the smooth function is locally a smooth function, by the Inverse Function Theorem. Since the function is only two-to-one, it is locally invertible. Hence the image is a smooth manifold.

30. In (a), the function $F(x, y) = x^2 + y^2 - 1$ is smooth near the point $(x_0, y_0)$, which has $F(x_0, y_0) = 0$, and the assumption is that $\frac{\partial F}{\partial x}(x_0, y_0) \neq 0$. That is, the 1-by-1 matrix with entry $\frac{\partial F}{\partial x}(x_0, y_0) \neq 0$ is invertible. The theorem says that in a suitable rectangular neighborhood $I \times J$ of $(x_0, y_0)$ with $I \subseteq \mathbb{R}^1$ and $J \subseteq \mathbb{R}^1$, each $y$ value yields a unique $x$ value with $F(x, y) = 0$ and the resulting function $x = f(y)$ for $x \in I$ is smooth and satisfies $F(f(y), y) = 0$ for all $y$ in $J$. Then the open subset $I \times J$ of $\mathbb{R}^2$ contains the embedded graph of a smooth function, as in Problem 25.

In (b), the same procedure is to be applied to the function $F(x_1, \dots, x_{n+1}) - 1$ and the point $((x_1)_0), \dots, (x_{n+1})_0)$ on $S^n$ under the assumption that

$$\Big(\frac{\partial F}{\partial x_1}\Big)((x_1)_0, \dots, (x_{n+1})_0),$$

namely $2(x_1)_0$, is nonzero. The Implicit Function Theorem yields a rectangular open neighborhood $I \times J$ of $((x_1)_0, \dots, (x_{n+1})_0)$ with $I \subseteq \mathbb{R}^1$ and $J \subseteq \mathbb{R}^n$ such that each value of $(x_2, \dots, x_{n+1})$ in $J$ yields a unique $x_1$ in $I$ with $F(x_1, \dots, x_{n+1}) = 0$ and the resulting function $x_1 = f(x_2, \dots, x_{n+1})$ is smooth and satisfies

$$F(f(x_2, \dots, x_{n+1}), x_2, \dots, x_{n+1}) = 0 \quad \text{for all } (x_2, \dots, x_{n+1}) \in J.$$

Then the open subset $I \times J$ of $\mathbb{R}^{n+1}$ contains the embedded graph of a smooth function, as in Problem 25.

In (c), fix $x_0$, and suppose that $k$ of the columns of $J(x_0)$ are linearly independent. Possibly by permuting the variables, we may assume that the first $k$ of the columns are linearly independent. Regard $F$ as a function of $n$ variables whose entries $(F_1, \dots, F_k)$ are members of $\mathbb{R}^k$. The assumption is that the matrix $\left\{\frac{\partial F_i}{\partial x_j}(x_0)\right\}$ is nonsingular. The Implicit Function Theorem yields a rectangular set $I \times J \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}$ centered at $((x_1)_0, \dots (x_n)_0)$ and a smooth function $f(x_{k+1}, \dots, x_n)$ defined in $J$ such that for each $(x_{k+1}, \dots, x_n)$ in $J$, there is a unique $(x_1, \dots, x_k)$ in $I$ with $F(x_1, \dots, x_n) = 0$ and the resulting function $(x_1, \dots, x_k) = f(x_{k+1}, \dots, x_n)$ is smooth and satisfies $F(f(x_{k+1}, \dots, x_n), x_{k+1}, \dots, x_n) = 0$ for all $(x_2, \dots, x_{n+1})$ in $J$. Then the open subset $I \times J$ of $\mathbb{R}^n$ contains the embedded graph of a smooth function, as in Problem 25.

## Chapter II

1. Straightforward calculation.

2. Two ways of proving this result that generalize to all dimensions are to make use of Corollary 1.16 of the present text and to proceed via row reduction of matrices as outlined in Section III.10 of *Basic Real Analysis.*

For dimension 3 an argument is available that makes use of cross product, as follows: We compute the volume of the parallelepiped spanned by $u$, $v$, and $w$ as the area of the base spanned by $u$ and $v$, times the height. The area of the base we know to be $|u \times v| = |u||v||\sin\theta|$. The height is the magnitude of the projection of $w$ in the direction perpendicular to the base, i.e., in the direction of $u \times v$. Thus the height is $\left|\dfrac{w \cdot (u \times v)}{|u \times v|^2}\,(u \times v)\right| = \dfrac{|w \cdot (u \times v)|}{|u \times v|}$. Then the product of the base and height is $|w \cdot (u \times v)|$, which is the determinant in question.

3. The first, fourth, and fifth are equal. The second, third, and sixth are the negative of these.

4. For (a), div $F = 2xy$ and curl $F = (8y - 3z^2)\mathbf{i} - (x^2 + 3x)\mathbf{k}$.
For (b), div $F = 2 + 2x^3y$ and curl $F = (4z - 7)\mathbf{j} + (3x^2y^2)\mathbf{k}$.

5. Without loss of generality we may assume that $M$ is connected. If a smooth $m-1$ form $\eta$ exists with $d\eta = \omega$, then Stokes's Theorem says that $\int_{\partial M} \eta = \int_M d\eta = \int_M \omega$. In a connected compatible chart $\alpha = (x_1, \dots, x_m)$, $\alpha^*(\omega)$ can be written as $F_\alpha\, dx_1 \wedge \dots \wedge dx_m$ for some nowhere-vanishing smooth function $F_\alpha$. Then $F_\alpha$ does not change sign, and $\int_M \omega$ is not zero. Consequently $\int_{\partial M} \eta \neq 0$. But this contradicts Theorem 2.1 since $\partial M$ is a smooth manifold without boundary.

6. For (a), we can take $\omega = (dx_1 \wedge dx_2) + (dx_3 \wedge dx_4)$.

For (b), we have $\omega = d\eta$ with $\eta = \alpha \wedge d\alpha \wedge \cdots \wedge d\alpha$ since $d^2 = 0$. Then the previous problem shows that $\omega$ has to vanish somewhere.

7. In (a), the value of $d\omega$ is the sum of $(\partial/\partial x)\big(x(x^2+t^2+z^2)^{-3/2}\big)\,(dx \wedge dy \wedge dz)$ and two similar terms. The coefficient of $dx \wedge dy \wedge dz$ is

$$\begin{aligned}(x^2+y^2+z^2)^{-3/2} &+ x(-3/2)(x^2+y^2+z^2)^{-5/2}(2x)\\ &= (x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2-3x^2)(x^2+y^2+z^2)^{-1}.\end{aligned}$$

The contributions from the other two terms are similar except that $x$ is to be replaced by $y$ and then $z$. The sum of the three terms is then

$$(x^2+y^2+z^2)^{-5/2}(3(x^2+y^2+z^2)-3x^2-3y^2-3z^2) = 0.$$

In (b), let $M$ be the "inside" of $T$. We can apply Stokes's Theorem (Theorem 4.7) to $T$ since $\omega$ is smooth everywhere inside and on $T$. Then we have $\int_T \omega = \int_M d\omega = 0$.

8. Part (a) is a restatement of Problem 7a.

In (b), the Divergence Theorem gives $\int_S F \cdot d\mathbf{S} = 0$ since $\operatorname{div} F = 0$. The orientation on $S$ is given by an outward normal from $M$, which is then outward on $S_1$ and in toward the origin on $S_a$. Hence $0 = \int_S F \cdot d\mathbf{S} = \int_{S_1} F \cdot d\mathbf{S} - \int_{S_a} F \cdot d\mathbf{S}$.

9. Take $\omega = \frac{1}{n}\sum_{j=1}^{n}(-1)^{j-1}dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$ and $f(x_1,\dots,x_n) = (x_1^2+\cdots+x_n^2)^{n/2}$.

10. In Section 3 the paragraph beginning "The traditional procedure" is irrelevant and can be omitted. In the statement of Proposition 2.6, $(-1)^m\alpha^*(dx_1 \wedge \cdots \wedge dx_{m-1})$ is to be replaced by $-\alpha^*(dx_1 \wedge \cdots \wedge dx_m)$. (Note the sign!) The proof of Proposition 2.6 is unchanged down to the paragraph beginning "Thus we have constructed." For the case $m = 1$, we still have $F_{\alpha_p}$ as it is, positive or negative. The orientation at $p$ is still the sign of $F_{\alpha_p}(0)$.

In Section 4, formulas $(*)$ and $(**)$ are unchanged. In the paragraph beginning "On $\partial\mathbb{H}^m$," some changes are needed. We have

$$\omega = F_1(0, x_2, \dots, x_m)\, dx_2 \wedge \cdots \wedge dx_m.$$

For the case $m \geq 2$ the proof becomes, "Since $-dx_2 \wedge \cdots \wedge dx_m$ is positively oriented in the orientation of the boundary that we are using, application of Theorem 1.29 gives

$$\begin{aligned}\int_{\partial\mathbb{H}^m} \omega &= -\int_{\partial\mathbb{H}^m} F_1(0, x_2, \dots, x_m)\, dx_m \cdots dx_2\\ &= -\int_{a_2}^{b_2} \cdots \int_{a_m}^{b_m} F_1(0, x_2, \dots, x_m)\, dx_m \cdots dx_2. \qquad (\dagger)\end{aligned}$$

For $m = 1$, we get $\int_{\partial\mathbb{H}^m} \omega = -F_1(0)$. So $(\dagger)$ holds for all $m \geq 1$.

Formula (††) is still valid, and we still do the integration in the variable $x_r$ first. For $r \geq 1$, we get 0 from the inside integral. For $r = 1$, the inside integral is

$$\int_0^c \left(\frac{\partial F_1}{\partial x_1}\right) dx_1 = F_1(c, x_2, \dots, x_m) - F_1(0, x_2, \dots, x_m)$$

with $F(c, x_2, \dots, x_m) = 0$ by the support condition. Therefore (††) boils down to

$$-\int_{a_2}^{b_2} \cdots \int_{a_m}^{b_m} F_1(0, x_2, \dots, x_m)\, dx_m \cdots dx_2,$$

which equals (†). Thus we get (‡), and the remainder of the proof is unchanged.

11. We can parametrize the surface by using $s$ and $t$ as parameters, with $s$ standing for $x$ and $t$ standing for $y$. Then the parametrization is $(s,t) \mapsto \begin{pmatrix} s \\ t \\ s^2+t^2 \end{pmatrix}$ with derivative $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2s & 2t \end{pmatrix}$. Then we have $\dfrac{\partial(x,y)}{\partial(s,t)} = 1$, $\dfrac{\partial(y,z)}{\partial(s,t)} = -2s$, and $\dfrac{\partial(z,x)}{\partial(s,t)} = -2t$. The integrand $F = x\mathbf{i} \cdot d\mathbf{S}$ is $x\,dy \wedge dz = x(-2s)\,ds\,dt = -2s^2\,ds\,dt$. There is no natural orientation on the surface, but we are told to orient the surface by using an outward/downward vector. That is, we are to consider the basis of the tangent space at a point of the surface, include an outward/downward vector before it (a vector with third component negative), and see whether our parametrization is consistent with this basis of $\mathbb{R}^3$. To fix the ideas, take $(s,t) = (0,0)$. Then the basis we choose of $\mathbb{R}^3$ can be $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. The matrix formed from these basis vectors has determinant $-1$, and our parametrization is the opposite of what we need. Let us therefore start over, using $(s,t) \mapsto \begin{pmatrix} t \\ s \\ s^2+t^2 \end{pmatrix}$ with derivative $\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2s & 2t \end{pmatrix}$ as parametrization. Then $\dfrac{\partial(y,z)}{\partial(s,t)} = 2t$. With this parametrization the integrand becomes $F = x\mathbf{i} \cdot d\mathbf{S} = x\,dy \wedge dz = x(2t)\,ds\,dt = 2t^2\,ds\,dt$. The integration extends over the set where $s^2 + t^2 \leq 4$. Switching to polar coordinates in the $s$-$t$ plane shows that the integral is $\int_0^2 \int_0^{2\pi} 2r^2(\sin^2\theta)\,r\,dr\,d\theta = \pi \int_0^2 2r^3\,dr = \boxed{8\pi}$.

As it should, this orientation gives minus the answer we would get with the opposite orientation. Had we not taken the orientation into account properly, we would have integrated $-2s^2\,ds\,dt$ over the set where $s^2 + t^2 \leq 4$ and gotten $-8\pi$ as the answer.

12. The boundary curve of $S$ is given by the subset of points $(x,y,z)$ that satisfy both conditions, namely $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$, and have $z \geq 0$. Substitution gives $z^2 = 3$. Thus the intersection is the circle with $z = \sqrt{3}$ and $x^2 + y^2 = 1$. Stokes's Theorem says that the integral is equal to $\int_C F \cdot d\mathbf{s}$, but we have to orient $C$ properly. Since the orientation of $S$ is upward, this situation is like looking at the ordinary unit circle in the $x$-$y$ plane. The circle is therefore to be traversed with $S$ on the left, and the parametrization can be taken as $t \mapsto (\sqrt{3}\cos t, \sqrt{3}\sin t, \sqrt{3})$.

The derivative is $(-\sqrt{3}\,\sin t, \sqrt{3}\,\cos t, 0)$. On the circle the value of $F$ in terms of the parameter $t$ is $(yz, 0, xy) = (3\sin t, 0, \cos t \sin t)$ Thus the integral is

$$= \int_0^{2\pi} (3\sin t, 0, 3\cos t \sin t) \cdot (-\sqrt{3}\,\sin t, \sqrt{3}\,\cos t, 0)\,dt$$
$$= \int_0^{2\pi} -3\sqrt{3}\sin^2 t\,dt = \boxed{-3\pi\sqrt{3}}$$

13. A direct attack on the line integral leads to an unpleasant term $e^{3\sin t}$ because of the presence of $e^z$ of $F$. In preparation for using the Kelvin–Stokes Theorem, direct computation gives $\operatorname{curl} F = (x, -2y, y)$ with the $e^z$ gone. By the Kelvin–Stokes Theorem the integral equals $\int_S (\operatorname{curl} F)\cdot d\mathbf{S}$ when $S$ is any oriented smooth surface with boundary curve $C$, provided the orientations match properly. An example of such a surface is the disk given by $x^2 + y^2 \le 9$ and $y = 4$ with a suitable orientation. By the same token the surface integral equals the line integral $\int_C G \cdot d\mathbf{s}$, where $G = (-yz, 0, xy)$, since $\operatorname{curl} G = \operatorname{curl} F$. (In changing $F$ into $G$, we can drop pure $x$ terms from the first entry, pure $y$ terms from the second entry, and pure $z$ terms from the third entry without changing the curl.) Since $\mathbf{s}'(t) = (-3\sin t, 0, 3\cos t)$, the given line integral is

$$= \int_0^{2\pi} \big[(-4(3\sin t))(-3\sin t) + 4(3\cos t)(3\cos t)\big]\,dt$$
$$= \int_0^{2\pi} (36\sin^2 t + 36\cos^2 t)\,dt = \int_0^{2\pi} 36\,dt = \boxed{72\pi}$$

14. The boundary is the circle $C$ in the plane $z = 0$ with $x^2 + y^2 = 16$. Since $S$ is oriented upward, the induced orientation on $C$ is clockwise (with the hemisphere on the left). Thus $C$ can be parametrized as $t \mapsto (4\cos t, 4\sin t, 0)$ with derivative $-4\sin t, 4\cos t, 0)$. The given integral is therefore

$$= \int f \cdot d\mathbf{s} = \int_0^{2\pi} \big(y(-4\sin t) - x(4\cos t) + 0\big)\,dt = \int_0^{2\pi} (-16\sin^2 t - 16\cos^2 t)\,dt,$$

which equals $\boxed{-32\pi}$

15. In (a), the circle can be parametrized as $\theta \mapsto \begin{pmatrix} \cos\theta \\ \sin\theta \\ 2-\sin\theta \end{pmatrix}$ for $0 \le \theta \le 2\pi$. We are given $F(x, y, z) = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$, and we have

$$d\mathbf{s} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} + (-\cos\theta)\,\mathbf{k}.$$

Then

$$\int_C F \cdot d\mathbf{s} = \int_0^{2\pi} (-\sin^2\theta)(-\sin\theta) + (\cos\theta)(\cos\theta) + (2 - \sin\theta)^2(-\cos\theta)\,d\theta = \pi.$$

In (b), the filled ellipse is to be oriented upward. We can parametrize it as $(r, \theta) \mapsto \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 2-r\sin\theta \end{pmatrix}$ with derivative $\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ -\sin\theta & -r\cos\theta \end{pmatrix}$. Then

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r, \qquad \frac{\partial(x, z)}{\partial(r, \theta)} = r(\sin^2\theta - \cos^2\theta), \qquad \frac{\partial(y, z)}{\partial(r, \theta)} = 0.$$

Direct calculation gives curl $F = (1+2y)\mathbf{k}$, and then curl $F \cdot d\mathbf{S} = (1+2y)\frac{\partial(x,y)}{\partial(r,\theta)} = r(1+2r\sin\theta) = r + 2r^2\sin\theta$. Thus the integral is $\int_0^1\int_0^{2\pi}(r+2r^2\sin\theta)\,d\theta\,dr = 2\pi\int_0^1 r\,dr = \pi$.

16. The leftmost inequality sign follows from the fact that $\omega$ is nowhere vanishing, the argument being like the one for Problem 5. The first of the three equalities follows because the fact that $r$ is a retraction shows up on the level of pullbacks as meaning that $r^*$ is the identity on forms located where $r$ is the identity, i.e., on $\partial B$. The second equality is by Stokes's Theorem, Theorem 4.7. The third equality is by Proposition 1.24, which says that exterior derivative commutes with pullback.

17. In the previous problem, there is some virtue in making explicit the role of the inclusion $i : \partial B \to \overline{B}$ is the computation. The fact that $r$ is a retraction means that $f \circ i = 1_{\partial B}$, and this translates into the identity $i^* f^* = 1$ on forms of each degree. The computation is less ambiguous if it is written as

$$0 < \int_{\partial B}\omega = \int_{\partial B} i^* r^*(\omega) = \int_B dr^*(\omega) = \int_B r^*(d\omega).$$

Remembering that pullbacks preserve degree and that $r^*$ therefore carries $\Omega^k(\partial B)$ into $\Omega^k(B)$ for each $k$, we can track down the degrees of the various forms in the computation. The $\omega$ on the left is in $\Omega^{n-1}(\partial B)$, $i^*r^*(\omega)$ is in $\Omega^{n-1}(\partial B)$, $dr^*(\omega)$ is in $\Omega^n(B)$ by Stokes's Theorem, and $r^*(d\omega)$ is in $\Omega^n(B)$. Since $r^*(d\omega)$ ends up in $\Omega^n(B)$, $r^*$ must have been acting on something in $\Omega^n(\partial B)$. This space is 0 since $\partial B$ has dimension $n-1$, and thus $r^*(d\omega) = r^*(0) = 0$.

18. For any point $p$ in $B$, the fact that $f(p) \neq p$ implies that there is a unique line passing through $p$ and $f(p)$. This line meets the sphere $\partial B$ in two points, and we define $r(p)$ to be the point that is closer to $f(p)$. (To complete the definition, we define $r$ to be the identity on points of $\partial B$.) Let us write the definition of $r$ is symbols, and then we can see that $f$ is smooth. The parametrically defined line $t \mapsto (1-t)p + tf(p)$ passes through $p$ when $t=0$ and passes through $f(p)$ when $t=1$. From the geometry it is evident that it meets $\partial B$ twice, once for some negative value of $t$ and once for some value of $t$ greater than 1. We seek an expression for the value of $t$ greater than 1. Thus we set $|(1-t)p + tf(p)|^2 = 1$ and solve the resulting quadratic equation for $t$. The coefficient of $t^2$ is

$$|p|^2 - 2p\cdot f(p) + |f(p)|^2 = |p - f(p)|^2,$$

and this is positive since $f(p) \neq p$. The constant term is $|p|^2 - 1$, which is negative since $p$ is in $B$. Thus the two roots $t$ have opposite sign, and our desired root $t$ is the one with the plus sign in the quadratic formula. Consequently we can obtain an explicit formula for $r(p)$, and its dependence on $p$ is smooth if $f$ is smooth. The function $r$ is a smooth retraction, which the previous problem shows cannot exist. Therefore $f$ must have a fixed point.

19. Regard $f$ as extended to $\mathbb{R}^n$ by extending it as 0 outside $\overline{B}$. Choose a member $\varphi \geq 0$ of $C^\infty_{\text{com}}(\mathbb{R}^n)$ of total integral 1, let $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$ for $\varepsilon > 0$, and convolve the scalar-valued function $\varphi_\varepsilon$ with each entry of $f$. Then $\varphi_\varepsilon * f$ converges uniformly to $f$ on $\mathbb{R}^n$ as $\varepsilon$ tends to 0. Thus the sequence $\{f_k\}$ may be taken to be the sequence of restrictions to $\overline{B}$ of the functions $\varphi_{1/k} * f$.

20. We are assuming that $\{f_k\}$ is a sequence of smooth functions carrying $\overline{B}$ to itself such that $f_k(x_k) = x_k$ for all $k$ and such that $\{f_k\}$ converges uniformly to $f$ on $\overline{B}$. The Bolzano-Weierstrass Theorem produces a limit point $x_0$ in $\overline{B}$ for the sequence $\{x_k\}$. Passing to a subsequence and renumbering, we may assume that $\lim_k x_k = x_0$. Then we have

$$|f(x_0) - x_0| \leq |f(x_0) - f(x_k)| + |f(x_k) - f_k(x_k)| + |f_k(x_k) - x_k| + |x_k - x_0|.$$

On the right side, the first term tends to 0 by continuity of $f$, the second term tends to 0 by the uniformity of the convergence, the third term is 0 because $x_k$ is a fixed point of $f_k$, and the fourth term tends to 0 since $\lim_k x_k = x_0$. Since the left side is independent of $k$, it must be 0.

## Chapter III

1. For (a), let $S$ be the set of vertices. We proceed by induction on the cardinality $V$ of $S$, the base case of the induction being the case $V = 4$ of a tetrahedron. For a tetrahedron the assertion is clear. Let a polyhedron be given with $n \geq 5$ faces, and assume that a triangulation exists whenever a compact convex polyhedron has $\leq n - 1$ faces. We shall attempt to introduce a plane that divides $S$ into two proper but overlapping subsets; if we can do this, then by induction we can do the triangulation for the polyhedron associated to each subset of vertices, and the union of the triangulations will be a triangulation of the given polyhedron. We fix attention on any three vertices and consider the unique plane that contains them. Let this plane be the set where some linear functional $L$ is 0. One subset of $S$ will consist of those vertices for which $L \geq 0$, and the other subset will consist of those vertices for which $L \leq 0$. We have seen that we are done if both these subsets are proper.

Thus suppose that one or the other of the subsets is all of $S$. Then the plane that passes through our three vertices is completely on one side of our polyhedron and those three vertices must span a face. In other words, we have associated a unique face to to each triple of vertices. On the other hand, if a face is given, then the vertices of that (triangular) face are a triple of vertices. We conclude the $F$ equals the number of triples of vertices, which is $\frac{1}{6}V(V-1)(V-2)$.

Meanwhile to each edge we can associate two vertices, and distinct edges yield distinct pairs of vertices. Thus $E \leq \frac{1}{2}V(V-1)$. Substituting into Euler's formula, we obtain $\frac{1}{6}V(V-1)(V-2) + V = F + V = E + 2 \leq \frac{1}{2}V(V-1) + 2$, and we are led to the inequality $V^3 - 6V^2 + 11V - 12 \leq 0$. The derivative of the polynomial $P(V)$ on the left is $3V^2 - 12V + 11$, whose larger root is $\frac{1}{6}(12 + \sqrt{23})$, which is less than 4. Thus $P(V)$ is an increasing function for $V \geq 4$. Computation gives $P(4) = 0$. Therefore $P(V) > 0$ for $V > 4$, and we cannot have our required inequality $P(V) \leq 0$ for $V > 4$. Tracing back, we see we are forced to conclude that when $V \geq 5$, it is possible to divide $S$ into two proper subsets by some plane and thereby to complete the induction.

2. For each $p$ in $M$, let $(M_p, \alpha_p)$ be a compatible chart about $p$ for the manifold-with-corners $M$ of dimension $m$; here $M_p$ is an open neighborhood of $p$, and $\alpha_p(M_p)$ is open in $\mathbb{Q}^m$. We may assume that no point of $M_p$ has larger index than $p$ does. Now let $F : M \to \mathbb{R}^m$ be an embedding. Since $F$ is continuous and $M$ is compact, $F(M)$ is bounded. Since $F$ is an embedding, $F$ is a homeomorphism of $M_+$ onto its image in $\mathbb{R}^m$. Define $U$ to be the open set $F(M_+)$, let $B = U^{\text{cl}} - U$, and let $E$ be the image under $F$ of all points in $M$ of index $\geq 2$. We are to see that $U \cup (B - E)$ is a smooth manifold-with-boundary, that $E$ is compact, and that $E$ has $m-1$ dimensional Minkowski content 0 in $\mathbb{R}^m$. Proposition 3.6c shows that the set of points of index $\geq 2$ is closed in $M$. Hence it is compact, and its image $E$ in $\mathbb{R}^m$ is compact. We have arranged that $F(M_+) = U$, and hence $F$ carries the set of points of index 1 onto $B - E$.

For each point $p$ in $M$ of index 0 or 1, the open subset $M_p$ of $M$ consists completely of points of index 0 or 1. Hence $F$ carries $M_p$ into $U \cup (B - E)$. The pairs $(F(M_p), \alpha_p \circ F^{-1})$ form an atlas for $U \cup (B - E)$ and exhibit $U - (B - E)$ as a manifold-with-boundary.

Finally we are to see that $E$ has $m - 1$ dimensional Minkowski content 0 in $\mathbb{R}^m$. This step follows from Corollary 3.12.

3. This is a routine adaptation of the argument for Example 3. The condition for the first partial derivative of the $(i,\, j)^{\text{th}}$ entry of $x$ to vanish is that the $(i,\, j)^{\text{th}}$ minor of $x$ should vanish. The set where all these 3-by-3 minors vanish is the set of matrices of rank $\leq 2$. Call this set $E$.

We shall exhibit $E$ as the union of 16 compact subsets of vector subspaces of $\mathbb{R}^{16}$ of dimension 12. Each of these will have 15 dimensional Minkowski content 0; then we can conclude that $E$ has 15 dimensional Minkowski content 0, and the set where $\det x \leq 0$ is a Whitney domain.

Consider a matrix $x$ for which the upper left 2-by-2 determinant is nonzero. The vector subspace of $\mathbb{R}^{16}$ corresponding to this choice of entries has dimension $8 + 2 + 2 = 12$, and $x$ lies in this subspace. Thus all matrices of rank 2 lie in the union of 16 vector subspaces of $\mathbb{R}^{16}$ of dimension 12. The matrices of rank $< 2$ lie in this same finite union, and we see that $E$ has 15 dimensional Mnkowski content 0. Thus the set where $\det x \leq 0$ is a Whitney domain in $\mathbb{R}^{16}$.

4. All of them. In (a), the respective first partial derivatives are $-yz$, $-xz$, and $2z - xy$. If these are simultaneously all 0, then $z = 0$ and also $x = 0$ or $y = 0$; also the converse is true. Thus $U$ is the set where $z(z - xy) < 0$, i.e., the set where $z$ and $z - xy$ are nonzero quantities of the same sign. Also $B$ is the set where $z = 0$ or

$z = xy$, and $E$ is the set where $z = 0$ and $xy = 0$.

In (b), write $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so that $\det(x) = ad - bc$. The set $U$ in question is where $\operatorname{Re}(ad - bc) < 0$. There are eight variables, namely the real and imaginary parts of $a, b, c, d$. If all eight first partial derivatives are 0, we are led to $a = b = c = d = 0$. Thus $b$ is the set where $\operatorname{Re}(ad - bc) = 0$, and $E = \{0\}$.

In (c), we proceed somewhat as in Algebraic Example 3 in Section 5. We study the set of 4-by-4 skew-symmetric matrices with $\det x \le 0$. We want to know where $\det x = 0$, and we want to identify the singular set. We can use each entry function above the diagonal as coordinates. The partial derivative in question with respect to the variable in the first row and second column is

$$\frac{d}{dt} \det \begin{pmatrix} 0 & x_{12}+t & x_{13} & x_{14} \\ -x_{12}-t & 0 & -x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & a_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{pmatrix} \Bigg|_{t=0}$$

In this expression will appear constant terms, terms with $t$, and terms with $t^2$. We use the multilinearity of the determinant to isolate the cofficient of $t$ and find that it equals the sum of two 3-by-3 determinants. Some of the terms cancel, and we find that the derivative at $t = 0$ is the determinant of the 2-by-2 matrix in positions 3 and 4. At any singular point all such derivatives at $t = 0$ have to be 0. The bottom line is that the only singular point is $x = 0$. So again $E = \{0\}$.

5. For (a), $V$ is the intersection of two closed balls. Each of them is a manifold-with-boundary. Then for each point where one or both of the inequalities are strict has an open neighborhood of the kind in a manifold-with-boundary. Each point where both equalities hold has an open neighborhood diffeomorphic to an open neighborhood of $(1, 0, 0)$ in $\mathbb{Q}^3$, and thus we have a manifold-with-corners.

For (b), we are working with $F = x^2\mathbf{i}$, for which $\operatorname{div} F = 2x$. The Divergence Theorem (Theorem 3.7) gives $\int_S x^2 y\, dy \wedge dx = \int_V 2x\, dx\, dy\, dz$. Since $V$ is symmetric about 0 in the $x$ variable and the integrand is odd in the $x$ variable, the integral is 0.

6. For $F = 3y\mathbf{i} + 2x\mathbf{j} + (z - 8)\mathbf{k}$, $\operatorname{div} F = 1$. Thus the given surface integral equals the volume of the tetrahedron that is decribed. The maximum values of $x$, $y$, and $z$ subject to $4x + 2y + z = 8$ with all variables $\ge 0$ are $x = 2$, $y = 4$, and $z = 8$. The volume in question is 1/6 of the volume of a parallelepiped with sides 2, 4, and 8. It is therefore $64/6 = \boxed{32/3}$.

7. Here $F = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\operatorname{div} F = 3$. Thus the integral equals $3 \cdot 7 = \boxed{21}$.

8. For this $F$, $\operatorname{div} F = 5$. Since the volume of a closed half ball of radius 2 is $\frac{2}{3}\pi 2^3$, the integral equals $5 \cdot (\frac{2}{3}\pi 2^3) = \boxed{80\pi/3}$.

9. In (b),

$$\begin{aligned}
U &= \{(x, y, z) \,\big|\, (x, y) \in M_+ \text{ and } f(x, y) < z < g(x, y)\}, \\
B &= \{(x, y, z) \,\big|\, (x, y) \in \partial M \text{ and } f(x, y) \le z \le g(x, y)\} \\
&\qquad \cup \{(x, y, z) \,\big|\, (x, y) \in M_+ \text{ and } z = f(x, y)\} \\
&\qquad \cup \{(x, y, z) \,\big|\, (x, y) \in M_+ \text{ and } z = g(x, y)\}, \\
E &= \{(x, y, z) \,\big|\, (x, y) \in \partial M \text{ and } z = f(x, y)\} \\
&\qquad \cup \{(x, y, z) \,\big|\, (x, y) \in \partial M \text{ and } z = g(x, y)\}.
\end{aligned}$$

10. In (a), we apply the Fundamental Theorem of Calculus on each subinterval $I_j$, completely ignoring the other subintervals, and there is no problem. Then we add the results and obtain $\int_a^b f'(t)\,dt = \sum_{j=1}^k \int_I f'(t)\,dt = \sum_{j=1}^k \int_{I_j} f_j'(t)\,dt = \sum_{j=1}^k (f_j(t_j) - f_j(t_{j-1})) = f_k(t_k) - f_0(t_0) = f(b) - f(a)$ because the (finite) series before the next-to-last equality sign telescopes.

In (b) and (c), we can indeed interpret the $j^{\text{th}}$ equality as saying the 0 form $f_j$ and the 1 form $df_j = f_j'(t)\,dt$ together satisfy $\int_{\{a_j, b_j\}} f_j = \int_{I_j} df_j$ under a certain orientation. Combining these equalities into a single equality for $f$ requires a certain consistency for the orientations, so that the series in (a) can be seen to telescope at the last step. The orientations on the two-point sets $\{a_j, b_j\}$ are the induced orientations from the various intervals $[a_j, b_j]$, and these are arranged so that each intermediate point $a_1, \dots, a_{k-1}$ occurs with opposite orientations the two times it occurs.

When this framework is applied to a closed triangle—that is, when the $t$ interval is regarded as parametrizing the edge of the triangle—consistent orientations are obtained by orienting the triangle and giving each edge the induced orientation. In this case the expression $f(b) - f(a)$ on the right is 0, since $a = b$. Thus the theorem is that the integral of the derivative is 0; in other words, the result is a version of Theorem 2.1.

11. The definition of a piecewise smooth 1 form on the (closed) faces and edges of a tetrahedron can be taken to be that it is continuous function from the union of the faces and edges of the tetrahedron whose restriction to each face is a smooth 1 form on the closed face. Stokes's Theorem applies to each face as a manifold-with-corners, and we obtain the usual formula $\int_{\text{edges}} \omega = \int_{\text{face}} d\omega$. The 3 dimensional part of the tetrahedron is not present, but if it were and if we were to orient it, then we could use the induced orientation on each face. With this choice when we take all the faces into account, we again have cancellation in pairs for the contributions from the lower dimensional integrals, and the conclusion is that $\sum \int_{\text{faces}} d\omega = 0$.

12. No.

13. Since $E$ is compact and $F$ is continuous, $F(E)$ is compact. Choose compatible charts $(M_{\alpha_1}, \alpha_1), \dots, (M_{\alpha_r}, \alpha_r)$ in $M$ such that $E \subseteq M_{\alpha_1} \cup \cdots \cup M_{\alpha_r}$, and choose by Lemma 1.26b an open cover $\{P_{\alpha_1}, \dots, P_{\alpha_r}\}$ of $E$ such that $P_{\alpha_i}^{\text{cl}} \subseteq M_{\alpha_i}$ for each $i$.

Then $E = (P^{\rm cl}_{\alpha_1} \cap E) \cup \cdots \cup (P^{\rm cl}_{\alpha_r} \cap E)$ exhibits $E$ as the union of the respective compact subsets $P^{\rm cl}_{\alpha_i} \cap E$ of $M_{\alpha_i} \cap E$. The set $\alpha_i(P^{\rm cl}_{\alpha_i} \cap E)$ is a compact subset of $\alpha_i(M_{\alpha_i} \cap E)$ and by hypothesis has $\ell$ dimenional Minkowski content 0 in $\mathbb{R}^m$. It is enough to show that the compact set $F(P^{\rm cl}_{\alpha_i} \cap E) = (F \circ \alpha_i^{-1})(\alpha_i(M_{\alpha_i} \cap E))$ has $\ell$ dimensional Minkowski content 0 in $N$.

In other words we may assume from the outset that $M$ is an open subset of $\mathbb{R}^m$, that we are given a compact subset $E$ of $M$ of $\ell$ dimensional Minkowski content 0, and that we are to show that $F(E)$ has $\ell$ dimensional Minkowski content 0 in $N$.

Choose charts $(N_{\beta_1}, \beta_1), \ldots, (N_{\beta_s}, \beta_s)$ in $N$ so that $F(E) \subseteq N_{\beta_1} \cup \cdots \cup N_{\beta_s}$, and then choose by Lemma 1.26b an open cover $\{Q_{\beta_1}, \ldots, Q_{\beta_s}\}$ of $F(E)$ such that each $Q^{\rm cl}_j$ is compact and $Q^{\rm cl}_{\beta_j} \subseteq N_{\beta_j}$ for each $j$.

For each $p$ in $E$, choose an open neighborhood $M_p$ of $p$ such that $F(M_p)$ is contained in a single $Q_{\beta_j}$. These open neighborhoods cover $E$, and finitely many of them, say $M_{p_1}, \ldots, M_{p_t}$, suffice to cover $E$. Choose by Lemma 1.26b an open cover $\{R_{p_1}, \ldots, R_{p_t}\}$ of $E$ such that $R^{\rm cl}_{p_k} \subseteq M_{p_k}$ for each $k$.

The restriction $F\big|_{M_{p_k}}$ of $F$ is smooth from $M_{p_k}$ into some $Q_{\beta_j}$, say $Q_{\beta_{j(k)}}$. When it is followed by $\beta_{j(k)}$, the result is a smooth function from an open subset of $\mathbb{R}^m$ into a Eucldiean space. Proposition 3.11 applies to this function and shows that it carries compact sets of $\ell$ dimensional Minkowski content 0 into compact sets of $\ell$ dimensional Minkowski content 0. From the inclusions

$$(\beta_{j(k)} \circ F)(R_{p_k}) \subseteq (\beta_{j(k)} \circ F)(R^{\rm cl}_{p_k}) \subseteq \beta_{j(k)}(Q_{\beta_{j(k)}}),$$

we see that $(\beta_{j(k)} \circ F)(R^{\rm cl}_{p_k})$ has $\ell$ dimensional Minkowski content 0 in Euclidean space. Thus $F(R^{\rm cl}_{p_k})$ has $\ell$ dimensional Minkowski content 0 in $N$. We combine this fact with the chain of inclusions

$$F(E) \subseteq F(R_{p_1} \cup \cdots \cup R_{p_t}) = F(R_{p_1}) \cup \cdots \cup F(R_{p_t}) \subseteq F(R^{\rm cl}_{p_1}) \cup \cdots \cup F(R^{\rm cl}_{p_t}),$$

and we conclude that $F(E)$ has $\ell$ dimensional Minkowski content 0 in $N$.

14. Arguing as in Problem 13, we see that it is enough to see that the smooth image in $N$ of any compact subset $E$ of a Euclidean space $\mathbb{R}^d$ of dimension $d \leq n-2$ has $n-1$ dimensional Minkowski content 0. A compact subset $E$ of $\mathbb{R}^d$ has $d$ dimensional Minkowski content equal to its Lebesgue measure, and then $E$ has $d+1$ dimensional Minkowski content equal to 0. Since $d+1 \leq n-1$, $E$ has $n-1$ dimensional Minkowski content 0. Problem 13 then allows us to conclude that that the smooth image of $E$ in any smooth manifold of dimension $\geq n-1$ has $n-1$ dimensional Minkowski content 0.

15. This is similar to Problem 2. The relevance of the assumption of compactness is in proving that the (closed) set of points of index $\geq 2$ is compact.

16. This equivalence is essentially the content of Proposition 3.10. In one direction suppose that $E$ has $\ell$ dimensional Minkowski content 0 and therefore that

$\lim_{\delta\downarrow 0} \delta^\ell N(E,\delta) = 0$. Then for any $\epsilon > 0$, there is a $\delta_0$ such that $\delta < \delta_0$ implies $\delta^\ell N(E,\delta) < \epsilon$. Take $k = N(E,\delta)$, and let $B_1, \ldots, B_k$ have $\operatorname{diam}(B_i) < \delta$. Then $k\delta^\ell < \epsilon$, and $E$ is of zero $\ell$ extent. In the converse direction suppose $E$ is of zero $\ell$ extent. Let $\epsilon > 0$ be given, and choose $\zeta_0$ according to that condition. Whenever $\zeta < \zeta_0$ is given, choose $k$ so that $E \subseteq B_1 \cup \cdots \cup B_k$, $\operatorname{diam}(B_i) \le \zeta$, and $k\zeta^\ell < \epsilon$. Then $\zeta^\ell N(E,\zeta_0) < \epsilon$, and $\zeta_0^\ell N(E,\zeta_0) \le \epsilon$. In other words, $\lim_{\zeta_0\downarrow 0} \zeta_0^\ell N(E,\zeta_0) = 0$, and then $E$ has $\ell$ dimensional Minkowski content 0 by Proposition 3.10.

17. As in Section 5, let

$$N_{\text{sep}}(E,\delta) = \begin{Bmatrix} \text{maximum number of points of } E \\ \text{at distance} \ge \delta \text{ from one another} \end{Bmatrix}.$$

Suppose we have a configuration of $N_1$ points $x_1$ of $E_1$ that are at distance $\ge \delta$ from one another, and suppose also that we have a configuration of $N_2$ points $x_2$ of $E_2$ that are at distance $\ge \delta$ from one another. Then the corresponding set of points $(x_1, x_2)$ in $E_1 \times E_2$ has the property that any two distinct members of the product set have

$$|(x_1,x_2) - (x_1', x_2')| \ge \max\{|x_1 - x_1'|, |x_2 - x_2'|\} \ge \delta.$$

Therefore there exist $N_1N_2$ points of $E_1 \times E_2$ at distance $\ge \delta$ from one another, and the definition of $N_{\text{sep}}$ gives

$$N_{\text{sep}}(E_1 \times E_2, \delta) \le N_1 N_2.$$

Taking the minimum over all such configurations allows us to conclude that

$$N_{\text{sep}}(E_1 \times E_2, \delta) \le N_{\text{sep}}(E_1,\delta) N_{\text{sep}}(E_2,\delta).$$

Combining ths inequality with the first two conclusions of Lemma 3.9 yields

$$\begin{aligned} N(E_1 \times E_2, \delta) &\le N_{\text{sep}}(E_1 \times E_2, \delta) \\ &\le N_{\text{sep}}(E_1,\delta) \\ &\le N_{\text{sep}}(E_2,\delta) \\ &\le N(E_1,\delta/2) N(E_2,\delta/2), \end{aligned}$$

and the result follows.

18. We know that $a$ dimensional Minkowski content coincides with Lebesgue measure for compact subsets of $\mathbb{R}^a$. Also Lemma 3.9 shows that $|E^\delta|$ is comparable in size to $\delta^a N(E,\delta)$. As $\delta$ tends to 0, $|E^\delta|$ tends to $|E|$ by complete additivity of Lebesgue measure, and this limit is finite since $E$ is compact. Thus $\delta^a N(E,\delta)$ is bounded as $\delta$ tends to 0.

19. In both parts of the problem, Problem 17 gives

$$N(E_1 \times E_2, \delta) \le N(E_1, \delta)N(E_2, \delta). \qquad (*)$$

We multiply through by $\delta^{\ell_1+\ell_2}$ for (a) and by $\delta^{\ell_1+m_2}$ for (b). Then we let $\delta$ tend to 0.

In (a), Proposition 3.10 shows that $\delta^{\ell_1}N(E_1, \delta)$ and $\delta^{\ell_2}N(E_2, \delta)$ tend to 0. By $(*)$, $\delta^{\ell_1+\ell_2}N(E_1 \times E_2, \delta)$ tends to 0. Thus the converse direction of Proposition 3.10 shows that $E_1 \times E_2$ has $\ell_1 + \ell_2$ dimensional Minkowski content 0.

For (b), we argue in the same way except that we use Problem 18 to see that $\delta^{a_2}N(E_2, \delta)$ is bounded as $\delta$ tends to 0. This bounded quantity is multiplied by $\delta^{\ell_1}N(E_1, \delta)$, which tends to 0, and the product thus tends to 0. We conclude that $\delta^{\ell_1+m_2}N(E_1 \times E_2, \delta)$ tends to 0, and it follows that $E_1 \times E_2$ has $\ell_1 + m_2$ dimensional Minkowski content 0.

20. For (a), we are to show that $(U, B, E)$ has the properties of a Whitney domain in $\mathbb{R}^{m_1+m_2}$. The set $U$ is open in $\mathbb{R}^{m_1+m_2}$ because its factors are open in $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, and $B$ is closed and is the boundary of $U$ because $B$ equals $U^{\text{cl}} - U$. The set $U$ is bounded in $\mathbb{R}^{m_1+m_2}$ because $U_1$ is bounded in $\mathbb{R}^{m_1}$ and $M$ is compact in $\mathbb{R}^{m_2}$. The set $E$ is compact as the product of two compact sets. What needs to be shown is that $E$ has $m_1 + m_2 - 1$ dimensional Minkowski content 0.

It is enough to prove that each of $E_1 \times M$ and $B_1 \times \partial M$ has $m_1 + m_2 - 1$ dimensional Minkowski content 0. Consider $E_1 \times M$. Since $E_1$ has $m_1 - 1$ dimensional Minkowski content 0 and $M$ is compact in $\mathbb{R}^{m_2}$, Problem 19b shows that $E_1 \times M$ has $m_1 + m_2 - 1$ dimensional Minkowski content 0.

Consider $B_1 \times \partial M$. The subset $B_1$ of $\mathbb{R}^{m_1}$ by assumption is a closed bounded portion of the set in $\mathbb{R}^{m_1}$ where a nonzero real-valued polynomial in $m_1$ variables equals 0. Problem 10 of Chapter VI of *Basic Real Analysis* shows that the compact set $B_1$ has $m_1$ dimensional Lebesgue measure 0. It therefore has $m_1$ dimensional Minkowski content 0. The set $\partial M$ is a compact manifold of dimension $m_2 - 1$. Using the style of argument in Problems 13 and 14 and applying Problem 19, we see that $B_1 \times \partial M$ is a compact set of $m_1 + m_2 - 1$ dimensional Minkowski content 0.

At this writing, the author does not know the answer to (b).