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A Sequel to
Basic Real Analysis

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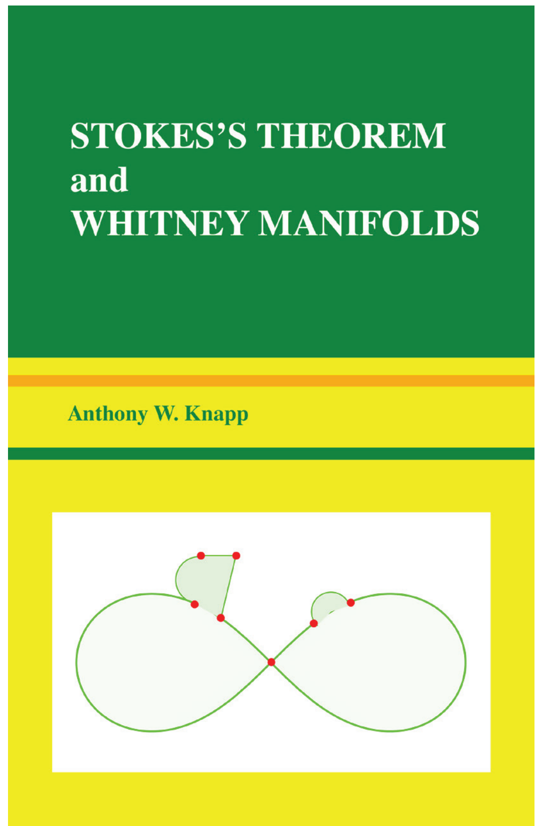

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Cover: An example of a Whitney domain in two-dimensional space. The green portion is a manifold-with-boundary for which Stokes's Theorem applies routinely. The red dots indicate exceptional points of the boundary where a Whitney condition applies that says Stokes's Theorem extends to the whole domain.

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CHAPTER III

Whitney's Setting for Stokes's Theorem

Abstract. This chapter looks for a single setting in which Stokes's Theorem applies at once to all situations of practical interest. It begins by developing the theory in the setting of manifolds-with-corners and continues with a theory in a more general setting studied by H. Whitney.

Section 1 introduces the model space $\mathbb{Q}^m$ for $m \geq 2$, in terms of which manifolds-with-corners are defined. The section contains one result that is relatively hard to prove: the index of a point of $\mathbb{Q}^m$ is taken to be the number of coordinates that are equal to 0, and it is shown that any diffeomorphism between open sets in $\mathbb{Q}^m$ maps points of one index into points of the same index. Consequently the notion of index is well defined for the points of a manifold-with-corners. Other definitions concerning manifolds translate easily into corresponding definitions for manifolds-with-corners. These include smooth real-valued function, support, germ, tangent space, cotangent space, smooth differential forms, pullbacks of differential forms, and the derivative of a smooth map between manifolds-with-corners.

Section 2 introduces strata, the stratum $S_k(M)$ consisting of all points of index k in a manifold-with-corners M. Strata have a number of useful properties, one of which is that the strata of index 0 and 1 combine to yield a manifold-with-boundary.

Section 3 gives a version of the Stokes's Theorem for manifolds-with-corners, saying $\int_{\partial M} \omega = \int_M d\omega$ as usual. In this equality the integral on the left is over the stratum of all points of index 1, and the integral on the right is over the stratum of all points of index 0. Simple examples show that this theorem is not a trivial consequence of the theorem about manifolds-with-boundary when applied to the manifold-with-boundary consisting of all points of index 0 and 1 in M.

Section 4 establishes a version of the Divergence Theorem due to Whitney that applies to any bounded region of $\mathbb{R}^m$ for $m \geq 2$ when most of the topological boundary behaves as it does for a manifold-with-boundary and when the set of exceptional points of the topological boundary is small in a specific technical sense. Such a region will be called a Whitney domain. If the set of exceptional points is finite, then it is small in the technical sense.

Section 5 examines in some detail the technical condition in Section 4. That condition becomes: the set of exceptional points is compact and either is empty or has $m-1$ dimensional Minkowski content 0. It is shown that the condition that a compact set has ℓ dimensional Minkowski content 0 is intrinsic to the set as a subset of a Euclidean space and does not depend on its embedding. Furthermore any function from one Euclidean space to another that satisfies a Lipschitz condition always carries compact subsets of ℓ dimensional Minkowski content 0 to compact sets of ℓ dimensional Minkowski content 0. Consequently the notion "ℓ dimensional Minkowski content 0" is well defined for compact subsets of smooth manifolds and is preserved under smooth mappings into Euclidean spaces. The section concludes with examples of Whitney domains constructed from the zero loci of polynomials.

Section 6 extends the scope of Stokes's Theorem to Whitney manifolds, a class of spaces that includes all manifolds-with-corners and that allows all Whitney domains as additional model cases. The result is that the Stokes formula applies in what seems to be the full set of practical situations of interest to mathematicians, physicists, and engineers.

1. Definition and Examples of Manifolds-with-Corners

Smooth manifolds of dimension $m \geq 0$, as introduced in Chapter I, were defined as separable Hausdorff spaces that are locally modeled on open subsets of $\mathbb{R}^m$. In similar fashion smooth manifolds-with-boundary of dimension $m \geq 1$, as introduced in Chapter II, were defined as separable Hausdorff spaces that are locally modeled on open subsets of the closed half space

$$\mathbb{H}^m = \{(x_1, \ldots, x_{m-1}, x_m) \in \mathbb{R}^m \mid x_m \geq 0\}.$$

In the first part of this chapter, we work with smooth manifolds-with-corners of dimension $m \geq 2$ as separable Hausdorff spaces that are locally modeled on open subsets of the closed generalized quadrant

$$\mathbb{Q}^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_j \geq 0 \text{ for } 1 \leq j \leq m\}.$$

The **open subsets** of $\mathbb{Q}^m$ are understood to be those subsets that are relatively open in the relative topology from $\mathbb{R}^m$. The goal of the first three sections of this chapter is to prove an extension of Stokes's Theorem to manifolds-with-corners. At the least such a theorem will simultaneously handle balls and rectangular solids in $\mathbb{R}^m$. The failure of the theorems of Chapters I and II to handle balls and rectangular solids at the same time was a weakness of the earlier theory that we shall now be able to remedy. We can do much better, and we begin the development of an improved theory in Section 4.

Before coming to the formal definition of smooth manifold-with-corners, we need to establish some definitions concerning smooth functions on $\mathbb{Q}^m$, just as we did with $\mathbb{H}^m$ in Section II.2. A real-valued function f defined on an open subset U of $\mathbb{Q}^m$ will be said to be **smooth** if there is a smooth function F defined an open subset V of $\mathbb{R}^m$ such $U = V \cap \mathbb{Q}^m$ and f is the restriction of F to U. The extending function F need not, of course, be unique. With this definition of smoothness in place, we can define the space $\mathcal{C}_p(\mathbb{Q}^m)$ of germs of smooth functions at points p of $\mathbb{Q}^m$ and the tangent space $T_p(\mathbb{Q}^m)$ at p.

We write $\mathbb{Q}^m_+$ for the interior of $\mathbb{Q}^m$, namely the subset

$$\mathbb{Q}^m_+ = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_j > 0 \text{ for } 1 \leq j \leq m\}.$$

and we write $\partial\mathbb{Q}^m$ for the topological boundary, namely the subset

$$\partial\mathbb{Q}^m = \{(x_1, \ldots, x_m) \in \mathbb{Q}^m \mid x_j = 0 \text{ for at least one } j \text{ with } 1 \leq j \leq m\}.$$

The definitions of $\mathcal{C}_p$ and T_p are not new for p in $\mathbb{Q}^m_+$, but for p in $\partial\mathbb{Q}^m$, they are. We obtain facts about $\mathcal{C}_p$ and T_p in the same way as in Section I.1.

If U_1 and U_2 are two open subsets of $\mathbb{Q}^m$, a **smooth map** $F : U_1 \to U_2$ is function whose m component functions are all smooth real-valued functions on

U_1. The derivative $(DF)_p : T_p(U_1) \to T_{F(p)}(U_2)$ of the smooth map F at a point is defined just as in Section I.1. The smooth map F is a **diffeomorphism** if it is a homeomorphism with inverse $G : U_2 \to U_1$ such that the m component functions of each of F and G are smooth real-valued functions on U_1 and U_2, respectively. The composition of smooth maps is smooth, and the derivative of the composition is the composition of the derivatives. It follows that at each point the derivative of a diffeomorphism is an invertible linear function.

In the study of manifolds-with-boundary, we distinguished two kinds of points, manifold points and boundary points, and the distinction was straightforward. In a corresponding but more subtle fashion for manifolds-with-corners, we define[1] the **index** of a point $(x_1, \ldots, x_m)$ in $\mathbb{Q}^m$ to be the number of indices j for which $x_j = 0$. The points in $\mathbb{Q}^m_+$ have index 0 and the points in $\partial\mathbb{Q}^m$ have index ≥ 1. For this notion to be usable with a general manifold-with-corners, we need Proposition 3.1 below, whose proof will make use of a lemma.

Proposition 3.1. If $F : U \to V$ is a diffeomorphism of one nonempty open subset of $\mathbb{Q}^m$ onto another, then every $p \in U$ has the property that the index of p equals the index of $F(p)$.

Lemma 3.2. Let $A = (a_{ij})_{i,j=1}^m$ be a square matrix with real entries. If there is an integer k with $1 \leq k \leq m$ such that $a_{ij} = 0$ whenever $i \leq k$ and $j \geq k$, then $\det A = 0$.

PROOF OF LEMMA 3.2. The proof is by induction on k simultaneously for all m. The base case of the induction is $k = 1$. In this case, $a_{ij} = 0$ for $i = 1$ and all j. In other words, the first row of A is 0. Hence $\det A = 0$.

Suppose that the lemma has been proved for the integer $k - 1 \geq 1$ and that we are to consider a matrix A for the integer k. We expand $\det A$ in cofactors about the first row, obtaining an alternating sum of terms with a coefficient a_{1j} that multiplies the determinant of a matrix of size $m - 1$. The upper left entry of that matrix is a_{22} for the first term and is a_{21} for the subsequent terms. Since the coefficient a_{1j} is 0 for $j \geq k$, we need only consider the first $k - 1$ terms in the expansion. Each of those terms corresponds to a matrix of the form in the lemma but with k replaced by $k - 1$. By inductive hypothesis, each such determinant is 0. Therefore $\det A = 0$, and the induction is complete. $\square$

PROOF OF PROPOSITION 3.1. Possibly replacing F by F^{-1}, we see that it is enough to prove that the index I of $F(p)$ is $\leq$ the index J of p. It will simplify the ideas if we think of U and V as lying in distinct copies of $\mathbb{Q}^m$, so that the order of the variables in U does not affect the order of the variables in V. Let us write p

[1] Some authors use the term "depth" in place of "index."

as $(x_1, \dots, x_m)$ and F as $(F_1, \dots, F_m)$, and let us concentrate on a single point p of $\mathbb{Q}^m$, say $p = p_0 = (x_{1,0}, \dots, x_{m,0})$. The function F being a diffeomorphism, the m-by-m Jacobian matrix A of the derivative $DF(p_0)$ is invertible.

We reorder the variables of U so that the first J of the entries of p_0 are 0 and the others are > 0. Then we reorder the variables of V so that the first I of the entries of $F(p_0)$ are 0 and the others are > 0. Consider the restriction of $F_1, \dots, F_I$ to points $(x_{1,0}, \dots, x_{J,0}, x_{J+1}, \dots, x_m)$ as a function of several variables $(x_{J+1}, \dots, x_m)$. This function is ≥ 0 everywhere in a Euclidean neighborhood of $(x_{J+1,0}, \dots, x_{m,0})$ and takes on its minimum value 0 at $(x_{J+1,0}, \dots, x_{m,0})$. Thus the first partial derivatives of $F_1, \dots, F_I$ with respect to $(x_{J+1}, \dots, x_m)$ must be 0 at any point where the minimum value is attained. In symbols,

$$\Big(\frac{\partial F_i}{\partial x_j}\Big)(p_0) = 0 \quad \text{for } i \leq I \text{ and } j \geq J+1. \tag{$*$}$$

Arguing by contradiction, suppose that $I > J$. If $i \leq I$ and $j \geq I$, then we have $j \geq I > J$ and hence $j \geq J + 1$. In view of $(*)$, the Jacobian matrix A of $DF(p_0)$, whose $(i, j)^{\text{th}}$ entry is $a_{ij} = \big(\partial F_i/\partial x_j\big)(p_0)$, has $a_{ij} = 0$ for $i \leq I$ and $j \geq I$. Taking $k = I$ in Lemma 3.2, we see that the matrix A has $\det A = 0$, in contradiction to the fact that A is invertible. This contradiction shows that we must after all have had $I \leq J$. $\square$

Now we can introduce manifolds-with-corners. Let M be a separable Hausdorff topological space, and fix an integer $m \geq 2$. For purposes of working with manifolds-with-corners, a **chart** (M_α, α) on M of **dimension** m is a homeomorphism α of a nonempty open subset M_α of M onto an open subset $\alpha(M_\alpha)$ of $\mathbb{Q}^m$; the chart is said to be **about** a point p in M if p is in the domain M_α of α. When it is convenient to do so, we can restrict attention to charts (M_α, α) for which M_α is connected.

A smooth **manifold-with-corners** of dimension m is a separable Hausdorff space M with a family $\mathcal{F}$ of charts (M_α, α) of dimension m such that

- (i) any two charts (M_α, α) and (M_β, β) in $\mathcal{F}$ are (smoothly) **compatible** in the sense that $\beta \circ \alpha^{-1}$, as a mapping of the open subset $\alpha(M_\alpha \cap M_\beta)$ of $\mathbb{Q}^m$ to the open subset $\beta(M_\alpha \cap M_\beta)$ of $\mathbb{Q}^m$, is a diffeomorphism,
- (ii) the family of compatible charts (M_α, α) is an **atlas** in the sense that the open sets M_α cover M, and
- (iii) the family $\mathcal{F}$ is maximal among families of compatible charts on M.

In the presence of an understood atlas, a chart will be said to be **compatible** if it is compatible with all the members of the atlas.

Because of Proposition 3.1, we can unambiguously transfer the definition of "index" from $\mathbb{Q}^m$ to any smooth manifold-with-corners M: if (M_α, α) is a chart about a point p in M, then the **index** of p in M is defined to be the index of $\alpha(p)$ in $\mathbb{Q}^m$. The points of index 0 are the **manifold points**, the points of index ≥ 1 are sometime called boundary points, and the points of index ≥ 2 are sometimes called corner points.

As with smooth manifolds in the sense of Chapter I and with smooth manifolds-with-boundary in Chapter II, any atlas of compatible charts for a smooth manifold-with-corners can be extended in one and only one way to a maximal atlas of compatible charts. Also if U is any nonempty open subset of an m dimensional smooth manifold-with-corners M, then U inherits the structure of a smooth manifold-with-corners as follows: first define an atlas of U to consist of the intersection of U with all members of the atlas for M, using the restrictions of the various functions α, and then discard occurrences of the empty set.

We turn to examples. Some of these will be examples of (smooth) manifolds-with-corners, and some will be nearly-but-not-quite examples of manifolds-with-corners. For some of the latter, there will be a simple way of subdividing or triangulating the given space that exhibits it as a finite union of manifolds-with-corners. In any case the theorem in Section 3 is going to be that the Stokes formula, $\int_{\partial M}\omega = \int_M d\omega$, holds for all manifolds-with-corners. In this equality the integral on the left side is carried on the points of index 1, and the integral on the right side is carried on the points of index 0. Our decompositions of some of the near manifolds-with-corners as finite unions of genuine manifolds-with-corners will have the Stokes formula holding on each piece, and we shall be able to add these formulas for the pieces to obtain the Stokes formula for the union.

EXAMPLES.

(1) Any smooth manifold-with-boundary of dimension $\geq$ 2 is a smooth manifold-with-corners, there being no corner points. Any filled compact convex polygon in dimension 2 is a manifold-with-corners, as a consequence of the definition.

(2) No manifold-with-corners has any phantom corner points, in which a boundary point can be interpreted either as a corner or not. In dimension 2, for example, the boundary changes direction at each corner point, and there are no angles of 0 or 360 degrees. This is a consequence of Proposition 3.1 and the fact that index is well defined. See Figure 3.1.

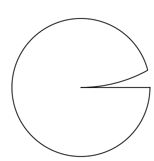

FIGURE 3.1. Manifolds-with-corners have no angles of 0 or 360 degrees.

(3) A manifold-with-corners has a certain local convexity to it. In dimension 2, for example, no interior angle of more than π can occur at a boundary point. This is a consequence of the fact that $\mathbb{Q}^m$ has this property at its boundary points.

(4) In dimension 2, a space that looks like the filled space in Figure 3.2, although not itself a manifold-with-corners (according to Example 3), can be subdivided into two adjacent pieces that are manifolds-with-corners by inserting an auxiliary line that becomes part of the boundary of each piece. (The auxiliary line is dashed in the figure.) An orientation on the set of manifold points yields by restriction an orientation on each of the two adjacent components and then yields an induced orientation on the boundaries of each piece. Since a single reflection is involved in passing from the induced orientation for the one component to the induced orientation for the other component, the two orientations on the auxiliary line will cancel in the computation of integrals over the boundary. Thus the validity of the Stokes formula $\int_{\partial M} \omega = \int_M d\omega$ for each of the constituents will imply the validity of the Stokes formula for the whole space.

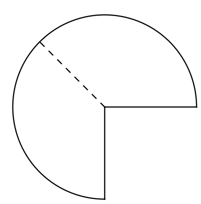

FIGURE 3.2. Triangulation available for angles greater than π.

(5) A filled closed cube in $\mathbb{R}^3$ is a smooth manifold-with-corners. The interior points have index 0, the points on the interiors of the six faces have index 1, the points on the interiors of the eight edges have index 2, and the eight vertices have index 3. The subset consisting of the faces, edges, and vertices is not a manifold-with-corners because no open neighborhood of a vertex is diffeomorphic to an open subset of any $\mathbb{Q}^m$.

(6) A filled closed tetrahedron in $\mathbb{R}^3$ is a manifold-with-corners, but a filled closed square pyramid in $\mathbb{R}^3$ is not. In the latter case, the pyramid can be subdivided into two adjacent pieces that are manifolds-with-corners (tetrahedra actually) by inserting an auxiliary triangle whose base is a diagonal of the square

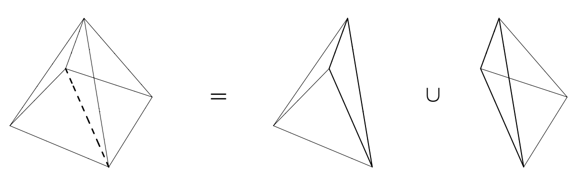

FIGURE 3.3. Square pyramid subdivided into two tetrahedra. The vertex of the pyramid appears at the top of each solid.

base of the pyramid and whose vertex is the vertex of the pyramid. See Figure 3.3. The relevant diagonal of the base is shown dashed. The auxiliary triangle becomes part of the boundary of each piece. As in Example 4, the two induced orientations on the added triangle are opposite and then cancel when computing integrals over boundaries. Thus the validity of the Stokes formula $\int_{\partial M} \omega = \int_M d\omega$ for each of the constituents will imply the validity of the Stokes formula for the whole space.

(7) A solid cylinder in $\mathbb{R}^3$ is a manifold-with-corners. Its surface, which consists of two closed disks for the ends and the product of a circle and a closed interval for the side of the cylinder, is a manifold-with-boundary. A solid cone in $\mathbb{R}^3$ such as $z^2 \leq x^2 + y^2 \leq 25$ is not a manifold-with-corners because the cone point at the origin has no open neighborhood diffeomorphic to an open subset of $\mathbb{Q}^3$; no simple way is evident for decomposing this solid cone into the union of nonoverlapping manifolds-with-corners.

(8) Whenever M is a smooth manifold-with-corners, then the points of index 0 form a smooth manifold, the points of index 0 or 1 form a smooth manifold-with-boundary, and the points of index 0 through 2 form a smooth manifold-with-corners. This example will be amplified in the next section when we introduce "strata" for smooth manifolds-with-boundary.

(9) The numeral 8 in Figure 3.4, once it has been filled, is not a manifold-with-corners because it has no neighborhood of the crossing point that is diffeomorphic to an open subset of $\mathbb{Q}^2$. However, the top half of the filled numeral is a manifold-with-corners, there being just the one corner at the crossing point. Similarly the bottom half is a manifold-with-corners. The whole space is thus the union of two manifolds-with-corners whose intersection is simply the crossing point. Accordingly the Stokes formula applies to each half. Since the crossing point has index 2 in both cases, it plays no role in integrations. Thus the validity of the Stokes formula for each piece will imply the validity of the Stokes formula for the whole filled numeral.

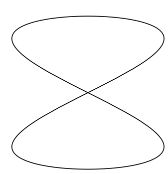

FIGURE 3.4. Numeral 8 centered at the origin, to be regarded as filled.

Finally we are in a position to introduce the notion of a smooth function and various related constructs for smooth manifolds-with-corners. A **smooth real-valued function** $f : M \to \mathbb{R}$ on the smooth manifold-with-corners of dimension m is by definition a function such that for each $p \in M$ and each compatible chart (M_α, α) about p, the function $f \circ \alpha^{-1}$ is smooth as a function from the open subset $\alpha(M_\alpha)$ of $\mathbb{Q}^m$ into $\mathbb{R}$. This is the expected definition, and there are no surprises. A smooth real-valued function is necessarily continuous.

If E is a nonempty open subset of M, the space of smooth real-valued functions on E will be denoted by $C^\infty(E)$. The space $C^\infty(E)$ is an associative algebra over $\mathbb{R}$ under the pointwise operations, and it contains the constants. The **support** of a real-valued function is, as always, the closure of the set where the function is nonzero. We write $C^\infty_{\text{com}}(E)$ for the subset of $C^\infty(E)$ of functions whose support is a compact subset of M.

Relative to a point p of the manifold-with-corners M, we define a **germ** at p, the **tangent space** $T_p(M)$ at p, and the **cotangent space** at p in the same way as in the manifold case. For the manifold points of M, the definition is completely unchanged. The only difference occurs in the case of boundary points: when matters are referred back to the model space $\mathbb{Q}^m$, the open sets of $\mathbb{Q}^m$ do not need to be open in the underlying Euclidean space $\mathbb{R}^m$. The space $\mathcal{C}_p(M)$ of germs at p is an associative algebra over $\mathbb{R}$ with identity.

The nature of $T_p(M)$ and that of $T_p^*(M)$ are unchanged from the manifold case. If (M_α, α) is a chart about p, and if $\alpha = (x_1, \dots, x_m)$, then a basis of $T_p(M)$ consists of the m first partials $\left[\partial/\partial x_j\right]$ evaluated at p. If p has j^{th} coordinate 0 in $\mathbb{Q}^m$, then $\left[\partial/\partial x_j\right]_p$ can be computed as a one-sided partial derivative. Examples of members of $T_p^*(M)$ are the differentials of smooth functions at p, the differential of f at p being defined by $(df)_p(L) = Lf$ for L in $T_p(M)$, just as in the manifold case.

We can then go on to define differential 1 forms, differential k forms, and smoothness of differential forms. There are no surprises. The notion of **pullback** of a differential form is still meaningful.

The derivative DF of a smooth function between manifolds-with-corners is defined just as in the case of manifolds. Let $F : M \to N$ be a smooth function from a smooth manifold-with-corners M of dimension m into a smooth manifold-with-corners N of dimension n. For any $p \in M$, the function F allows any germ $g \in \mathcal{C}_{F(p)}(N)$ to be pulled back to a germ $g \circ F$ in $\mathcal{C}_p(M)$. Then any tangent vector L in $T_p(M)$ is carried into a tangent vector $(DF)_p(L)$ in $T_{F(p)}(N)$ by the formula $(DF)_p(L)(g) = L(g \circ F)$. The result is a linear function $(DF)_p : T_p(M) \to T_{F(p)}(N)$ called the **derivative** of F at p.

The final preparatory step for working with manifolds-with-corners is to make smooth partitions of unity be available. We proceed exactly as at the end of Section II.2, beginning with analogs of Lemma 2.3 and 2.4.

Lemma 3.3. If U is a nonempty open subset of a smooth manifold-with-corners M and if f is in $C^\infty_{\text{com}}(U)$, then the function F defined on M so as to equal f on U and to equal 0 off U is in $C^\infty_{\text{com}}(M)$ and has support contained in U.

REMARK. This is proved in the same way that Lemma 1.2 was proved for smooth manifolds. The argument makes use of the Hausdorff property of M.

Lemma 3.4. Suppose that p is a point in a smooth manifold-with-corners M, that (M_α, α) is a compatible chart about p, and that K is a compact subset of M_α

containing p. Then there is a smooth function $f : M \to \mathbb{R}$ with compact support contained in M_α such that f has values in $[0, 1]$ and f is identically 1 on K.

REMARK. Except for changes in notation, this is proved in the same way as Lemma 2.4.

The notion of a **smooth partition of unity** of a manifold-with-corners M **subordinate** to the finite open cover $\{U_i\}$ of a compact subset K of M works just as in the case of smooth manifolds-with-boundary. The statement is as follows.

Proposition 3.5. Let M be a smooth manifold-with-corners, let K be a nonempty compact subset, and let $\{U_i \mid 1 \leq i \leq r\}$ be a finite open cover of K. Then there exist functions f_i in $C^\infty(M)$ for $1 \leq i \leq r$, taking values between 0 and 1 such that each f_i is identically 0 off a compact subset of U_i and $\sum_{i=1}^{r} f_i$ is identically 1 on K.

REMARK. Except for changes in notation, this is proved in the same way as Proposition 2.5.

2. Index and Strata

Let M be a smooth manifold-with-corners of dimension m. If p is in M and (M_α, α) is a chart about p, we have defined the index of p to be the number of integers k for which the member $\alpha(p)$ of $\mathbb{Q}^m$ has k^{th} coordinate 0. Proposition 3.1 showed that this number is independent of the chart, hence depends only on M and p. It is denoted by $\text{index}_M(p)$. It satisfies $0 \leq \text{index}_M(p) \leq m$.

The set M_+ of all points p of M with $\text{index}_M(p) = 0$ is a smooth manifold of dimension m, and we have defined those points to be **manifold points**. The remaining points, those with $\text{index}_M(p) \geq 1$, are sometimes called boundary points, and those with $\text{index}_M(p) \geq 2$ are sometimes called corner points.

We define $S_k(M) = \{p \in M \mid \text{index}_M(p) = k\}$ for $0 \leq k \leq m$, calling it the **stratum** of points in M of index k. It is plain that M is the disjoint union of its strata. Strata satisfy the additional conditions listed in the following proposition.

Proposition 3.6. If M is a smooth manifold-with-corners of dimension m, then its strata are such that

(a) each nonempty stratum $S_k(M)$ has the structure of smooth manifold of dimension $m - k$,
(b) for each k for which $S_k(M)$ is nonempty, the union of all strata $S_l(M)$ for $l \leq k$ is a manifold-with-corners of dimension m,
(c) the closure of $S_k(M)$ is the union of all strata $S_l(M)$ for $l \geq k$,
(d) M is a smooth manifold if and only $S_k(M)$ is empty for all $k > 0$, and

(e) M is a smooth manifold-with-boundary if and only $S_k(M)$ is empty for all $k > 1$, and in this case the boundary is $S_1(M)$.

REMARKS. An example to bear in mind is that of a solid cube M in $\mathbb{R}^3$. The stratum $S_0(M)$ is the interior, $S_1(M)$ is the union of the six faces but without edges and vertices, $S_2(M)$ is the union of the eight edges but without the vertices, and $S_3(M)$ is the set of eight vertices. It might seem unfortunate that $S_1(M) \cup S_2(M) \cup S_3(M)$ and $S_2(M) \cup S_3(M)$ are not manifolds-with-corners, but such features will not affect us because our concern is only with Stokes's Theorem.

WARNING. Although those unfortunate features do not concern us, they do affect some authors who have different goals. Often those authors will change one or another definition in the theory to achieve their purposes. For example, counting each vertex twice allows one to make $S_2(M) \cup S_3(M)$ into the disjoint union of four closed intervals; in this way $S_2(M) \cup S_3(M)$ becomes a manifold-with-boundary. It is therefore necessary always to be alert to an author's definitions of manifold-with-corners and related concepts.

PROOF. In (a) for the case that $M = \mathbb{Q}^m$, $S_k(\mathbb{Q}^m)$ is the set of points that lie on exactly k hyperplanes $\{x_i = 0\}$. This is a smooth manifold, being diffeomorphic to the disjoint union of Euclidean spaces of dimension $m - k$. For general M, if p is a point in $S_k(M)$ and (M_α, α) is a compatible chart of dimension m about $p \in M$, then the set $S_k(M) \cap M_\alpha$ and the restriction of α form a chart of dimension $m - k$ about $p \in S_k(M)$. These charts in $S_k(M)$ are compatible and provide an atlas for $S_k(M)$.

Conclusions (b), (d), and (e) follow directly from the definitions.

In (c), the result is clear for the case that M is $\mathbb{Q}^m$ or is a nonempty open subset of $\mathbb{Q}^m$. Hence if (M_α, α) is a compatible chart for M, then the closure of $S_k(M_\alpha)$ in M_α is the union of all strata $S_l(M_\alpha)$ for $l \geq k$. Consequently the closure of $S_k(M)$ in M contains the union of all strata $S_l(M_\alpha)$ for $l \geq k$. This being so for all α, the closure of $S_k(M)$ contains the union of all strata $S_l(M)$ for $l \geq k$. Arguing by contradiction, suppose that the closure contains a point p that is not in the union. This p must be a limit point of $S_k(M)$. Choose a chart (M_α, α) about p. Since the complement of M_α in M is closed, p must be a limit point of $S_k(M_\alpha)$. By what we have already proved, p must be in some $S_l(M_\alpha)$ for $l \geq k$. Then also p lies in the larger set $S_l(M)$, in contradiction to the assumption that p is not in the union of the $S_l(M)$ for $l \geq k$. $\square$

3. Stokes's Theorem for Manifolds-with-Corners

A version of Stokes's Theorem is valid for manifolds-with-corners, the formula being $\int_{\partial M} \omega = \int_M d\omega$ as usual. Proposition 3.1b, which says that $M_+ \cup S_1(M)$ is a manifold-with-boundary, gives us the proper framework. The integration over M is really to be an integral over the manifold M_+, and the integration over ∂M is to be an integration over $S_1(M)$. Just as M_+ is a dense open manifold in M, so too $S_1(M)$ is a dense open manifold of the topological boundary ∂M of M, according to Propositions 3.6c and 3.6a.

Accordingly it is a reasonable question to ask why Stokes's Theorem for manifolds-with-corners is not just a special case of Theorem 2.7. The answer is that Theorem 2.7 assumes that the given $m-1$ form ω has compact support in the manifold-with-boundary. An example will illustrate. Let M be the closed filled unit square in $\mathbb{R}^2$. This is a compact manifold-with-corners, but the associated manifold-with boundary omits the four corners, each of which has index 2. Theorem 2.7 thus asks that the given ω have compact support in the space consisting of the square with the four corners deleted. Running through the usual argument would thus show us that the formula of Green's Theorem, namely,

$$\int_{\partial M} P\,dx + Q\,dy = \int_M \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dx\,dy,$$

is valid whenever P and Q are smooth functions on the square that vanish in a neighborhood of each of the corners. Attempting to derive the theorem for general smooth P and Q on the square from this special case requires a passage to the limit that is more difficult to justify than the complete proof of Stokes's Theorem for manifolds-with-corners that we give later in this section. We shall not abandon the thought of handling matters by a passage to the limit, however, but shall merely postpone consideration of it until Section 4. A close look at the passage to the limit lies behind the theory of Whitney's that we develop starting in Section 4.

Let M be an m dimensional manifold-with-corners with $m \geq 2$, let ∂M be its boundary, and let M_+ be its subset of manifold points. We shall say that M is **orientable** (or **oriented**) if M_+ is orientable (or oriented). This definition is meaningful because M_+ is a smooth manifold. Then $S_1(M)$ acquires an **induced orientation** as in Section II.3, since $M_+ \cup S_1(M)$ is a manifold-with-boundary.

Theorem 3.7. Let M be an oriented manifold-with-corners of dimension $m \geq 2$, regard its boundary ∂M as $S_1(M)$, and give the boundary the induced orientation. If ω is any smooth $m-1$ form on M of compact support, then

$$\int_{\partial M} \omega = \int_M d\omega.$$

PROOF. The model space is $\mathbb{Q}^m$, and we first prove the theorem in this special case. The smooth $m-1$ form ω necessarily has an expansion

$$\omega = \sum_{j=1}^m F_j\,dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_m,$$

the circumflex pointing to a missing term. All the coefficient functions F_j are smooth, and we have

$$d\omega = \sum_{j=1}^{m} \frac{\partial F_j}{\partial x_j}\, dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_m = \sum_{j=1}^{m} (-1)^{j-1} \frac{\partial F_j}{\partial x_j}\, dx_1 \wedge \cdots \wedge dx_m.$$

Since the support of ω is compact, we may assume that each F_j vanishes outside $[0, R]^m$ for some number R. Theorem 1.29 gives

$$\int_{\mathbb{Q}^m} d\omega = (-1)^{j-1} \sum_{j=1}^{m} \int_{[0,R]^m} \frac{\partial F_j}{\partial x_j}(x_1, \dots, x_m)\, dx_1 \cdots dx_m$$

because $\mathbb{Q}^m$ has the standard orientation for $\mathbb{R}^m$. We can evaluate the j^{th} integration by the Fundamental Theorem of Calculus, obtaining

$$\begin{aligned}\int_0^R \frac{\partial F_j}{\partial x_j}(x_1, \dots, x_m)\, dx_j &= F_j(x_1, \dots, R, \dots, x_m) - F_j(x_1, \dots, 0, \dots, x_m) \\ &= -F_j(x_1, \dots, 0, \dots, x_m).\end{aligned}$$

Therefore

$$\int_{\mathbb{Q}^m} d\omega = (-1)^{j-1} \sum_{j=1}^{m} \int_{[0,R]^{m-1}} (-F_j)(x_1, \dots, 0, \dots, x_m)\, dx_1 \cdots \widehat{dx_j} \cdots dx_m. \tag{$*$}$$

To compute $\int_{\partial M} \omega = \int_{S_1(\mathbb{Q}^m)} \omega$, we have to sort out the orientation of each component of $S_1(\mathbb{Q}^m)$. There are m components, the j^{th} one being given by

$$Z_j = \{(x_1, \dots, x_m) \mid x_j = 0 \text{ and all other } x_i > 0\}.$$

To orient Z_1, for example, we take an outward pointing vector like $(-1, 0, \dots, 0)$, follow it by the standard basis for the subspace where $z_1 = 0$, and see what is needed to transform it into the standard basis of the whole space. The change requires one sign change and the identity permutation, and hence Z_1 has the opposite orientation from the standard one. For Z_j, we argue similarly, and its orientation is $(-1)^j$ times the standard one. Meanwhile, dx_j equals 0 on Z_j, and only one term of ω survives in the integration. Thus Theorem 1.29 gives

$$\begin{aligned}\int_{S_1(\mathbb{Q}^m)} \omega &= \sum_{j=1}^{m} \int_{Z_j} \omega \\ &= (-1)^j \sum_{j=1}^{m} \int_{[0,R]^{m-1}} F_j(x_1, \dots, 0, \dots, x_m)\, dx_1 \cdots \widehat{dx_j} \cdots dx_m.\end{aligned} \tag{$**$}$$

From $(*)$ and $(**)$, we conclude that

$$\int_{\mathbb{Q}^m} d\omega = \int_{S_1(\mathbb{Q}^m)} \omega, \tag{$\dagger$}$$

and the proof of the theorem is complete when $M = \mathbb{Q}^m$.

To handle the general case, we proceed in the same manner as in the proof of Theorem 2.7: About each point p in M of the compact support S of ω, we choose a *positive* compatible chart (M_α, α). Since the sets M_{α_j} form an open cover of the compact set S, we can choose a finite subcover $\{M_{\alpha_1}, \ldots, M_{\alpha_k}\}$. By Proposition 3.5 (instead of Proposition 2.5), let $\{\psi_1, \ldots, \psi_k\}$ be a smooth partition of unity of M subordinate to this finite open cover. For $1 \leq i \leq k$, the $m-1$ form $\psi_i \omega$ is compactly supported in M_{α_i}, and the $m-1$ form $(\alpha^{-1})^*(\psi_i\omega)$ is compactly supported in $\alpha_i(M_{\alpha_i}) \subseteq \mathbb{Q}^m$. Let us extend it to all of $\mathbb{Q}^m$ by setting it equal to 0 off $\alpha_i(M_{\alpha_i}) \subseteq \mathbb{Q}^m$, leaving its name unchanged. Then

$$\begin{aligned}
\int_M d(\psi_i\omega) = \int_{M_{\alpha_i}} d(\psi_i\omega) &= \int_{\alpha_i(M_{\alpha_i})} (\alpha_i^{-1})^*(d(\psi_i\omega)) && \text{by Theorem 1.29} \\
&= \int_{\mathbb{Q}^m} (\alpha_i^{-1})^*(d(\psi_i\omega)) && \text{after extension by 0} \\
&= \int_{\mathbb{Q}^m} d((\alpha_i^{-1})^*(\psi_i\omega)), && \text{by Proposition 1.24} \\
&= \int_{\partial\mathbb{Q}^m} (\alpha_i^{-1})^*(\psi_i\omega) && \text{by } (\dagger) \\
&= \int_{\partial M_{\alpha_i}} \psi_i\omega = \int_{\partial M} \psi_i\omega && \text{by Theorem 1.29.}
\end{aligned}$$

Summing over i from 1 to k and using the fact that $\sum_{i=1}^{k} \psi_i$ is identically 1, we obtain

$$\int_M d\omega = \sum_{i=1}^{k} \int_M d(\psi_i\omega) = \int_{\partial M} \Big(\sum_{i=1}^{k} \psi_i\omega\Big) = \int_{\partial M} \omega,$$

and the proof of the general case is complete. □

4. Whitney's Generalization of the Divergence Theorem

Although Theorem 3.7 handles many situations of practical interest for Stokes's Theorem, it by no means handles all. In Section 1 we saw at least five examples of spaces of geometric interest that could almost be handled by Theorem 3.7 but did not fit the hypotheses completely. In four of those examples, we identified ad hoc techniques that reduced those examples to ones that could be handled directly. Those techniques all essentially amounted to introducing a specific triangulation to subdivide the space into simpler spaces for which Theorem 3.6 could apply directly.[2] The sum of the Stokes formulas for the simpler spaces yielded the Stokes formula for the given space.

Working with triangulations is hard and asks for more of a geometric grasp of the space globally than we often have. In addition, we had no technique at all for handling the circular cone in Example 7. So we need a new device. The new device will come down to justifying the kind of passage to the limit that we tried to avoid early in the previous section. The main theorem that will incorporate that passage to the limit is Hassler Whitney's form of Stokes's Theorem.

In this section let us concentrate on situations where the underlying manifold or generalized manifold is a subset of $\mathbb{R}^m$ of full dimension m. This is the core of the problem. Effectively we are thus working on generalizing the m dimensional Divergence Theorem, which handled this case for manifolds-with-boundary and manifolds-with-corners when the space in question can be realized as a subset of $\mathbb{R}^m$. We shall see how one theorem of Whitney's handles all situations in $\mathbb{R}^m$ without further effort. We postpone to Section 6 a general theorem about cases of Stokes's Theorem that are not embedded in $\mathbb{R}^m$.

An example to keep in mind is the one in Example 7 that we could not handle, namely that of a filled ice-cream cone. So that we can concentrate on the vertex, let us think of the cone as infinite in size. The thought that suggests itself is that we might be able to handle the cone as a manifold-with-corners if we were to remove some of it near the vertex, and perhaps then we could pass to the limit.

Thus we return to the question we set aside early in Section III.3. If we have an exceptional set E on the boundary that we do not have tools to handle, can we discard the exceptional set so as to obtain a noncompact manifold-with-boundary, apply Theorem 2.7 to any compactly supported $m-1$ form ω on the manifold-with-boundary, and then pass to the limit to eliminate the support restriction on ω? Whitney's answer is yes as long as the exceptional set is not too large in a technical sense.

To fix the ideas, let U be a nonempty bounded open set in $\mathbb{R}^m$ with (compact) topological boundary B, and let E be a compact subset of B that we think of as small and exceptional. We shall impose conditions on B so that $(B-E)\cup U$ is a noncompact manifold-with-boundary. We are to be given a smooth $m-1$ form ω on $B\cup U$, with smoothness meaning as usual that in an open neighborhood of each point of $B\cup U$, ω extends to a smooth $m-1$ form on the open neighborhood.

[2] We have not sought techniques for handling general roughness of the differential forms that are involved. We work only with smooth forms and regard rough ones as not of practical interest.

We want to prove the Stokes formula $\int_{(B-E)} \omega = \int_U d\omega$ without making any assumption about the support of ω.

Write $D(x, E)$ for the distance from a point x in $\mathbb{R}^m$ to the compact set E. The key to quantifying the smallness of E is the order of magnitude of the Lebesgue measure of the open set where $D(x, E) < \delta$ when $\delta > 0$ is small; we may think of this open set as a thickened version of E. For example if $m = 2$ and E is a one-point set, then the set where $D(x, E) < \delta$ is a disk of radius δ, whose measure is $\pi\delta^2$. Still in $\mathbb{R}^2$ if E instead is a line segment of length 1, then the set where $D(x, E) < \delta$ has the shape of a filled racetrack, and its measure is $2\delta + \pi\delta^2$. In other words a one-point set leads us to the order of magnitude of δ^2, whereas a line segment leads us to the order of magnitude of δ. This distinction is what will allow us to handle each missing corner of a square in Green's Theorem, but we would not be able to handle a whole missing side.

More generally let $|A|$ be the Lebesgue measure of a Borel subset A of $\mathbb{R}^m$. Whitney's generalization of the Divergence Theorem in dimension m, given as Theorem 3.8 below,[3] will say that the condition

$$\lim_{\delta \downarrow 0} \delta^{-1} |\{x \in \mathbb{R}^2 \mid D(x, E) < \delta\}| = 0$$

is just the right hypothesis to allow us to ignore the exceptional set E and treat the whole generalized manifold as an ordinary manifold-with-boundary. We shall investigate sets E with this property in the next section.

In the meantime let us observe that a one-point set E in dimension m always satisfies this condition because $\delta^{-1}|\{x \in \mathbb{R}^2 \mid D(x, E) < \delta\}|$ is approximately a constant times δ^{m-1} for small δ. We already saw a number of cases in Section 1 where E consists of just a single point, and we shall recall them after proving the theorem. They will furnish our first examples where the theorem applies.

Theorem 3.8. Let U be a nonempty bounded open set in $\mathbb{R}^m$ with $m \geq 2$, let B be its topological boundary, and let E be a closed subset of B. Suppose further that $(B - E) \cup U$ is a smooth manifold-with-boundary of dimension m in the following sense:

> to each point p of $B - E$, there exists a unit vector $v(p)$ such that if axes in $\mathbb{R}^m$ are chosen with $v(p)$ in the x_1 direction, then the set of points of $B - E$ in some neighborhood of p is given by a smooth function $x_1 = h(x_2, \dots, x_n)$ and the set of points of U in this neighborhood is given by the inequality $x_1 < h(x_2, \dots, x_m)$.

[3] This is Theorem 14A in Whitney's book listed in the Selected References. The theorem here is what Whitney's published theorem says in case the differential form ω has no smoothness problems up to and including the boundary. The published theorem allows the differential form to have a certain amount of roughness.

Suppose further that E has the property that either E is empty or

$$\lim_{\delta \downarrow 0} \delta^{-1} |\{x \in \mathbb{R}^m \mid D(x, E) < \delta\}| = 0. \tag{$*$}$$

Let U be given the standard orientation from $\mathbb{R}^m$, and let $B - E$ be given the induced orientation. If ω is a compactly supported smooth $m - 1$ form on $B \cup U$, then the Stokes formula holds in the sense that

$$\int_{B-E} \omega = \int_U d\omega. \tag{$**$}$$

REMARKS.

(1) Let us refer to the triple (U, B, E) in the theorem as a **Whitney domain** in $\mathbb{R}^m$. In his book Whitney himself referred to such triples with U connected as "standard domains." In our case, on the one hand, we want to treat certain examples with U disconnected, such as a filled numeral 8, as Whitney domains, and on the other hand, connectedness plays no role in the proof of Theorem 3.8. Thus we have deviated from Whitney's treatment and dropped the hypothesis of connectedness.

(2) The inset condition in the theorem describes certain charts about points of $B - E$ and tells how to orient them relative to the orientation of the underlying space $\mathbb{R}^m$. It really amounts to the condition that $(B - E) \cup U$ is a smooth manifold-with-boundary, saying in addition that all the charts are positively oriented with the induced orientation.

(3) If E is empty, condition $(*)$ is to be ignored, and the theorem still applies. In this case it amounts to the m dimensional Divergence Theorem for the compact smooth manifold-with-boundary $M = U \cup B$ and is a special case of Theorem 2.7.

(4) The condition that ω is smooth is to be understood to mean that about each point of $U \cup B$, the differential form ω extends to a smooth differential form in an open set of $\mathbb{R}^m$. Concretely this means that in a neighborhood of the point, ω has an expansion $\sum_{j=1}^{m} F_j(x, \dots, x_m)\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_m$ with each F_j smooth in a neighborhood of the point.

PROOF. Fix a closed ball X in $\mathbb{R}^m$ large enough to contain all the points of interest. We shall approximate ω by smooth forms ω_k that have compact support in $(B - E) \cup U$, apply Theorem 2.7 to each ω_k, and then pass to the limit.

Let I_k be the indicator function of the subset of $x \in X$ where $D(x, E) \geq 2^{-k}$, i.e., let $I_k(x)$ be 1 on that subset and 0 off the subset. Let J_k be the indicator function of the set of x where $D(x, E) < 2^{-k}$. Then $I_k(x) = 1 - J_k(x)$ for $x \in X$. Fix a smooth function $\varphi \geq 0$ on $\mathbb{R}^m$ that is supported on the closed unit ball and has $\int_{\mathbb{R}^m} \varphi\, dx = 1$, and let $\varphi_{k+1} = 2^{(k+1)m} \varphi(2^{(k+1)} x)$. The function φ_{k+1} is ≥ 0, has total integral 1, and is supported on the ball where $|x| \leq 2^{-(k+1)}$.

The function $I_k * \varphi_{k+1}$ is smooth and vanishes off the set where $D(x, E) \geq 2^{-(k+1)}$. The differential form $\omega_k = (I_k * \varphi_{k+1})\omega$ is smooth on $B \cup U$ and vanishes off the set where $D(x, E) \geq 2^{-(k+1)}$. Consequently it is a compactly supported form on the manifold-with-boundary $(B - E) \cup U$, and Theorem 2.7 applies to it. The theorem gives

$$\int_{B-E} \omega_k = \int_U d\omega_k$$

for all k. We shall prove that

$$\lim_k \int_{B-E} \omega_k = \int_{B-E} \omega \qquad \text{and} \qquad \lim_k \int_U d\omega_k = \int_U d\omega, \tag{†}$$

and then we will have proved $(**)$ and the theorem.

Let us examine the difference

$$\begin{aligned} \omega - \omega_k &= \omega(1 - (I_k * \varphi_{k+1})) \\ &= \omega(1 - (1 - J_k) * \varphi_{k+1}) \\ &= \omega(J_k * \varphi_{k+1}). \end{aligned} \tag{††}$$

The function $J_k * \varphi_{k+1}$ vanishes off the set where $D(x, E) > 2^{-(k-1)}$ and is ≤ 1 everywhere. Thus $\lim \omega_k = \omega$ pointwise in the complement of E, and dominated convergence applies to yield the first formula of (†).

Toward the second formula of (†), let us use Proposition 1.23a to write

$$d\omega - d\omega_k = d(J_k * \varphi_{k+1}) \wedge \omega + (J_k * \varphi_{k+1})\, d\omega$$

and

$$\Big| \int_U d\omega - \int_U d\omega_k \Big| \leq \Big| \int_U d(J_k * \varphi_{k+1}) \wedge \omega \Big| + \Big| \int_U (J_k * \varphi_{k+1})\, d\omega \Big|. \tag{‡}$$

The easy term to handle in (‡) is the second term. In it the form $d\omega$, being smooth on $B \cup U$, is integrable on U, and we saw in the previous paragraph that $J_k * \varphi_{k+1}$ tends to 0 pointwise off E, always being ≤ 1. Thus

$$\lim_k \textstyle\int_U (J_k * \varphi_{k+1}) d\omega = 0$$

by dominated convergence, i.e., the second term of (‡) tends to 0.

The first term of (‡) involves a sum of terms $(\partial/\partial x_j)(J_k * \varphi_{k+1})(dx_j \wedge \omega)$. Since $dx_j \wedge \omega$ is smooth on the compact set $B \cup U$, integration with it operates as the product of a bounded function by Lebesgue measure. Thus to show that the first term of (‡) tends to 0, it is enough to show that the integral of the coefficient function $(\partial/\partial x_j)(J_k * \varphi_{k+1})$ with respect to Lebesgue measure tends to 0. Let us abbreviate $\partial/\partial x_j$ as ∇_j and consider the coefficient $\nabla_j(J_k * \varphi_{k+1}) = J_k * \nabla_j \varphi_{k+1}$.

By the chain rule, $\nabla_j \varphi_{k+1}(x) = 2^{(k+1)} 2^{m(k+1)} (\nabla_j \varphi)(2^{(k+1)x})$, and we can write this as $\nabla_j \varphi_{k+1}(x) = 2^{k+1}(\nabla_j \varphi)_{k+1}$ if we continue to use subscript notation for dilations by powers of 2. With $\|\cdot\|_1$ denoting the L^1 norm with respect to Lebesgue measure, we have

$$\nabla_j(J_k * \varphi_{k+1}) = J_k * \nabla_j \varphi_{k+1} = J_k * 2^{k+1}(\nabla_j \varphi)_{k+1}$$

and
$$\begin{aligned}\|\nabla_j(J_k * \varphi_{k+1})\|_1 &\le 2^{k+1}\|J_k\|_1 \|(\nabla_j \varphi)_{k+1}\|_1 \\ &= 2^{k+1}\|\nabla_j \varphi\|_1 \big|\{x \in \mathbb{R}^m \mid D(x, E) < 2^{-k}\}\big|.\end{aligned}$$

The right side is a multiple of $\delta^{-1}\big|\{x \in \mathbb{R}^m \mid D(x, E) < \delta\}\big|$ for $\delta = 2^{-k}$, and it tends to 0 by hypothesis $(*)$. Thus the first term of (‡) tends to 0, and this completes the proof of the second formula of (†). $\square$

EXAMPLES. We have observed that the exceptional set certainly satisfies condition $(*)$ if it consists of just finitely many points, provided $m \ge 2$. The following were potential examples in dimension 2 in this situation that were mentioned in Section 1. We now see that they are all Whitney domains and that Theorem 3.8 is therefore applicable:

(1) any manifold-with-corners of dimension $m = 2$ embedded in $\mathbb{R}^2$, and in particular any filled compact convex polygon in dimension 2,

(2) any filled simple polygon in dimension 2, convex or not,

(3) any filled simple region in dimension 2 with finitely many curved sides even if those curved sides make angles of 0, 180, or 360 degrees with one another,

(4) a filled numeral 8 in $\mathbb{R}^2$.

In the next section we shall examine condition $(*)$ more closely, and we shall be led to examples with more complicated exceptional sets.

5. Sets with ℓ Dimensional Minkowski Content Zero

Let us examine more closely the condition $(*)$ on the exceptional set E so that Theorem 3.8 applies, namely that

$$\lim_{\delta \downarrow 0} \delta^{-1} |\{x \in \mathbb{R}^m \mid D(x, E) < \delta\}| = 0.$$

For $0 \le \ell \le m$ and $m \ge 1$, we define the ℓ **dimensional Minkowski content** of a nonempty compact set E in $\mathbb{R}^m$ to be

$$\mathcal{M}^\ell(E) = \lim_{\delta \downarrow 0} \big|\{x \in \mathbb{R}^m \mid D(x, E) < \delta\}\big| \big/ (\alpha_{m-\ell}\delta^{m-\ell})$$

if this limit exists. Here $\alpha_{m-\ell}$ is the $m-\ell$ dimensional volume of the unit ball in $\mathbb{R}^{m-\ell}$ if $\ell < m$, and we take $\alpha_{m-\ell}$ to be 1 if $\ell = m$. If the limit does not exist, then one refers to the lim sup and lim inf as the "upper ℓ dimensional Minkowski content" and "lower ℓ dimensional Minkowski content" of E, respectively. If $\ell = m$, the m dimensional Minkowski content of a compact set exists and equals the Lebesgue measure of the set.

In the setting of Theorem 3.8, ℓ equals $m-1$, and the assumption $(*)$ in the theorem is that the limit exists and equals 0. *Thus the assumption* $(*)$ *is that the* $m-1$ *dimensional Minkowski content of* E *is* 0.

In what follows it will simplify statements to adopt the convention that the ℓ dimensional Minkowski content of the empty set is 0.

It is useful to keep in mind the following example. With $\ell \le m$, suppose that E is an ℓ dimensional cube of side 1 positioned in m dimensional space as the product $[0,1]^\ell \times \{0\}^{m-\ell}$. To compute the volume of the δ neighborhood of E, we can integrate 1 over that neighborhhood. The integration then extends in each of the first ℓ variables over an interval of length between $1+\delta$ and $1+2\delta$, while in the last $m-\ell$ variables it extends over the ball of radius δ centered at the origin in $\mathbb{R}^{m-\ell}$, whose volume is $\alpha_{m-\ell}$. The result of the integration thus has to be something between $\alpha_{m-\ell}\delta^{m-\ell}(1+\delta)^\ell$ and $\alpha_{m-\ell}\delta^{m-\ell}(1+2\delta)^\ell$. Dividing by $\alpha_{m-\ell}\delta^{m-\ell}$ and letting δ tend to 0, we obtain $\mathcal{M}^\ell(E) = 1$. Thus the ℓ dimensional cube E in $\mathbb{R}^m$ has ℓ dimensional Minkowski content 1; the Minkowski content of that cube is 0 in dimensions larger than ℓ and is infinite in dimensions smaller than ℓ.

The set function $\mathcal{M}^\ell(E)$ is not asserted to be defined on all compact subsets of $\mathbb{R}^m$, but when it is defined, it is anyway nonnegative, and it has the property that if A and B are compact sets, then

$$\mathcal{M}^\ell(E \cup F) \le \mathcal{M}^\ell(E) + \mathcal{M}^\ell(F)$$

with equality if E and F are disjoint. In fact, the containment

$$\{x \in \mathbb{R}^m \mid D(x, E \cup F) < \delta\} \subseteq \{x \in \mathbb{R}^m \mid D(x, E) < \delta\} \cup \{x \in \mathbb{R}^m \mid D(x, F) < \delta\}$$

is valid for all δ; if E and F are disjoint and nonempty, then they are at a positive distance δ_0 from one another and the above containment is an equality for $\delta \le \delta_0$.

Because of condition $(*)$ in Theorem 3.8, our main interest is in what happens when the ℓ dimensional Minkowski content exists and equals 0 for a compact subset of $\mathbb{R}^m$ when $\ell \le m$. Let us record three easy facts about that situation:

(1) If E_1 and E_2 are compact in $\mathbb{R}^m$ with $E_2 \subseteq E_1$ and if E_1 has ℓ dimensional Minkowski content 0, then so does E_2.

(2) If E_1 and E_2 are compact in $\mathbb{R}^m$ with E_1 and E_2 having ℓ dimensional Minkowski content 0, then the same thing is true of $E_1 \cup E_2$.

(3) If the compact set E in $\mathbb{R}^m$ has finite ℓ dimensional Minkowski content, then E has k dimensional Minkowski content 0 for every k with $\ell < k \le m$, as follows by comparing the definitions of $\mathcal{M}^k(E)$ and $\mathcal{M}^\ell(E)$.

Let us pause and assess what this little theory tells us for Theorem 3.8. A Whitney domain in $\mathbb{R}^m$ was defined in effect as the closure of a nonempty bounded open set U in $\mathbb{R}^m$ such that the topological boundary B can be written as $B = (B - E) \cup E$, where $U \cup (B - E)$ is an m dimensional manifold-with-boundary and E is a compact subset of B of $m - 1$ dimensional Minkowski content 0.

However E is defined as a compact subset of B, the hope is that E has dimension $m-2$ or less and that consequently it has $m-1$ dimensional Minkowski content 0. The reality is that E is often hard to deal with. Accordingly we shall introduce some tools for working with the notion of ℓ dimensional Minkowski content 0.

To begin with, the definition of Minkowski content of a nonempty compact set E supplies a value that depends on external information about E. We shall establish an equivalent definition that depends only on internal information about E. Define

$$E^\delta = \{x \in \mathbb{R}^m \bigm| D(x, E) > \delta\} \qquad \text{and} \qquad B^\delta = \{x \in \mathbb{R}^m \bigm| |x| < \delta\}.$$

Since E is compact, only finitely many open balls of radius $< \delta$ are needed to cover E. Let

$$N(E, \delta) = \begin{Bmatrix} \text{minimum number of open balls of} \\ \text{diameter} < \delta \text{ needed to cover } E \end{Bmatrix}$$

and

$$N_{\mathrm{sep}}(E, \delta) = \begin{Bmatrix} \text{maximum number of points of } E \\ \text{at distance} \geq \delta \text{ from one another} \end{Bmatrix}.$$

Lemma 3.9. For E compact and nonempty in $\mathbb{R}^m$,

(a) $N(E, \delta) \leq N_{\mathrm{sep}}(E, \delta)$.
(b) $N_{\mathrm{sep}}(E, \delta) \leq N(E, \delta/2)$,
(c) $|E^\delta| \leq N(E, \delta)|B_\delta|$, and
(d) $|E^\delta| \geq N_{\mathrm{sep}}(E, \delta)|B_{\delta/2}|$.

PROOF. Write $B_r(x)$ for the open ball of all points y in $\mathbb{R}^m$ with $|y - x| < r$.

For (a), if $k = N_{\mathrm{sep}}(E, \delta)$, choose a set $S = \{x_1, \dots, x_k\}$ of points of E such that $|x_i - x_j| \geq \delta$ for all $i \neq j$. The balls $B_{2\delta}(x_1), \dots, B_{2\delta}(x_k)$ must cover E because otherwise some point y of E has $|x_i - y| \geq 2\delta$ for all i and $S \cup \{y\}$ is a set of $k + 1$ points of E at distance $\geq \delta$ from one another. Thus some system of k open balls of radius 2δ covers E. Shrinking each of these balls a sufficiently small amount still leaves them covering E but having radius $< 2\delta$, therefore diameter $< \delta$. The number $N(E, \delta)$ is by definition $\leq$ this number k, and therefore $N(E, \delta) \leq N_{\mathrm{sep}}(E, \delta)$.

For (b), if $k = N_{\mathrm{sep}}(E, \delta)$, choose a set $S = \{x_1, \dots, x_k\}$ of points of E with $|x_i - x_j| \geq \delta$ for all $i \neq j$. If $\mathcal{C}$ is a collection of balls $B_{r_1}(y_1), \dots, B_{r_n}(y_n)$ of radius $< \delta/4$ that cover E, then we can associate to each index j of the members of S some index $i = i(j)$ of the members of $\mathcal{C}$ such that $B_{r_i}(y_i)$ contains x_j. No two members of S can be in any single $B_{r_i}(y_i)$ because the diameter of $B_{r_i}(u_i)$ is less than $2r_i$, which is less than $\delta/2$. Thus the function $j \mapsto i(j)$ is one-one from S into $\mathcal{C}$. This proves that the number of balls is $\geq$ the number of points in S. Hence the minimum possible number of balls is $\geq N_{\text{sep}}(E, \delta)$.

For (c), if $k = N(E, \delta)$, let $\mathcal{C} = \{B_{r_1}(x_1), \dots, B_{r_k}(x_k)\}$ be a collection of k open balls of radius $< \delta/2$ in $\mathbb{R}^m$ that cover E. If x is in E^δ, the compactness of E implies that there is a point y in E with $|x - y| = \delta$. The point y must lie in some $B_{r_j}(x_j)$, and thus $|x - x_j| \leq |x - y| + |y - x_j| \leq \delta/2 + r_j < \delta$. Thus the collection of balls $B_\delta(x_1), \dots B_\delta(x_k)$ covers E^δ, and we must have $|E^\delta| \leq k|B_\delta|$, as asserted.

For (d), if $k = N_{\text{sep}}(E, \delta)$, choose a set $S = \{x_1, \dots, x_k\}$ of points of E such that $|x_i - x_j| \geq \delta$ for all $i \neq j$. The balls $B_{\delta/2}(x_j)$ are pairwise disjoint and lie completely in E^δ. Thus $|E^\delta| \geq |B_{\delta/2}(x_1)| + \cdots + |B_{\delta/2}(x_k)| = k|B_{\delta/2}| = N_{\text{sep}}(E, \delta)|B_{\delta/2}|$. □

Proposition 3.10. If $\ell \leq m$, a nonempty compact set E in $\mathbb{R}^m$ has ℓ dimensional Minkowski content equal to 0 if and only if

$$\lim_{\delta \downarrow 0} \delta^\ell N(E, \delta) = 0,$$

where

$$N(E, \delta) = \begin{Bmatrix} \text{minimum number of open balls of} \\ \text{diameter} < \delta \text{ needed to cover } E \end{Bmatrix}.$$

REMARK. In view of parts (a) and (b) of Lemma 3.9, it would be equivalent to write the condition as $\lim_{\delta \downarrow 0} \delta^\ell N_{\text{sep}}(E, \delta) = 0$. This equality depends only on E as a metric space and does not make use of any embedding. However, we will find the formulation of the condition as $\lim_{\delta \downarrow 0} \delta^\ell N(E, \delta) = 0$ to be more useful.

PROOF. Applying (a), (d), and (c) of Lemma 3.9 in turn, we obtain

$$N(E, \delta)|B_{\delta/2}| \leq N_{\text{sep}}(E, \delta)|B_{\delta/2}| \leq |E^\delta| \leq N(E, \delta)|B_\delta| = 2^m N(E, \delta)|B_{\delta/2}|,$$

and thus

$$2^{-m}\delta^m N(E, \delta)|B_1| \leq |E^\delta| \leq \delta^m N(E, \delta)|B_1|.$$

The proposition follows. □.

A function F from a nonempty subset of $\mathbb{R}^a$ into $\mathbb{R}^b$ is said to satisfy a **Lipschitz condition** on a set E with constant C if $|F(x) - F(y)| \leq C|x - y|$ for all x and y in E. It follows from Taylor's Theorem with integral remainder[4] that any smooth function from an open convex set in $\mathbb{R}^a$ into $\mathbb{R}^b$ satisfies a Lipschitz condition when restricted to any compact subset of the domain.

[4] Theorem 3.11 of *Basic Real Analysis*.

Proposition 3.11. Let F be a function from a compact subset E of $\mathbb{R}^a$ into $\mathbb{R}^b$ that satisfies a Lipschitz condition, and suppose that $\ell \geq 0$ is an integer. If E has ℓ dimensional Minkowski content equal to 0 in $\mathbb{R}^a$, then $F(E)$ has ℓ dimensional Minkowski content equal to 0 in $\mathbb{R}^b$.

REMARK. No relationship between a and b is assumed.

PROOF. Decomposing F as the composition of a dilation followed by a function satisfying a Lipschitz condition with Lipschitz constant 1, we see that it is enough to prove the corollary in the case that the Lipschitz constant is 1. Under this assumption let E be a compact subset of $\mathbb{R}^a$ that has ℓ dimensional Minkowski content 0. We may assume that E is nonempty. In view of Proposition 3.10, we are assuming that $\lim_\delta \delta^\ell N(E, \delta) = 0$, and we want to prove that $\lim_\delta \delta^\ell N(F(E), \delta) = 0$.

Let E be covered by N open balls of diameter $< \delta$, say

$$E \subseteq B_{r_1}(x_1) \cup \cdots \cup B_{r_k}(x_k).$$

Then

$$F(E) \subseteq F(B_{r_1}(x_1)) \cup \cdots \cup F(B_{r_k}(x_k)),$$

and the right side is

$$\subseteq B_{r_1}(F(x_1)) \cup \cdots \cup B_{r_k}(F(x_k))$$

because F satisfies a Lipschitz condition with Lipschitz constant 1. This shows that

$$N(F(E), \delta)) \leq N(E, \delta),$$

and Proposition 3.11 follows from Proposition 3.10. $\square$

Proposition 3.11 allows us to introduce a well defined notion of ℓ dimensional Minkowski dimension 0 for compact subsets of any smooth manifold of dimension $\geq \ell$ and to show that smooth mappings of these manifolds into any Euclidean space of dimension $\geq \ell$ carry these sets into compact sets of ℓ dimensional Minkowski content 0 in the Euclidean space. The details are as follows.

Corollary 3.12. Let M be a smooth manifold of dimension m, let $\ell \geq 0$ be an integer, and let E be a nonempty compact subset of M. Suppose that $\{M_\alpha, \alpha)\}$ is an atlas for M such that some finite open cover $\{M_{\alpha_1}, \ldots, M_{\alpha_r}\}$ of E has the property that for each j with $1 \leq j \leq r$, each compact subset S of $M_{\alpha_j} \cap E$ has $\alpha_j(S)$ of ℓ dimensional Minkowski content 0 in $\mathbb{R}^n$. Then for every (M_β, β) in the atlas, each compact subset T of $M_\beta \cap E$ has $\beta(T)$ of ℓ dimensional Minkowski content 0 in $\mathbb{R}^n$.

REMARKS.

(1) When a finite open cover of E exists as in the lemma, we say that the compact subset E of M has ℓ **dimensional Minkowski content 0**.

(2) With this definition the finite union of compact subsets of ℓ dimensional Minkowski content 0 in a smooth manifold of dimension $n \geq \ell$ has ℓ dimensional Minkowski content 0. This is a consequence of the corresponding fact about compact subsets of Euclidean space.

(3) With only cosmetic changes in the proof, this corollary remains valid if "smooth manifold" in the statement is replaced by "smooth manifold-with-boundary" or "smooth manifold-with-corners."

PROOF. Fix the open cover $\{M_{\alpha_1}, \ldots, M_{\alpha_r}\}$ of E, and choose by Lemma 1.26b an open subcover $\{P_{\alpha_1}, \ldots, P_{\alpha_r}\}$ of E such that $P_{\alpha_j}^{\text{cl}} \subseteq M_{\alpha_j}$ for each j. Suppose that M_β is any member of the atlas and that T is a compact subset of $M_\beta \cap E$. Then $T = (P_{\alpha_1}^{\text{cl}} \cap T) \cup \cdots \cup (P_{\alpha_r}^{\text{cl}} \cap T)$ exhibits T as the union of respective compact subsets $P_{\alpha_j}^{\text{cl}} \cap T$ of $M_{\alpha_j} \cap E$. The set $\alpha_j(P_{\alpha_j}^{\text{cl}} \cap T)$ is a compact subset of $\alpha_j(M_{\alpha_j} \cap E)$ and by hypothesis has ℓ dimensional Minkowski content 0 in $\mathbb{R}^m$.

Let us apply Proposition 3.11 to the smooth mapping $F = \beta \circ \alpha_j^{-1}$, which is a diffeomorphism from the open set $\alpha_j(M_{\alpha_j} \cap M_\beta)$ onto the open set $\beta(M_{\alpha_j} \cap M_\beta)$. Since $\alpha_j(P_{\alpha_j}^{\text{cl}} \cap T)$ is a compact subset of $\alpha_j(M_{\alpha_j} \cap M_\beta)$ of ℓ dimensional Minkowski content 0, its image $\beta(P_{\alpha_j}^{\text{cl}} \cap T)$ under F is a compact subset of $\beta(M_{\alpha_j} \cap M_\beta) \subseteq \mathbb{R}^m$ of ℓ dimensional Minkowski content 0. Taking the union over j, we see that $\beta(T)$ has ℓ dimensional Minkowski content 0 in $\mathbb{R}^m$. $\square$

It is now easy to extend certain results about ℓ dimensional Minkowski content 0 from Euclidean space to smooth manifolds. Some extensions of this kind appear in the problems at the end of the chapter.

GEOMETRIC EXAMPLES.

(1) The above results allow us to see that various polyhedral sets meet condition $(*)$ for exceptional sets in Theorem 3.8. A filled square pyramid in $\mathbb{R}^3$ has four vertices, eight edges, five faces, and the solid part. The Stokes formula involves the solid part and the faces. All other potential contributions are compact of dimension ≤ 1, which is two less than the ambient dimension, and there are only finitely many of them. Corollary 3.14 says that each of them has 2 dimensional Minkowski content 0, and the finite union of compact sets of 2 dimensional Minkowski content 0 has Minkowski content 0. Therefore condition $(*)$ in Theorem 3.8 is satisfied, and the Stokes formula holds for a solid square pyramid.

(2) More generally any closed convex **polytope** in $\mathbb{R}^m$, i.e., the generalization to dimension m of a closed convex polyhedron in $\mathbb{R}^3$, fits this description. Aside from the solid and the faces, all other potential contributions can be taken to

be compact of dimension $\leq m-2$, and there are only finitely many of them. Corollary 3.14 says that each of them has $m-1$ dimensional Minkowski content 0, and the finite union of compact sets of $m-1$ dimensional Minkowski content 0 has Minkowski content 0. Thus again condition $(*)$ is satisfied, and theorem 3.8 applies.

(3) In any manifold-with-corners of dimension m that is embedded in Euclidean space $\mathbb{R}^m$, the exceptional set that arises in Theorem 3.8 consists of all points of index ≥ 2, i.e., of all corner points. The subset of corner points that lies within the support of a given smooth $m-1$ form is compact, and Corollary 3.14 says that this subset satisfies condition $(*)$. Thus Theorem 3.7 is a special case of Theorem 3.8 if the given manifold-with-corners embeds in $\mathbb{R}^m$. A filled ice cream cone in $\mathbb{R}^3$ is an example. The full version of Stokes's Theorem that we give in the next section will apply to all Whitney manifolds and in particular will apply to all smooth manifolds-with-corners, whether embedded in $\mathbb{R}^m$ or not.

A further class of examples for Theorem 3.8 is of an algebraic nature and arises from zero loci of real polynomials in several real variables $x_1, \ldots, x_m$. We shall assume that a given polynomial F is a function of m variables and is irreducible over $\mathbb{R}$. Guided by Theorem 3.8, we consider the region in $\mathbb{R}^m$ where $F < 0$. The statement of that theorem gives us a clue what to expect with the boundary. At a point on the topological boundary, if $\partial F/\partial X_j$ is nonzero for some j, then the Implicit Function Theorem allows us to solve locally for x_j in terms of the other variables, obtaining a smooth function f of $m-1$ variables, and locally a part of the boundary of the region will be the graph of f with the part of $\mathbb{R}^m$ below the graph corresponding to the interior of the region under study. The subset of the boundary for which this condition holds is thus part of the boundary of a manifold-with-boundary in the familiar sense. The subset of the boundary for which the condition fails is called the **singular set** of F and is taken as the exceptional set E in the theorem.

When we are applying Theorem 3.8, it is helpful for our regions in $\mathbb{R}^m$ to be bounded, so that integrals are well defined, and we think of intersecting our set of interest with a large closed ball $\{x \bigm| |x| \leq C\}$ for some C. Since the goal is to have a theorem for differential forms of compact support, we take always take C large enough so that every point of the support has $|x| \leq C$, and the part of the boundary where $|x| = C$ does not enter the Stokes formula. The adjustment of requiring $|x| \leq C$ results in temporarily enlarging the boundary so that some points with $|x| = C$ are included. These new boundary points are uninteresting for our current purposes, since they play no role at the end.[5]

[5]This description is not quite good enough. To avoid problems from the sharp edge of the region where $|x| = C$, we actually work with the region where $F < 0$ and a specific smooth auxiliary function in $C^\infty_{\text{com}}(\mathbb{R}^m)$ is > 0. The auxiliary function can be taken to be $\varphi(C^{-1}x)$, where φ is a function ≥ 0 in $C^\infty_{\text{com}}(\mathbb{R}^m)$ that is identically 1 for $|x| \leq \frac{1}{2}$ and is identically 0 for $|x| \geq 1$. This auxiliary function is smooth and equals 0 for $|x| \geq C$.

ALGEBRAIC EXAMPLES.

(1) With two variables, let $F(x, y) = x^4 + y^4 - 1$. Our region becomes the set where $x^4 + y^4 < 1$. This is a bounded region of $\mathbb{R}^2$. The respective first partial derivatives of F are $4x^3$ and $4y^3$, and they do not simultaneously vanish at any point of our locus. Thus the exceptional set E is empty, and Theorem 3.8 for this case reduces to the ordinary Divergence Theorem for $\mathbb{R}^2$, hence to Green's Theorem in the plane if we adjust notation suitably. The business with introducing a large closed ball is unnecessary since our region is already bounded.

(2) Let the underlying space be $\mathbb{R}^4$, which we can identify with the space of 2-by-2 real matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ if we want. We take F to be the determinant function $ad - bc$, and we consider the set of all matrices x for which $\det x \leq 0$. To a first approximation the open set U in Theorem 3.8 will be the set of all matrices x for which $\det x < 0$, and the topological boundary B will be the set where $\det x = 0$. However, we are not interested in effects from considering large matrices, and we therefore consider only those matrices x for which $|x| \leq C$ for some positive constant C, where $|x|^2$ is the sum of the squares of the entries. Thus the actual U is the set of x with $\det x < 0$ and $|x| < C$. The actual topological boundary B consists of an interesting part where $\det x = 0$ and $|x| < C$ and an uninteresting part where $|x| = C$. The first partial derivatives of det are d, $-b$, $-c$, and a, respectively, and they vanish simultaneously only when $x = 0$. The point with $x = 0$ happens to be one of the points on the locus $\det x = 0$. Thus the singular set consists of $x = 0$ alone.

Thus the interesting part of the boundary B consists of the all points where $\det x = 0$. Points x in its nonsingular part have $x \neq 0$, and the exceptional set E consists of 0 alone.[6] Since a one-point set satisfies condition $(*)$ of Theorem 3.8, (U, B, E) is a Whitney domain, and the Stokes formula is applicable in this situation.

(3) Let the underlying Euclidean space be $\mathbb{R}^9$ realized as the space of all 3-by-3 real matrices. We study the set where $\det x \leq 0$. Again we want to know where $\det x = 0$, and we want to identify the singular set. Each matrix entry function $x \mapsto x_{ij}$ is a coordinate function, and we want to examine the first partial derivative $\partial(\det x)/\partial x_{ij}$. Thus let e_{ij} be the matrix for which $x_{ij}(e_{ij}) = \delta_{ij}$. By definition,

$$\frac{\partial(\det x)}{\partial x_{ij}} = \frac{d}{dt}\det(x + tx_{ij})\Big|_{t=0} = \lim_{t\to 0} t^{-1}(\det(x + te_{ij}) - \det x).$$

Since det is an alternating multilinear function of its columns, the expression within the outer parentheses on the right equals the determinant of a matrix

[6] The uninteresting part of B consists of all points with $|x| = C$. No point with $|x| = C$ has all four first partial derivatives equal to 0, and therefore the singular set for this example is completely contained in the interesting part of the boundary.

that equals x in all but the j^{th} column and there equals the j^{th} column of te_{ij}. Expanding the determinant by cofactors, we see that the limit collapses to $(-1)^{i+j}$ times the $(i, j)^{\text{th}}$ minor[7] of x. So the partial derivative that we seek is just a 2-by-2 minor of x. The set where all first partial derivatives vanish is exactly the set where all 2-by-2 minors are 0, which is the set of all matrices of rank at most 1. The condition on the minors implies that $\det x = 0$, and consequently the singular set of the locus where $\det x = 0$ is the set of matrices of rank ≤ 1. Let E be this set.

We shall want to apply Theorem 3.8. The open set U has dimension 9, and the nonsingular part $B - E$ of the boundary has dimension 8. What we might expect is that somehow the singular set E has dimension at most 7, and then condition $(*)$ ought to be satisfied in the theorem. The set E is not a manifold, however, and some care is needed. What we really want is for the compact set E to have 8 dimensional Minkowski content 0. To see this, we shall write E as the union of 9 compact subsets of 5 dimensional vector subspaces of $\mathbb{R}^9$, and each of these compact subsets will have 8 dimensional Minkowski content 0; then Remark 2 after Corollary 3.12 will allow us to conclude that E has 8 dimensional Minkowski content 0. For the moment fix attention on the first row and column of matrices, and consider a member x of E with $x_{11} \neq 0$. Since each x in E has rank ≤ 1, the second and third columns of this x must be multiples of the first column. The set of matrices for which the second and third columns are multiples of the first is a linear subspace of $\mathbb{R}^9$ of dimension 5, and x lies in this subspace. (The first column contributes 3 to the dimension, and each multiple contributes one more.)

We can argue similarly with each of the nine pairs of indices (i, j), not just $(1, 1)$. If a member x of E has $x_{ij} \neq 0$, then x lies in a certain (different) 5 dimensional vector subspace of $\mathbb{R}^9$. The member 0 of E lies in all of these subspaces. The conclusion is that E lies in the union of nine specific subspaces of $\mathbb{R}^9$ of dimension 5. The intersection of E with each subspace is closed, hence compact, and thus E is exhibited as the finite union of compact sets lying in 5 dimensional subspaces. We have seen that any compact subset of $\mathbb{R}^k$ has $m-1$ dimensional Minkowski content 0 if $k < m-1$. Here we have $k = 5$ and $m = 9$, and the conclusion is that E has 8 dimensional Minkowski content 0.

Therefore condition $(*)$ is met, and Theorem 3.8 applies. Once again we are skipping lightly over the uninteresting part of the boundary where $|x| = C$. We may do so because we are interested only in differential forms of compact support.[8] Anyway the Stokes formula applies to differential forms of degree 9 with U as the set of 3-by-3 matrices of negative determinant, with $B - E$ as the

[7]The $(i, j)^{\text{th}}$ **minor** of an n-by-n matrix is the determinant of the matrix of size $n-1$ obtained by deleting the i^{th} row and j^{th} column.

[8]In this example, there are matrices in the set E that lie on the sphere $|x| = C$, but they can be ignored because of the smoothing technique mentioned in an earlier footnote.

set of 3-by-3 matrices x of rank 2, and with E as the set of 3-by-3 matrices of rank ≤ 1.

6. Whitney's Global Form of Stokes's Theorem

For the final stage in our work with Stokes's Theorem, we shall expand our repertory of model cases. Then we can piece together local results to get the global form of Stokes's Theorem that we seek. The setting will be a "Whitney manifold" of dimension m, an object that we define below. In the end we will allow three types of model cases: $\mathbb{R}^m$, $\mathbb{H}^m$, and Whitney domains in $\mathbb{R}^m$, Whitney domains having been defined in the remarks with Theorem 3.8. It would be enough to use Whitney domains themselves as the sole kind of model, but it will help us to include $\mathbb{R}^m$ and $\mathbb{H}^m$ so that we can easily handle manifold points and well behaved boundary points with models that do not involves Whitney domains.

There is one subtle qualitative difference between the settings of manifolds-with-boundary and manifolds-with-corners vs. the setting of Whitney manifolds. In the earlier settings, there were different kinds of points: manifold points and boundary points in the case of manifolds-with-boundary, and points of different index in the case of manifolds-with-corners. Telling one kind of point apart from another was a question intrinsic to the point. With a Whitney domain (U, B, E) and therefore also with Whitney manifolds, the distinction between different kinds of points is no longer intrinsic. Indeed, we shall still have manifold points corresponding to U, ordinary boundary points corresponding to $B - E$, and exceptional boundary points corresponding to E, but it is always possible to change the label of one boundary point in B from ordinary to exceptional without affecting the validity of Theorem 3.8. Thus identifying exceptional points depends at least partly on how we label them. In order to have a theory that parallels the theories of manifolds-with-boundary and manifolds-with-corners, it will be necessary to carry along this information about labels in some of our definitions. As we make the definitions, it will be helpful to keep one nontrivial example in mind.

EXAMPLE. The surface S of an ice-cream cone in $\mathbb{R}^3$. The curved part of the surface can be realized as

$$\{(x, y, z) \mid x^2 + y^2 = z^2 \text{ and } 0 \leq z \leq 1\},$$

let us say. The points $\{(x, y, z) \mid x^2 + y^2 = 1 \text{ and } z = 1\}$ can be taken to be ordinary points of the boundary, and the point $(0, 0, 0)$ is an exceptional point of the boundary. This example is not a smooth manifold-with-boundary because of the behavior near the origin, and it is not covered by Theorem 3.8 because the surface is not a subset of dimension 3 in $\mathbb{R}^3$. Thus at this stage we do not know whether the Stokes formula is valid for S or not. Theorem 3.12 will affirm that it is indeed valid. In the statement of the formula, the integration over the boundary turns out to be limited to the 1 dimensional part of the boundary; the point $(0, 0, 0)$ plays no role.

Fix an integer $m \geq 2$. A **local Whitney domain** in $\mathbb{R}^m$ is the intersection of a Whitney domain $W = (U, B, E)$ in $\mathbb{R}^m$ with an open set O of $\mathbb{R}^m$ under the assumption that $U \cap O$ is nonempty. The subset of $\mathbb{R}^m$ of interest is then $(U \cup B) \cap O$, and the triple is $W \cap O = (U \cap O, B \cap O, E \cap O)$. The set $B \cap O$ is relatively closed in the closure of U, and $E \cap O$ is relatively closed in $B \cap O$. Observe that the set $E \cap O$ need not be compact.

We shall define "Whitney manifolds" M of dimension m. Let M be a locally compact separable metric space, let ∂M be a closed subset of M, and let $\partial_0 M$ be a closed subset of ∂M. The points of ∂M will be called the **boundary points** of M, and the points of $\partial_0 M$ will be called the **exceptional points**. Either of ∂M or ∂M_0 is allowed to be empty. For purposes of defining M as a Whitney manifold, a **Whitney chart** (M_α, α) on M of **dimension** m is a homeomorphism α of a nonempty open subset M_α of M onto some local Whitney domain $W_\alpha \cap O_\alpha$ in $\mathbb{R}^m$, say with $W_\alpha = (U_\alpha, B_\alpha, E_\alpha)$, such that the restriction of α to $M_\alpha \cap \partial M$ is a homeomorphism onto $B_\alpha \cap O_\alpha$ and the restriction of α to $M \cap \partial_0 M$ is a homeomorphism onto $E_\alpha \cap O_\alpha$. The image of α is understood to be $(U_\alpha \cup B_\alpha) \cap O_\alpha$. When the local Whitney domain has no exceptional points, i.e., when $E_\alpha \cap O_\alpha$ is empty, a Whitney chart is just an ordinary chart.

The Whitney chart (M_α, α) is said to be **about** a point p in M if p is in the domain M_α of α.

On such a space M, two charts (M_α, α) and (M_β, β) for which $M_\alpha \cap M_\beta$ is nonempty will be said to be smoothly **compatible** if $\beta \circ \alpha^{-1}$, as a mapping of the subset $\alpha(M_\alpha \cap M_\beta)$ of $\mathbb{R}^m$ to the subset $\beta(M_\alpha \cap M_\beta)$ of $\mathbb{R}^m$, is smooth and its inverse $\alpha \circ \beta^{-1}$ is smooth. As usual, smoothness at a boundary point means that the function extends to a smooth function in a neighborhood of the boundary point.

The locally compact separable metric space M is said to be a **Whitney manifold** of dimension m if a system $\mathcal{F}$ of Whitney charts $(M_\alpha, \varphi_\alpha)$ on M of dimension m is specified such that

(i) any two charts (M_α, α) and (M_β, β) in $\mathcal{F}$ are smoothly **compatible**,
(ii) the system of compatible charts (M_α, α) is an **atlas** in the sense that the sets M_α together cover M, and
(iii) $\mathcal{F}$ is maximal among families of compatible charts on M.

The next step is to review for Whitney manifolds all the constructions of smooth functions, tangent spaces, differential forms, etc. that we did for smooth manifolds, then for smooth manifolds-with-boundary, and finally for smooth manifolds-with-corners and check that the whole theory goes through with no surprises. This step is repetitious, and we omit it.

Let M be a Whitney manifold of dimension m. We say that M is **oriented** if the smooth manifold $M - \partial M$ is oriented. In this case, $\partial M - \partial_0 M$ is the finite or countably infinite union of its open components, each of which is a connected smooth manifold of dimension $m - 1$. We give each component the orientation induced from $M - \partial M$, and the result is that $\partial M - \partial_0 M$ becomes an oriented smooth manifold of dimension $m - 1$.

Theorem 3.12. Suppose that M is an oriented Whitney manifold of dimension m with boundary ∂M and exceptional set $\partial_0 M$, and suppose further that $\partial M - \partial_0 M$ is given the induced orientation. If ω is a compactly supported smooth differential $m - 1$ form on M, then the Stokes formula holds for M in the sense that

$$\int_{\partial M - \partial_0 M} \omega = \int_{M - \partial M} d\omega.$$

REMARKS. This theorem is based on Theorem 18A of Whitney's book in the Selected References. What we have stated here is mostly formalism, the deep result being Theorem 3.8 above. However, Theorem 3.12 is not a tautology, since as we shall see, it does say something new about the surface of an ice-cream cone in $\mathbb{R}^3$.

The notion of a **smooth partition of unity** of a Whitney manifold M **subordinate** to the finite open cover $\{U_i\}$ of a compact subset K of M works just as in the cases of smooth manifolds, smooth manifolds-with-boundary, and smooth manifolds-with-corners. This step too requires a little checking, and we omit it. The statement is as follows.

Lemma 3.13. Let M be an Whitney manifold, let K be a nonempty compact subset, and let $\{U_i \mid 1 \leq i \leq r\}$ be a finite open cover of K. Then there exist functions f_i in $C^\infty(M)$ for $1 \leq i \leq r$, taking values between 0 and 1, such that each f_i is identically 0 off a compact subset of U_i and $\sum_{i=1}^{r} f_i$ is identically 1 on K.

PROOF OF THEOREM 3.12. About each point p in M of the compact support S of ω, we choose a *positive* compatible Whitney chart (M_α, α). This is possible since the positive compatible charts form an atlas, M being oriented. Since the sets M_{α_j} form an open cover of the compact set S, we can choose a finite subcover $\{M_{\alpha_1}, \dots, M_{\alpha_k}\}$. By Lemma 3.13 let $\{\psi_1, \dots, \psi_k\}$ be a smooth partition of unity of M subordinate to this finite open cover.

For each i with $1 \leq i \leq k$, $\alpha_i(M_{\alpha_i})$ is open in one of the model spaces $\mathbb{R}^m$, $\mathbb{H}^m$, or a Whitney domain (U, B, E), and $\psi_i \omega$ is compactly supported within that open subset of the model space. Since the model space is Hausdorff, the extension of $\psi_i \omega$ by 0 on the complement of $\alpha_i(M_{\alpha_i})$ is compactly supported and smooth on the whole model space.

For $1 \le i \le k$, the $m-1$ form $\psi_i\omega$ is compactly supported in M_{α_i}, and the $m-1$ form $(\alpha^{-1})^*(\psi_i\omega)$ is compactly supported in the open subset $\alpha_i(M_{\alpha_i})$ of one of the model spaces. Let us extend it to the whole model space by setting it equal to 0 off $\alpha_i(M_{\alpha_i})$, leaving its name unchanged. The computation is then the same in all cases, but the notation has to be interpreted a little differently when the model space is a Whitney domain. When the model space is $\mathbb{R}^m$ or $\mathbb{H}^m$, the computation is

$$\begin{aligned}
\int_M d(\psi_i\omega) = \int_{M_{\alpha_i}} d(\psi_i\omega) &= \int_{\alpha_i(M_{\alpha_i})} (\alpha_i^{-1})^*(d(\psi_i\omega)) && \text{by Theorem 1.29} \\
&= \int_{\text{model}} (\alpha_i^{-1})^*(d(\psi_i\omega)) && \text{after extension by 0} \\
&= \int_{\text{model}} d((\alpha_i^{-1})^*(\psi_i\omega)), && \text{by Proposition 1.24} \\
&= \int_{\partial(\text{model})} (\alpha_i^{-1})^*(\psi_i\omega) && \text{by Stokes for model} \\
&= \int_{\partial M_{\alpha_i}} \psi_i\omega = \int_{\partial M} \psi_i\omega && \text{by Theorem 1.29.}
\end{aligned}$$

In the above computation the first five integrations are understood to extend over the set of manifold points, not the full space indicated, and with that understanding we get the desired equality $\int_M d(\psi_i\omega) = \int_{\partial M} \psi_i\omega$. The expression "Stokes for model" refers to Theorem 2.1 or 2.7.

When the model space is a Whitney domain, the expression "Stokes for model" refers to Theorem 3.8. The first five lines of the above display again extend over the set of manifold points, and that is the way that Theorem 3.12 writes them. The integrations over the boundary extend only over the ordinary points of the boundary, according to Theorem 3.8, and an adjustment to the above notation needs to be made to take this fact into account.

In short we obtain the formula

$$\int_{M-\partial M} d(\psi_i\omega) = \int_{\partial M-\partial_0 M} \psi_i\omega$$

in every case. Summing over i from 1 to k and using the fact that $\sum_{i=1}^{k} \psi_i$ is identically 1, we obtain

$$\int_{M-\partial M} d\omega = \sum_{i=1}^{k} \int_{M-\partial M} d(\psi_i\omega) = \int_{\partial M-\partial_0 M} \Big(\sum_{i=1}^{k} \psi_i\omega\Big) = \int_{\partial M-\partial_0 M} \omega,$$

and the proof of the theorem is complete. $\square$

EXAMPLE. The surface S of an ice-cream cone in $\mathbb{R}^3$, continued. Let us see that the surface is a Whitney manifold of dimension 2 with just one Whitney chart. The image of the chart is the Whitney domain in $\mathbb{R}^2$ given by the punctured unit disk, with the puncture considered as an exceptional point of the boundary. Thus $U = \{(a, b) \in \mathbb{R}^2 \mid 0 < a^2 + b^2 < 1\}$, $B = \{(0, 0)\} \cup \{(a, b) \mid a^2 + b^2 = 1\}$, and $E = \{(0, 0)\}$. The chart is (S, φ) with the mapping $\varphi : S \to U \cup B$ given by

$$(a, b) = \varphi(x, y, z) = (x, y) \quad \text{for} \quad 0 \le x^2 + y^2 \le 1.$$

Since E consists of a single point, we have seen E satisfies the key hypothesis $(*)$ in Theorem 3.8, and therefore (U, B, E) is a Whitney domain.[9] The function φ is a homeomorphism of S onto the closed unit disk $U \cup B$. Since there is just one chart, no compatibility of charts needs to be proved. Theorem 3.12 applies. According to the theory, computations proceed just as with Green's Theorem for the unit disk; the exceptional point $(0, 0)$ plays no role in the integrations.

7. Problems

1. (a) A **compact convex polyhedron** in $\mathbb{R}^3$ is a compact set that does not lie in a single plane and that is the intersection of finitely many closed half planes. It has a number F of 2 dimensional faces, a number E of 1 dimensional edges, and a number V of 0 dimensional vertices. According to a formula due to Euler, these numbers are related by $F + V = E + 2$. Assume that the polyhedron is nondegenerate in the sense that no three vertices are collinear, and for simplicity assume that it is in "general position," which means that no four vertices are coplanar. Prove that the polyhedron can be triangulated, i.e., that it can be be written as the union of tetrahedra in such a way that each vertex of a tetrahedron is a vertex of the original polyhedron and that any two tetrahedra either are disjoint or intersect in a single face.

   (b) Deduce Stokes's Theorem for compact convex polyhedra in $\mathbb{R}^3$ from the result for tetrahedra, which is an instance of Theorem 3.7. Handle the necessary cancellation in the boundary integral in the same way as in Example 4 of Section 1.

2. Show that a compact manifold-with-corners of dimension m that is embedded in $\mathbb{R}^m$ is an example of a Whitney domain of dimension m, the exceptional set consisting of all points of index ≥ 2.

[9] It is possible to verify $(*)$ using the more sophisticated theory of Section 5 rather than the direct computation that appeared in Section 4. In the terminology of Section 5, E has 1 dimensional Minkowski content equal to 0 because, for example, it has finite nonzero 0 dimensional Minkowski content.

3. Guided by the third algebraic example in Section 5, show that a bounded portion of the subset of the space of 4-by-4 real matrices where $\det x \le 0$ can be made into a Whitney domain for which the exceptional set E is the set of all matrices of rank ≤ 2.

4. For which of the following functions and vector spaces of matrices does the procedure of the algebraic examples of Section 5 lead to a Whitney domain (U, B, E)? Describe B and E in each case.
   (a) $F(x, y, z) = z(z - xy)$ and the space $\mathbb{R}^4$,
   (b) $F(x) = \mathrm{Re}(\det(x))$ and the space of all 2-by-2 complex matrices.
   (c) $F(x) = \det(x)$ and the space of all skew-symmetric 4-by-4 real matrices,

Problems 5–9 concern the Divergence Theorem.

5. Let V be the solid in $\mathbb{R}^3$ given by
$$\{(x, y, z) \mid x^2 + y^2 + (z-2)^2 \le 4 \quad \text{and} \quad x^2 + y^2 + (z+1)^2 \le 1\}.$$
   (a) Check that V is a manifold-with-corners.
   (b) If S is the surface of V, evaluate $\int_S x^2\, dy \wedge dz$, where S is oriented via an outward pointing vector.

6. Evaluate $\int_S F \cdot d\mathbf{S}$, where $F = 3y\mathbf{i} + 2x\mathbf{j} + (z-8)\mathbf{k}$ and S is the surface of the solid in $\mathbb{R}^3$ bounded by the coordinate planes $x = 0$, $y = 0$, and $z = 0$, and by the plane $4x + 2y + z = 8$. Again S is oriented by an outward pointing vector.

7. Let S be the surface in $\mathbb{R}^3$ defined by
$$x^4 + y^4 + z^4 = a^4,$$
where $a > 0$ is chosen so that the region V enclosed by S has volume 7. Let $\omega = x\, dy \wedge dz + y dz \wedge dx + z dx \wedge dy$, and let S be oriented toward the outside. Evaluate the integral $\int_S \omega$.

8. Let $F(x, y, z)$ be the vector field
$$F(x, y, z) = z^2 \log(1 + y^2)\mathbf{i} + (5y + 2x^2)\mathbf{j} + (\cos^4 x + 3y)\mathbf{k}.$$
If S is the surface of the half ball where $x^2 + y^2 + z^2 \le 4$ and $z \ge 0$, compute $\int_S F \cdot d\mathbf{S}$ if S is oriented with an outward pointing vector.

9. Let M be a compact manifold-with-boundary embedded in $\mathbb{R}^2$, and suppose that $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$ are smooth functions such that $f < g$ everywhere.
   (a) Show that the subset
$$V = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in M \text{ and } f(x, y) \le z \le g(x, y)\}$$
   is a manifold-with-corners.
   (b) Identify subsets U, B, and E of $\mathbb{R}^3$ so that V can be viewed as the Whitney domain (U, B, E).

Problems 10-12 concern integration over piecewise C^∞ curves and other geometric objects that lend themselves to a canonical decomposition into pieces.

10. Let f be a (continuous) **piecewise smooth function** from a closed interval $I = [a, b]$ into $\mathbb{R}$. Specifically there is to be a partition, say

$$a = t_0 < t_1 < \cdots < t_k = b \quad \text{with } k \geq 1$$

such that $f(t)$ is a continuous function on $[a, b]$ and is of class C^∞ on each of $I_j = [t_{j-1}, t_j]$ for $1 \leq j \leq k$. Put $f_j = f|_{[t_{j-1}, t_j]}$ for $1 \leq j \leq k$.
   (a) Taking into account all the assumptions on f, verify that $\int_{I_j} f'(t)\,dt = f(t_j) - f(t_{j-1})$ for $1 \leq j \leq k$ and conclude that $\int_I f'(t)\,dt = f(b) - f(a)$.
   (b) Interpret the results of (a) via Section II.6d as saying that Stokes's Theorem holds for the 0 form $\omega = f$ on $[a, b]$ and the 1 form $d\omega = f'(t)\,dt$ even though ω is only piecewise smooth. (Educational note: In other words, Stokes's Theorem readily extends in $\mathbb{R}^1$ from smooth 0 forms to piecewise smooth 0 forms.)
   (c) Relate the cancellation that occurred in (a) to a question about orientations, and say what abstract hypothesis on orientations to impose in order to ensure this cancellation.

11. Proceeding similarly with objects in one higher dimension, introduce a notion of a piecewise smooth function on the faces and edges of a tetrahedron, and derive a version of Stokes's Theorem for the surface of a tetrahedron, the boundary integral being an integral of a 1 form on the union of the edges, all consistently oriented.

12. If the same procedure is followed with a square pyramid, is there any substantial difference in what happens?

Problems 13–19 primarily concern the notion of ℓ dimensional Minkowski content $\mathcal{M}^\ell(E)$.

13. Let $\ell \geq 0$ be an integer, and let $F : M \to N$ be a smooth mapping between smooth manifolds of dimension $\geq \ell$. Prove that if E is a compact subset of ℓ dimensional Minkowski content 0 in M, then $F(E)$ is a compact subset of ℓ dimensional content 0 in N. (The notion of ℓ dimensional Minkowski content 0 in the setting of a smooth manifold is defined in Corollary 3.12 and its remarks.)

14. Let M be a smooth manifold of dimension $m \geq 2$. Prove that the smooth image in M of any compact subset of a smooth manifold of dimension $\leq m - 2$ has $m - 1$ dimensional Minkowski content 0.

15. Show that any compact m dimensional manifold-with-corners, not necessarily embedded in $\mathbb{R}^m$, is an example of a Whitney manifold of dimension m.

16. In his book Whitney defined a set E in $\mathbb{R}^m$ to be of **zero ℓ extent** if the following is true: For each $\epsilon > 0$, there is some $\zeta_0 > 0$ such that for any $\zeta \leq \zeta_0$ there are balls $B_1, \dots, B_k$ for some k such that

$$E \subseteq B_1 \cup \cdots \cup B_k, \qquad \operatorname{diam}(B_i) \leq \zeta \text{ for all } i, \qquad k\zeta^\ell < \epsilon.$$

In his formulation of the result given as Theorem 3.8 here, he required that the exceptional set E be of zero $m-1$ extent. Prove that a nonempty compact set of $\mathbb{R}^m$ is of zero ℓ extent if and only if it has ℓ dimensional Minkowski content 0.

17. If E_1 and E_2 are nonempty compact subsets of $\mathbb{R}^{a_1}$ and $\mathbb{R}^{a_2}$, respectively, so that $E_1 \times E_2$ is a subset of $\mathbb{R}^{a_1+a_2}$, prove that

$$N(E_1 \times E_2, \delta) \leq N(E_1, \delta/2)N(E_2, \delta/2),$$

where $N(E, \delta)$ is as in Section 5.

18. If E is a nonempty compact subset of $\mathbb{R}^a$ and $N(E, \delta)$ is as in Section 5, prove that $\limsup_{\delta \downarrow 0} \delta^a N(E, \delta)$ is finite.

19. Suppose that E_1 and E_2 are compact subsets of $\mathbb{R}^{a_1}$ and $\mathbb{R}^{a_2}$, respectively, and suppose further that E_1 has ℓ_1 dimensional Minkowski content 0, where $\ell_1 \leq a_1$.
    (a) Prove that if E_2 is a compact subset of $\mathbb{R}^{a_2}$ with ℓ_2 dimensional Minkowski content 0, where $\ell_2 \leq a_2$, then $E_1 \times E_2$ is a compact subset of $\mathbb{R}^{a_1+a_2}$ of $\ell_1 + \ell_2$ dimensional Minkowski content 0.
    (b) Prove that if E_2 is a compact subset of $\mathbb{R}^{a_2}$, then $E_1 \times E_2$ is a compact subset of $\mathbb{R}^{a_1+a_2}$ of $\ell_1 + a_2$ dimensional Minkowski content 0.

20. Let (U_1, B_1, E_1) be a Whitney domain in $\mathbb{R}^{m_1}$, and let M be a compact smooth manifold-with-boundary of dimension m_2 in $\mathbb{R}^{m_2}$. Write M_+ for the set of manifold points in M and ∂M for the boundary.
    (a) Under the special assumption that (U_1, B_1, E_1) arises as in the geometric examples of Section 5 from a bounded portion of the subset of $\mathbb{R}^m$ where a real-valued polynomial F of m variables is < 0, prove that the product $(U, B, E) = (U_1, B_1, E_1) \times M$ has the natural structure of a Whitney domain in $\mathbb{R}^{m_1+m_2}$ if one defines

$$U = U_1 \times M_+,$$

$$B = (U_1 \times \partial M) \cup (B_1 \times M_+),$$

and

$$E = \big(E_1 \times M\big) \cup \big(B_1 \times \partial M\big).$$

    (b) Does the conclusion of (a) still hold without the special assumption that (U_1, B_1, E_1) arises from a bounded portion of the subset of $\mathbb{R}^m$ where a real-valued polynomial takes on negative values?