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Stokes's Theorem and Whitney Manifolds:

A Sequel to
Basic Real Analysis

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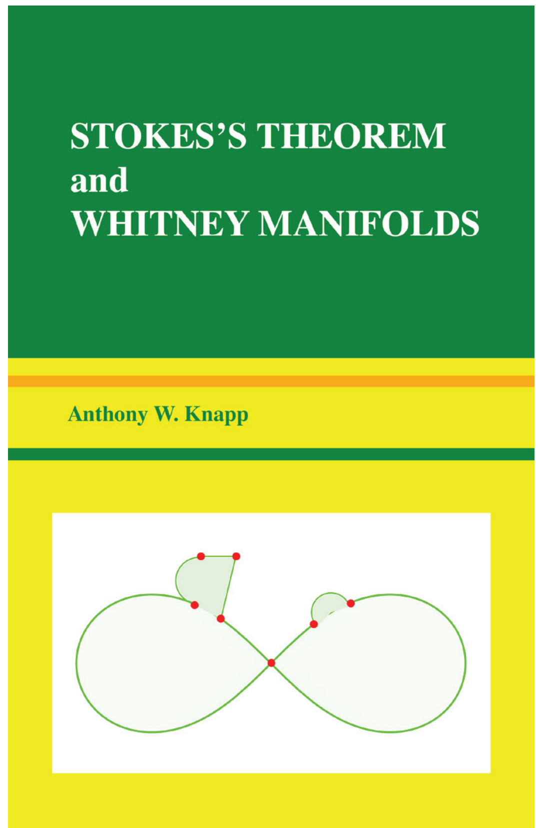

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Cover: An example of a Whitney domain in two-dimensional space. The green portion is a manifold-with-boundary for which Stokes's Theorem applies routinely. The red dots indicate exceptional points of the boundary where a Whitney condition applies that says Stokes's Theorem extends to the whole domain.

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CHAPTER II

Manifolds-with-Boundary

Abstract. This chapter introduces oriented manifolds-with-boundary, obtains Stokes's Theorem for them, and shows that the classical theorems of Green, Gauss-Ostrogradsky, and Kelvin–Stokes fit into this framework.

Section 1 introduces the subject by working with ordinary oriented smooth manifolds, i.e., those oriented smooth manifolds without boundary. Stokes's Theorem for this situation reduces to a theorem about compactly supported differential forms in Euclidean space.

Section 2 introduces smooth manifolds-with-boundary of dimension m, charts being homeomorphisms from nonempty open subsets of the manifold-with-boundary onto relatively open subsets of the closed half space $\mathbb{H}^m$ of $\mathbb{R}^m$. One distinguishes manifold points and boundary points and observes that the set of manifold points yields a smooth manifold. The section defines smoothness of real-valued functions and associated objects, and for this setting, it goes through much of the same kind of development that was done for manifolds in Chapter I.

Section 3 defines orientability of a smooth manifold-with-boundary to mean orientability of the smooth manifold of manifold points. If a smooth manifold-with-boundary is orientable, then so is its boundary, and a particular choice of orientation of the boundary, known as the induced orientation, is defined so that the signs will eventually work out properly in Stokes's Theorem.

Section 4 states and proves Stokes's Theorem for oriented smooth manifolds-with-boundary, handling the case of dimension $m = 1$ separately from the case of dimension $m \geq 2$.

Section 5 examines the meaning of Stokes's Theorem in the settings that give rise to three classical integration theorems—Green's Theorem, the Divergence Theorem, and the Kelvin–Stokes Theorem—and in the setting of line integrals independent of the path.

1. Stokes's Theorem for Manifolds without Boundary

This section establishes Stokes's Theorem for oriented "manifolds without boundary," which is to say, for oriented manifolds in the sense of Chapter I.[1] It will always be assumed that the differential forms that appear in integrals have compact

[1]Terminology differs among mathematicians—whether manifolds are restricted to the kind that was defined in Chapter I or whether manifolds can have embellishments, such as some kind of attached boundary. For this section we stick to the kind that was defined in Chapter I. Starting in Section 2, we shall work with "manifolds-with-boundary," which are not necessarily manifolds in the sense of Chapter I. Instead they come with extra points satisfying some special conditions. The use of hyphens in the name "manifold-with-boundary" will be a continuing reminder that a manifold-with-boundary is not necessarily a manifold.

support, i.e., that they are 0 outside of some compact subset of the manifold. On a compact manifold this condition is automatically satisfied.

All forms of Stokes's Theorem are local theorems in the following sense: The heart of the matter is to prove the theorem in a "model space," the model space for manifolds of dimension m being $\mathbb{R}^m$. The validity of the theorem in the model space and the local nature of the result imply the validity of the theorem in any chart. Finally an orientation allows for the results for single charts to be added up with the help of a partition of unity.[2] It is as if the manifold in question is divided into pieces, and then the proof of the theorem proceeds one piece at a time and the results added. The virtue of using a partition of unity is that the borders between the pieces are smoothed out so as to avoid technical problems arising from discontinuities.[3]

Theorem 2.1. If M is a smooth oriented manifold of dimension m, then every smooth $m-1$ form ω with compact support on M has

$$\int_M d\omega = 0.$$

REMARKS. The prototype for this theorem with M noncompact is the case that $M = \mathbb{R}^1$. In this case, d is the usual differentiation operator on functions (regarded as 0 forms), and the statement comes down to the assertion that $\int_{\infty}^{\infty} f'(x)\,dx = 0$ for any f in $C^\infty_{\text{com}}(\mathbb{R}^1)$. This assertion is immediate from the Fundamental Theorem of Calculus. The prototype for this theorem with M compact is the case that M is the circle. We may then think in terms of smooth periodic functions of period 2π on the line, and the statement comes down to the assertion that any smooth f of period 2π on the line has $\int_{-\pi}^{\pi} f'(x)\,dx = 0$. Again the Fundamental Theorem of Calculus applies, giving $f(\pi) - f(-\pi) = 0$ as the value of the integral.

PROOF. We shall use the same approach in proving each version of Stokes's Theorem—the one here for manifolds, the one in Section II.4 for manifolds-with-boundary, the one in Section III.3 for manifolds-with-corners, and the one in Section III.6 for Whitney manifolds. The main step is to prove the theorem for the model space, which in this case is $\mathbb{R}^m$.

Thus we consider the special case $M = \mathbb{R}^m$ with the standard orientation. We may assume that ω is not 0. The support S of ω being compact, we choose real

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[2] In other words, the only potential obstruction to extending Stokes's Theorem from a local result to a global result is the possible failure of the underlying manifold to be oriented.

[3] On the other hand, the shortcoming of using a partition of unity is that the method does not lend itself to actual computations.

numbers a_j and b_j for $1 \leq j \leq m$ such that all points $x = (x_1, \ldots, x_m)$ of S have $a_j < x_j < b_j$ for all j. The smooth $m-1$ form ω necessarily has an expansion

$$\omega = \sum_{r=1}^{m} F_r(x_1, \ldots, x_m)\, dx_1 \wedge \cdots \wedge \widehat{dx_r} \wedge \cdots \wedge dx_m,$$

the circumflex indicating a missing term. All the coefficient functions F_r are smooth and are equal to 0 off the compact set S. Then we have

$$\begin{aligned} d\omega &= \sum_{r=1}^{m} \sum_{s=1}^{m} \tfrac{\partial F_r}{\partial x_s}\, dx_s \wedge dx_1 \wedge \cdots \wedge \widehat{dx_r} \wedge \cdots \wedge dx_m \\ &= \sum_{r=1}^{m} (-1)^{r-1} \big(\tfrac{\partial F_r}{\partial x_r}\big)\, dx_1 \wedge \cdots \wedge dx_m. \end{aligned}$$

Hence the definition of integration of m forms on $\mathbb{R}^m$ gives

$$\int_{\mathbb{R}^m} d\omega = \sum_{r=1}^{m} (-1)^{r-1} \int_{\mathbb{R}^m} \big(\tfrac{\partial F_r}{\partial x_r}\big)\, dx_1 \cdots dx_m,$$

with the integral on the right side equal to an ordinary integral with respect to Lebesgue measure. Consider the r^{th} term of the sum on the right side. We can carry out the integration over $\mathbb{R}^m$ in any order, and we choose to do the x_r integration first. By the Fundamental Theorem of Calculus, that integral over x_r is

$$\begin{aligned} = \int_{\mathbb{R}^1} \big(\tfrac{\partial F_r}{\partial x_r}\big)\, dx_r &= \int_{a_r}^{b_r} \big(\tfrac{\partial F_r}{\partial x_r}\big)\, dx_r \\ &= F_r(x_1, \ldots, b_r, \ldots, x_m) - F_r(x_1, \ldots, a_r, \ldots, x_m). \end{aligned}$$

The right side is 0 because F_r vanishes off S. Therefore the r^{th} term is 0 for each r, and we conclude that

$$\int_{\mathbb{R}^m} d\omega = 0. \qquad (*)$$

The proof is now complete for the model case $\mathbb{R}^m$.

To handle the general case, we proceed as follows: About each point p in M of the compact support S of ω, we choose a *positive* compatible chart (M_α, α). Since the sets M_{α_j} form an open cover of the compact set S, we can choose a finite subcover $\{M_{\alpha_1}, \ldots, M_{\alpha_k}\}$. By Theorem 1.25 let $\{\psi_1, \ldots, \psi_k\}$ be a smooth partition of unity of M subordinate to this finite open cover. For $1 \leq i \leq k$, the $m-1$ form $\psi_i\omega$ is compactly supported in M_{α_i}. Then we have

$$\begin{aligned} \int_M d(\psi_i\omega) = \int_{M_{\alpha_i}} d(\psi_i\omega) &= \int_{\alpha_i(M_{\alpha_i})} (\alpha^{-1})^* d(\psi_i\omega) && \text{by Theorem 1.29 and positivity} \\ &= \int_{\alpha_i(M_{\alpha_i})} d((\alpha^{-1})^*(\psi_i\omega)) && \text{by Proposition 1.24.} \end{aligned}$$

Since $\psi_i \omega$ is compactly supported on M_{α_i}, the $m-1$ form $(\alpha^{-1})^*(\psi_i\omega)$ is compactly supported in $\alpha_i(M_{\alpha_i}) \subseteq \mathbb{R}^m$. Extending this form to be 0 on the remainder of $\mathbb{R}^m$ and leaving its name unchanged, we obtain $\int_{\alpha_i(M_{\alpha_i})} d((\alpha^{-1})^*(\psi_i\omega)) = \int_{\mathbb{R}^m} d((\alpha^{-1})^*(\psi_i\omega))$. The right side is 0 by the result $(*)$ for the model case. In other words,

$$\int_M d(\psi_i\omega) = 0 \quad \text{for all } i.$$

Summing over i from 1 to k and using the fact that $\sum_i \psi_i$ is identically 1, we obtain

$$0 = \sum_i \int_M d(\psi_i\omega) = \int_M d\Big(\sum_i \psi_i\omega\Big) = \int_M d\omega,$$

and the proof of the general case is complete. □

2. Elementary Properties and Examples

Smooth manifolds of dimension $m \geq 0$, as introduced in Chapter I, were defined as separable Hausdorff spaces that are locally modeled on open subsets of $\mathbb{R}^m$. In similar fashion the present section and the remainder of this chapter will work with smooth manifolds-with-boundary in dimension $m \geq 1$, which are separable Hausdorff spaces that are locally modeled on open subsets of the closed half space

$$\mathbb{H}^m = \{(x_1, \dots, x_{m-1}, x_m) \in \mathbb{R}^m \mid x_m \geq 0\}.$$

The **open subsets** of $\mathbb{H}^m$ are understood to be those subsets that are relatively open in the relative topology from $\mathbb{R}^m$. We write $\mathbb{H}^m_+$ for the interior of $\mathbb{H}^m$, namely the subset

$$\mathbb{H}^m_+ = \{(x_1, \dots, x_{m-1}, x_m) \in \mathbb{R}^m \mid x_m > 0\},$$

and we write $\partial\mathbb{H}^m$ for the boundary, namely the subset

$$\partial\mathbb{H}^m = \{(x_1, \dots, x_{m-1}, 0) \in \mathbb{R}^m\}.$$

Before coming to the formal definition of smooth manifold-with-boundary, we need to establish some definitions concerning smooth functions on $\mathbb{H}^m$. A real-valued function f defined on an open subset U of $\mathbb{H}^m$ will be said to be **smooth** if there is a smooth function F defined an open subset V of $\mathbb{R}^m$ such $U = V \cap \mathbb{H}^m$ and f is the restriction of F to U. The extending function F need not, of course, be unique.[4] With this definition of smoothness in place, we can define the space $\mathcal{C}_p(\mathbb{H}^m)$ of germs of smooth functions at points p of $\mathbb{H}^m$ and the tangent space $T_p(\mathbb{H}^m)$ at p. For $p \in \mathbb{R}^m_+$, these definitions are not new, but for $p \in \partial\mathbb{H}^m$, they are. We obtain facts about them in the same way as in Section I.1.

If U_1 and U_2 are two open subsets of $\mathbb{H}^m$, a **smooth map** $F : U_1 \to U_2$ is a continuous function whose m component functions are all smooth real-valued functions on U_1. The derivative $(DF)_p : T_p(U_1) \to T_{F(p)}(U_2)$ of the smooth map F at a point is defined just as in Section I.1. The smooth map F is a **diffeomorphism** if it is a homeomorphism with inverse $G : U_2 \to U_1$ such that the m component functions of each of F and G are smooth real-valued functions on U_1 and U_2, respectively. The composition of smooth maps is smooth, and the derivative of the composition is the composition of the derivatives. It follows that at each point the derivative of a diffeomorphism is an invertible linear function.

Although the components of a diffeomorphism F extend to be smooth functions on an open subset of $\mathbb{R}^m$ and similarly for G, no assertion is made about the extendability of the identities $F \circ G = 1$ and $G \circ F = 1$.

Let M be a separable Hausdorff topological space, and fix an integer $m \geq 1$. For purposes of working with manifolds-with-boundary, a **chart** (M_α, α) on M of **dimension** m is a homeomorphism α of a nonempty open subset M_α of M onto an open subset $\alpha(M_\alpha)$ of $\mathbb{H}^m$; the chart is said to be **about** a point p in M if p is in the domain M_α of α. When it is convenient to do so, we can restrict attention to charts (M_α, α) for which M_α is connected.

A smooth **manifold-with-boundary** of dimension m is a separable Hausdorff space M with a family $\mathcal{F}$ of charts (M_α, α) of dimension m such that

(i) any two charts (M_α, α) and (M_β, β) in $\mathcal{F}$ are (smoothly) **compatible** in the sense that $\beta \circ \alpha^{-1}$, as a mapping of the open subset $\alpha(M_\alpha \cap M_\beta)$ of $\mathbb{H}^m$ to the open subset $\beta(M_\alpha \cap M_\beta)$ of $\mathbb{H}^m$, is a diffeomorphism,

(ii) the family of compatible charts (M_α, α) is an **atlas** in the sense that the open sets M_α cover M, and

(iii) the family $\mathcal{F}$ is maximal among families of compatible charts on M.

In the presence of an understood atlas, a chart will be said to be **compatible** if it is compatible with all the members of the atlas.

As with smooth manifolds in the sense of Chapter I, any atlas of compatible charts for a smooth manifold-with-boundary can be extended in one and only one way to a maximal atlas of compatible charts. Also if U is any nonempty open subset of an m dimensional smooth manifold-with-boundary M, then U inherits the structure of a smooth manifold-with-boundary as follows: first define an atlas of U to consist of the intersection of U with all members of the atlas for M, using

[4] However, two extending functions F_1 and F_2 do have matching partial derivatives of all orders at every point of $U \cap \partial\mathbb{H}^m$, and we shall quietly make use of this fact.

the restrictions of the various functions α, and then discard occurrences of the empty set.

Later in this section we shall use charts to transfer our notions of tangent space, cotangent space, smooth function, smooth mapping, and derivative from $\mathbb{H}^m$ to general manifolds-with-boundary. But before we look at the details, let us underscore that manifolds-with-boundary are built from two distinct types of points.

The points of a smooth manifold-with-boundary divide into two distinct types—manifold points and boundary points. The **manifold points** are those points p for which there is a chart (M_α, α) about p with $\alpha(M_\alpha)$ contained in $\mathbb{H}^m_+$. The set of them will be denoted by M_+. The set M_+ is the union for all compatible charts (M_α, α) of the inverse image $\alpha^{-1}(\mathbb{H}^m_+)$, which is open in M by continuity of α. Thus M_+ is a nonempty open subset of M and is a smooth manifold of dimension m. The other points are called **boundary points**. One writes ∂M for the set of boundary points and calls ∂M the **boundary**.[5] As the complement of M_+ in M, it is a closed set.

Proposition 2.2. If M is a smooth manifold-with-boundary of dimension m, then

(a) each manifold point p has the property that every sufficiently small compatible chart (M_β, β) about p has $\beta(M_\beta)$ contained in $\mathbb{H}^m_+$,

(b) whenever (M_α, α) is a compatible chart for M, then its restriction to ∂M, namely $(M_\alpha \cap \partial M, \alpha\big|_{M_\alpha \cap \partial M})$, is a chart for ∂M of dimension $m-1$ as long as $M_\alpha \cap \partial M$ is nonempty,

(c) whenever (M_α, α) and (M_β, β) are two compatible charts for M that meet ∂M, then the charts $(M_\alpha \cap \partial M, \alpha\big|_{M_\alpha \cap \partial M})$ and $(M_\beta \cap \partial M, \beta\big|_{M_\beta \cap \partial M})$ are compatible for ∂M,

(d) ∂M becomes a smooth manifold of dimension $m-1$ if the atlas of charts is taken as the nonempty restrictions to ∂M of the charts in an atlas of compatible charts for M.

PROOF. For (a), suppose that (M_α, α) is a chart about p with $\alpha(M_\alpha) \subseteq \mathbb{H}^m_+$. If (M_β, β) is another chart about p, we are to show that $\beta(p)$ is in $\mathbb{H}^m_+$. Consider $\beta \circ \alpha^{-1}$ as a map from $\alpha(M_\alpha \cap M_\beta)$ to $\beta(M_\alpha \cap M_\beta)$. This map is smooth in the ordinary Euclidean sense with domain a Euclidean open set, it carries $\alpha(p)$ to $\beta(p)$, and its Jacobian determinant is nonzero at $\alpha(p)$. Therefore it carries a sufficiently small open set about $\alpha(p)$ onto a Euclidean open set about $\beta(p)$, by

[5]This notion of the boundary can differ from the set-theoretic notion, as Example 6 later in this section will show.

the Inverse Function Theorem.[6] The latter open set cannot be contained in $\mathbb{H}^m$ unless $\beta(p)$ lies in $\mathbb{H}^m_+$.

For (b), let (M_α, α) be a chart for M. In view of (a), α carries $M_\alpha \cap \partial M$ one-one onto $\alpha(M_\alpha) \cap \partial\mathbb{H}^m$. The set $M_\alpha \cap \partial M$ is relatively open in ∂M since M_α is open in M, and the set $\alpha(M_\alpha) \cap \partial\mathbb{H}^m$ is relatively open in $\partial\mathbb{H}^m$ since $\alpha(M_\alpha)$ is open in $\mathbb{H}^m$. The restrictions of α and α^{-1} are continuous. Thus $(M_\alpha \cap \partial M, \alpha\big|_{M_\alpha \cap \partial M})$ is a chart for ∂M; its dimension is $m - 1$ since the Euclidean space in question is $\partial\mathbb{H}^m$.

For (c), we are given (M_α, α) and (M_β, β) as in (b), we may assume that $M_\alpha \cap M_\beta$ is nonempty, and we are told that $\beta \circ \alpha^{-1} : \alpha(M_\alpha \cap M_\beta) \to \beta(M_\beta \cap M_\beta)$ and $\alpha \circ \beta^{-1} : \beta(M_\alpha \cap M_\beta) \to \alpha(M_\beta \cap M_\beta)$ are smooth. Put $\varphi = \beta \circ \alpha^{-1} = (\varphi_1, \dots, \varphi_m)$. The smoothness of φ means that each φ_j extends to a smooth real-valued function on an open neighborhood in $\mathbb{R}^m$ of its domain in $\mathbb{H}^m$. Then the restriction of φ_j to the intersection of that neighborhood with $\partial\mathbb{H}^m$ is certainly smooth, and hence $\beta \circ \alpha^{-1} : \alpha(M_\alpha \cap M_\beta \cap \partial M) \to \beta(M_\alpha \cap M_\beta \cap \partial M)$ is smooth. Similarly the restriction of $\alpha \circ \beta^{-1}$ is smooth. Thus the restrictions of the charts are compatible.

For (d), each nonempty restriction of a chart of dimension m for M is a chart of dimension $m - 1$ for ∂M by (b). These charts for ∂M are compatible with one another by (c), and they cover ∂M since the given charts cover M. Thus the charts for ∂M form an atlas. □

EXAMPLES.

(1) Any smooth manifold of dimension ≥ 1 is a smooth manifold-with-boundary, the boundary being the empty set.

(2) In dimension 1, any interval of $\mathbb{R}$, whether open or closed or half open, is a manifold-with-boundary; the boundary consists of those endpoints that are present. The circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is a manifold of dimension 1 without boundary. The definitions allow no flexibility to declare that some of the points of the circle are boundary points and the rest are manifold points.

(3) The closed ball $B^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_1^2 + \cdots + x_m^2 \leq 1\}$ is a manifold-with-boundary of dimension m, the boundary being the sphere $S^{m-1} = \{(x_1, \dots, x_m) \mid x_1^2 + \cdots + x_m^2 = 1\}$.

(4) The closed unit square $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$ is not a manifold-with-boundary because of the presence of the corners. If the four corners are removed from the set, then the result is a manifold-with-boundary, the boundary consisting of the remaining points on the four edges.

(5) A closed figure 6 in $\mathbb{R}^2$ is not a 1 dimensional manifold-with-boundary because the point where the 6 closes on itself does not satisfy the definitions.

[6] Theorem 3.17 of *Basic Real Analysis*.

(6) If U is an open subset of $\mathbb{R}^m$ whose topological boundary $U^{\text{cl}} - U$ is a smooth manifold of dimension $m-1$, then $M = (U^{\text{cl}} - U) \cup U = U^{\text{cl}}$ is a manifold-with-boundary of dimension m. However, if U is the union of the subsets where $|x| < 1$ and $1 < |x| < 2$ in $\mathbb{R}^2$, then the topological boundary of $U^{\text{cl}} - U$, which is two circles, is different from the boundary ∂U^{cl} of the manifold, which is one circle.

(7) It is often possible to define regions in Euclidean space parametrically or implicitly and end up with a manifold-with-boundary. In $\mathbb{R}^2$, for example, the image of a curve $t \mapsto (x(t), y(t))$ in the plane is smooth if $x(t)$ and $y(t)$ are smooth and if the Implicit Function Theorem can be invoked around each point of the image to realize the set in question locally as the graph of a smooth function, i.e., if $x'(t)$ and $y'(t)$ are nowhere simultaneously vanishing. When such a curve is closed, in the sense of taking the same value at the two endpoints of the domain of definition, and when it is simple, in the sense of being one-one except for the equality of values at the endpoints, it bounds a region of the plane. The region and the curve together form a manifold-with-boundary of dimension 2.

(8) The same considerations apply in higher dimensions. It is also of interest to define smooth manifolds-with-boundary in higher dimensional spaces by using parametric equations and invoking the Implicit Function Theorem. The Möbius band, given as Example 3 in Section I.6, is a smooth manifold of dimension 2 defined parametrically in $\mathbb{R}^3$ by two parameters. As it was defined in that section, it is a noncompact smooth manifold without boundary. If the domain in the t variable is taken to be $-1 \leq t \leq 1$ instead of $-1 < t < 1$, then we obtain a compact smooth manifold-with-boundary of dimension 2. The boundary can be seen to be connected in this case; topologically it is a circle.

A **smooth real-valued function** $f : M \to \mathbb{R}$ on the smooth manifold-with-boundary M of dimension m is by definition a function such that for each $p \in M$ and each compatible chart (M_α, α) about p, the function $f \circ \alpha^{-1}$ is smooth as a function from the open subset $\alpha(M_\alpha)$ of $\mathbb{H}^m$ into $\mathbb{R}$. A smooth real-valued function is necessarily continuous.

To verify that a real-valued function f on the smooth manifold-with-boundary M is smooth, it is sufficient, for each point in M, to check smoothness within only one compatible chart about that point. The reason is the compatibility of the charts: if (M_α, α) and (M_β, β) are two compatible charts about p, then $f \circ \beta^{-1}$ is the composition of the smooth function $\alpha \circ \beta^{-1}$, which is smooth between open subsets of $\mathbb{H}^m$, followed by the smooth real-valued function $f \circ \alpha^{-1}$.

If E is a nonempty open subset of M, the space of smooth real-valued functions on E will be denoted by $C^\infty(E)$. The space $C^\infty(E)$ is an associative algebra over $\mathbb{R}$ under the pointwise operations, and it contains the constants. The **support** of a real-valued function is, as always, the closure of the set where the function is nonzero. We write $C^\infty_{\text{com}}(E)$ for the subset of $C^\infty(E)$ of functions whose support is a compact subset of M.

Transferring our notions of tangent space, cotangent space, smooth function, smooth mapping, and derivative from $\mathbb{H}^m$ to general manifolds-with-boundary can be done by suitably adjusting the definitions and proofs that we gave above for $\mathbb{H}^m$. Some care is appropriate, however: Although functions on $\mathbb{H}^m$ can be viewed as restrictions to $\mathbb{H}^m$ of functions on $\mathbb{R}^m$, we have no such global extended space to use with a general manifold-with-boundary. See Figure 2.1.

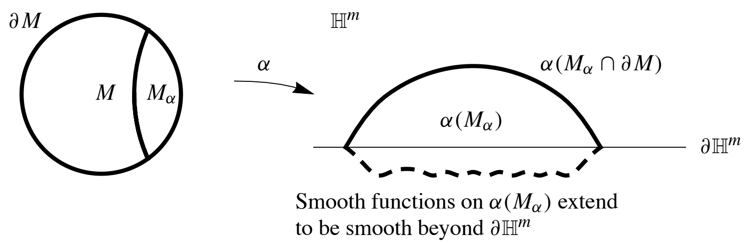

FIGURE 2.1. Nature of a chart about a boundary point.

If M is a smooth manifold-with-boundary of dimension m, we already have definitions of the tangent and cotangent spaces $T_p(M)$ and $T^*_p(M)$ at manifold points p, since M_+ is a smooth manifold. It is for boundary points p that we need to do something new. Thus let p be a boundary point. We define a **germ** at p to be an equivalence class of locally defined smooth real-valued functions in open neighborhoods of p. Arithmetic operations on germs mirror the corresponding operations on functions. The germs at p form an associative algebra $\mathcal{C}_p(M)$ over $\mathbb{R}$ with identity, just as in the manifold case. Derivations of $\mathcal{C}_p(M)$ are defined just as in the manifold case.

Observe, however, that in the case of a boundary point of $\mathbb{H}^m$, the open neighborhoods of boundary points are merely relatively open. They are somewhat one-sided and in particular are not open in $\mathbb{R}^m$.

The **tangent space** $T_p(M)$ at p is defined to be the real vector space of all derivations of $\mathcal{C}_p(M)$, just as it was in the manifold case in Section I.1. If a local coordinate system at p is given by means of a chart (M_α, α) with $\alpha = (x_1, \dots, x_m)$, then m examples of members of $T_p(M)$ are given by the derivations $\left[\frac{\partial}{\partial x_j}\right]_p$ defined by

$$\left[\frac{\partial f}{\partial x_j}\right]_p = \frac{\partial(f\circ\alpha^{-1})}{\partial u_j}\bigg|_{(u_1,\dots,u_m)=(x_1(p),\dots,x_m(p))} \qquad \text{for} \quad j = 1, \dots, m.$$

This is so even if p is a boundary point. In this case one or more of the partial derivatives may need to be interpreted as a one-sided partial derivative within

$\alpha(M_\alpha)$. Just as in the manifold case, the m derivations $\left[\frac{\partial}{\partial x_j}\right]_p$ form a vector-space basis of $T_p(M)$, regardless of whether p is a manifold point or a boundary point. Vector fields and smoothness of them are notions defined in the same way as in the manifold case.

The **derivative** DF of a smooth function $F : M \to N$ between manifolds-with-boundary is defined just as in the case of manifolds. If p is in M, then $(DF)(p)$ is a linear function from $T_p(M)$ to $T_{F(p)}(N)$. The **cotangent space** $T_p^*(M)$ is defined to be the dual of $T_p(M)$, just as in the manifold case. Differentials of smooth functions provide examples, the **differential** of f at p being defined by $(df)_p(L) = Lf$ for p in $T_p(M)$, just as in the manifold case. We can then go on to define differential 1 forms, differential k forms, and smoothness of differential forms. There are no surprises. The notion of **pullback** of a differential form is still meaningful.

The final preparatory step for working with manifolds-with-boundary is to make smooth partitions of unity be available. We begin with analogs of Lemmas 1.2 and 1.3.

Lemma 2.3. If U is a nonempty open subset of a smooth manifold-with-boundary M and if f is in $C^\infty_{\text{com}}(U)$, then the function F defined on M so as to equal f on U and to equal 0 off U is in $C^\infty_{\text{com}}(M)$ and has support contained in U.

REMARK. This is proved in the same way that Lemma 1.2 was proved for smooth manifolds. The argument makes use of the Hausdorff property of M.

Lemma 2.4. Suppose that p is a point in a smooth manifold-with-boundary M, that (M_α, α) is a compatible chart about p, and that K is a compact subset of M_α containing p. Then there is a smooth function $f : M \to \mathbb{R}$ with compact support contained in M_α such that f has values in [0, 1] and f is identically 1 on K.

PROOF. Let M have dimension m. The set $\alpha(K)$ is a compact subset of the open subset $\alpha(M_\alpha)$ of $\mathbb{H}^m$. Let U be an open subset of $\mathbb{R}^m$ such that $U \cap \mathbb{H}^m = \alpha(M_\alpha)$. Lemma 1.1 produces a function g in $C^\infty_{\text{com}}(U)$ with values in [0, 1] that identically 1 on $\alpha(K)$. Let f be the pullback of g to M_α; that is, let $f = g \circ \alpha^{-1}$. Extending f to be 0 on the complement of M_α in M and applying Lemma 2.3, we see that the extended f has the desired properties. $\square$

The notion of a **smooth partition of unity** of a manifold-with-boundary M **subordinate** to the finite open cover $\{U_i\}$ of a compact subset K of M works just as in the case of smooth manifolds without boundary. The statement is as follows.

Proposition 2.5. Let M be a smooth manifold-with-boundary, let K be a nonempty compact subset, and let $\{U_i \mid 1 \le i \le r\}$ be a finite open cover of K.

Then there exist functions f_i in $C^\infty(M)$ for $1 \le i \le r$, taking values between 0 and 1 such that each f_i is identically 0 off a compact subset of U_i and $\sum_{i=1}^{r} f_i$ is identically 1 on K.

REMARKS. Except for changes in notation, the proof is the same as for Theorem 1.25. Specifically Lemmas 1.26 and 1.27 are unchanged except that "manifold" in each of their statements is to be replaced by "manifold-with-boundary." Lemma 1.28 is unchanged except that "manifold" in its statement is to be replaced by "manifold-with-boundary" and the citation of Lemma 1.3 is to be replaced by a citation of Lemma 2.4. Then the proof of Theorem 1.25 goes through without further change.

3. Induced Orientation on the Boundary

Let M be an m dimensional manifold-with-boundary with $m \ge 1$, let ∂M be its boundary, and let M_+ be its subset of manifold points. We shall say that M is **orientable** (or **oriented**) if M_+ is orientable (or oriented). This definition is meaningful because M_+ is a smooth manifold. The point of this section is to address the question of determining an orientation on ∂M from an orientation on M_+. We postpone the case where ∂M has dimension 0, namely the case $m = 1$, until the last example of this section. Thus for now, let $m \ge 2$.

The goal of the exercise is to be able to prove Stokes's Theorem, which gives the formula $\int_{\partial M} \omega = \int_M d\omega$ for any compactly supported smooth $m-1$ form on a manifold-with-boundary M of dimension m. In the formula, the integral over M is really an integral over the set M_+ of manifold points. Both M_+ and ∂M are smooth manifolds, and they are disjoint. To make sense of the two integrals, we need orientations for M_+ and ∂M, and they need to be correlated in some way. As in the special case of Theorem 2.1, Stokes's Theorem is really a local matter in the presence of an orientation. It is therefore necessary to understand what is happening in the model space $\mathbb{H}^m$.

EXAMPLE. $M = \mathbb{H}^m$ as a manifold-with-boundary. The manifold points are those in $\mathbb{H}^m_+$, and the boundary points are those in $\partial\mathbb{H}^m$. A single chart suffices for the whole manifold-with-boundary. The atlas for M consists of this one chart; its restriction to $\partial\mathbb{H}^m$ gives us a single chart for ∂M, hence an atlas for ∂M. The subset M_+ of manifold points is $\mathbb{H}^m_+$, which is an open subset of $\mathbb{R}^m$. As such, it can inherit the standard orientation from $\mathbb{R}^m$, which is the one determined by the m form $dx_1 \wedge \cdots \wedge dx_m$. To obtain an orientation for ∂M, we cannot simply let x_m tend to 0 in the latter m form. Instead, we can proceed by declaring some nowhere-vanishing $m-1$ form on the Euclidean space $\partial\mathbb{H}^m$ to be positive. For example, we could declare that the orientation for $\partial\mathbb{H}^m$ is determined by $dx_1 \wedge \cdots \wedge dx_{m-1}$ since the variable x_m is constantly equal to 0 on ∂M. Unfortunately this choice leads to the Stokes formula only up to a sign; specifically it leads to the formula modified by the inclusion of a factor of $(-1)^m$ on one of the two sides. Another approach is to renumber the variables so that the special variable that gets put equal to 0 on ∂M is the first variable. The m form on M_+ is still the same, and the temptation is to declare that the orientation for the Euclidean space where $x_1 = 0$ is determined by $dx_2 \wedge \cdots \wedge dx_m$. As is shown in Problem 10 at the end of the chapter, this choice leads to the Stokes formula modified by a single factor of (-1) on one of the two sides. Or one could try some other way of relating ∂M to M_+ notationally, and one may expect that we are always led to a parity question, namely whether we get the desired Stokes formula or we get that formula except for a minus sign. There is a traditional procedure for orienting ∂M so that the signs come out correctly, and we tell what that is momentarily. The point is that the choice of procedure we make is rather arbitrary. The motivation is to make the signs come out right at the end, and any geometric justification is secondary.

The traditional procedure is to take an outward-pointing tangent vector to $\partial M = \mathbb{R}^m_0$ into account, considering it to be primary. The outward-pointing vector in question can be $-\big[\frac{\partial}{\partial x_m}\big]$. We follow it with the standard basis of tangent vectors to $\partial\mathbb{H}^m$, including them in their standard order and obtaining

$$(-\partial/\partial x_m, \partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_{m-1}).$$

Then we use the nowhere-vanishing differential m form $dx_1 \wedge \cdots \wedge dx_m$ on $\mathbb{H}^m$ to determine the alternating $m-1$ linear form on $\partial\mathbb{H}^m$ given by

$$(v_1, \ldots, v_{m-1}) \mapsto (dx_1 \wedge \cdots \wedge dx_m)(-\partial/\partial x_m, v_1, \ldots, v_{m-1}).$$

The value of this expression is

$$= \frac{1}{m!} \det \begin{pmatrix} dx_1\big(-\frac{\partial}{\partial x_m}\big) & dx_1(v_1) & \cdots & dx_1(v_{m-1}) \\ & \vdots & & \\ dx_{m-1}\big(-\frac{\partial}{\partial x_m}\big) & dx_{m-1}(v_1) & \cdots & dx_{m-1}(v_{m-1}) \\ dx_m\big(-\frac{\partial}{\partial x_m}\big) & dx_m(v_1) & \cdots & dx_m(v_{m-1}) \end{pmatrix}$$

$$= \frac{1}{m!} \det \begin{pmatrix} 0 & dx_1(v_1) & \cdots & dx_1(v_{m-1}) \\ \vdots & \vdots & \ddots & \\ 0 & dx_{m-1}(v_1) & \cdots & dx_{m-1}(v_{m-1}) \\ -1 & dx_m(v_1) & \cdots & dx_m(v_{m-1}) \end{pmatrix}$$

$$= \frac{(-1)^m}{m!} \det \begin{pmatrix} dx_1(v_1) & \cdots & dx_1(v_{m-1}) \\ \vdots & \ddots & \\ dx_{m-1}(v_1) & \cdots & dx_{m-1}(v_{m-1}) \end{pmatrix}$$

$$= (-1)^m m^{-1}(dx_1 \wedge \cdots \wedge dx_{m-1})(v_1, \dots, v_{m-1}),$$

and we see that the above form is nowhere vanishing on $\partial\mathbb{H}^m$. We take its equivalence class modulo everywhere positive functions to be the **induced orientation** on $\partial\mathbb{H}^m$. In other words, the induced orientation is determined by $(-1)^m(dx_1 \wedge \cdots \wedge dx_{m-1})$ up to a positive factor.

To work with this construction in the context of a general manifold-with-boundary M of dimension m, we shall make use of a particularly nice atlas of charts for M. This atlas consists of one compatible chart about each point of M. Distinct points are allowed to correspond to the same compatible chart.

For a manifold point p, we can use any chart about p that does not meet ∂M. For a boundary point p, we start from any compatible chart (M_α, α) about p such that M_α is connected. The charts are mutually compatible and cover M by construction. Thus the result is an atlas for M consistent with its manifold-with-boundary structure. The members of the atlas that do not meet ∂M are exactly the charts about boundary points.

The following proposition uses this constructed atlas on M to extend the notion of induced orientation from $\mathbb{H}^m$ and $\partial\mathbb{H}^m$ to M and ∂M.

Proposition 2.6. Let M be an m dimensional manifold-with-boundary, and suppose that $m \geq 2$ and that M_+ is oriented. Then the orientation on M_+ induces a nowhere-vanishing $m-1$ form η on ∂M with the property that on any connected positive chart (M_α, α) about a boundary point, η is the product of a positive function and the pullback $(-1)^m\alpha^*(dx_1 \wedge \cdots \wedge dx_{m-1})$.

REMARK. The orientation on ∂M obtained in this way, i.e., the equivalence class of η modulo everywhere-positive functions, is called the **induced orientation** for ∂M. It is uniquely determined by the orientation of M_+. A version of this proposition valid for $m = 1$ will be noted after the end of the proof.

PROOF. We start from the atlas for M constructed just before the statement of Proposition 2.6. It supplies one compatible chart about each point p of M. Applying Proposition 2.2 to this atlas, we obtain an atlas of compatible charts for ∂M by restriction, provided we discard those charts that do not meet ∂M. The ones that do not meet ∂M are all the charts about manifold points. Thus our construction has the property that the restrictions to ∂M of the charts about boundary points form an atlas of compatible charts for ∂M.

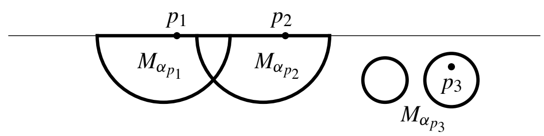

FIGURE 2.2. Some charts used in proving Proposition 2.6.

Since M_+ is orientable, Proposition 1.30 associates to each orientation of M_+ a nowhere-vanishing smooth m form on M. This m form is unique up to multiplication by a real-valued function that is everywhere positive. Fix such an ω for the given orientation of M_+. Before working with the atlas for ∂M, we shall make an adjustment to the charts about boundary points that we are including in the atlas for M.

For each p in ∂M, let $F_{\alpha_p} : \alpha_p(M_{\alpha_p}) \to \mathbb{R}$ be the smooth function such that

$$(\alpha_p^{-1})^*\omega = F_{\alpha_p}(x_1, \dots, x_m)\, dx_1 \wedge \cdots \wedge dx_m.$$

The right side is the local expression for ω in the image in $\mathbb{H}^m$ of the chart (M_{α_p}, α_p). Since ω is nowhere vanishing and $\alpha_p(M_{\alpha_p})$ is assumed to be connected, F_{α_p} has constant sign on $\alpha_p(M_{\alpha_p})$. If the constant sign is positive, we retain (M_{α_p}, α_p) for the adjusted atlas. If the sign is negative, we take advantage of the fact that $m > 1$ to redefine α_p by following it with the linear map $T : \mathbb{R}^m \to \mathbb{R}^m$ given by $T(x_1, x_2, \dots, x_m) = (-x_1, x_2, \dots, x_m)$. In this case we instead include $(M_{\alpha_p}, T \circ \alpha_p)$ in the adjusted atlas. Since

$$((T \circ \alpha_p)^{-1})^*\omega = ((\alpha_p^{-1}) \circ T^{-1})^*\omega = (T^{-1})^*(\alpha_p^{-1})^*\omega$$

by Proposition 1.18f and since $T^{-1} = T$, we have

$$((T \circ \alpha_p)^{-1})^*\omega = -F_{\alpha_p}(-x_1, x_2, \dots, x_m)\, dx_1 \wedge \cdots \wedge dx_m.$$

With this change the coefficient of $dx_1 \wedge \cdots \wedge dx_m$ is now everywhere positive on its domain $\alpha(M_{\alpha_p})$. This completes our adjustment to the charts about boundary points that we are including in our atlas.

Referring to (‡‡) in the proof of Theorem 1.29, we see that each function $\det \left\{ \dfrac{\partial y_i}{\partial x_j} \right\}_{i,j=1,\dots,m}$ arising from a coordinate change between two of the charts about boundary points in the adjusted atlas for M is positive on its domain.

We now want to interpret this information for the atlas on ∂M. The members of the atlas for ∂M are obtained by restricting to ∂M the charts about boundary points. We want to see that this atlas for ∂M exhibits ∂M as oriented.[7] Thus

[7] It indeed exhibits ∂M as oriented, but sadly the resulting orientation on ∂M is not quite the one we seek.

suppose we have two such charts, say (M_α, α) and (M_β, β), with the property that $M_\alpha \cap M_\beta \cap \partial M$ is not empty. It is enough to consider

$$\beta \circ \alpha^{-1}\big|_{\partial \mathbb{H}^m} : \alpha(M_\alpha \cap M_\beta \cap \partial M) \to \beta(M_\alpha \cap M_\beta \cap \partial M).$$

This is the same as the full map $\beta \circ \alpha^{-1}$ but restricted to the set where $x_m = 0$, and we know from Proposition 2.2a that $y_m = 0$ for such points. Since the m^{th} coordinate function is 0, the Jacobian matrix has all entries equal to 0 in its m^{th} row except for the diagonal entry, which is $\frac{\partial y_m}{\partial x_m}$. If we write $J(x_1, \ldots, x_m)$ for the full Jacobian determinant and $J_{\partial M}(x_1, \ldots, x_{m-1})$ for the Jacobian determinant of the upper left $m-1$ by $m-1$ block, we obtain

$$J(x_1, \ldots, x_{m-1}, 0) = \frac{\partial y_m}{\partial x_m}(x_1, \ldots, x_{m-1}, 0)\, J_{\partial M}(x_1, \ldots, x_{m-1}).$$

We have seen that the left side is everywhere positive. If we can show that $\frac{\partial y_m}{\partial x_m}(x_1, \ldots, x_{m-1}, 0)$ is everywhere ≥ 0, then it will follow that every Jacobian determinant $J_{\partial M}(x_1, \ldots, x_{m-1})$ is everywhere positive, and we will have proved that the adjusted atlas exhibits ∂M as oriented. But this is just one-variable calculus: the m^{th} component of y_m is ≥ 0 for $x_m \geq 0$ and takes the value 0 at $x_m = 0$; its first derivative must then be ≥ 0 at $x_m = 0$.

Thus we have constructed an orientation on ∂M. It is not exactly the orientation we seek. We define the **induced orientation** on ∂M to be the constructed orientation if m is even and to be the opposite of the constructed orientation if m is odd. In symbols if η_1 is a nonvanishing $m-1$ form on ∂M defining the constructed orientation, we can use $\eta = (-1)^m \eta_1$ in every case as a nowhere-vanishing $m-1$ form on ∂M defining the induced orientation. □

The proof of Proposition 2.6 breaks down when $m = 1$. The smooth function $F_{\alpha_p} : \alpha(M_{\alpha_p}) \to \mathbb{R}$ of the third paragraph of the proof still makes sense. Since $m = 1$, it involves just one variable:

$$(\alpha_p^{-1})^* \omega = F_{\alpha_p}(x)\, dx.$$

It is still true that the function F_{α_p} necessarily has constant sign on $\alpha_p(M_{\alpha_p})$. But if that sign is negative, no variables are available to make use of the reflection function T. Thus we leave F_{α_p} as it is, positive or negative, and we make no adjustment to the charts about boundary points. Nevertheless, the restrictions of these charts to ∂M still exhibit ∂M as oriented. We just take the orientation of a point p to be the sign of $F_{\alpha_p}(0)$, and there is no contradiction. Following through for $m = 1$ on the sign convention in our definition above of the induced orientation for $m > 1$, we define the contribution of a point p to the value of an integral over ∂M of a 0 form in the **induced orientation** to be $-F_{\alpha_p}(0)$.

4. Stokes's Theorem for Manifolds-with-Boundary

Now we come to Stokes's Theorem, working with an oriented manifold-with-boundary M of dimension $m \geq 1$. Proposition 2.6 has shown how to obtain an induced orientation of ∂M when starting from a given orientation of M.

Theorem 2.7. Let M be an oriented manifold-with-boundary of dimension $m \geq 1$, and give its boundary ∂M the induced orientation. If ω is any smooth $m-1$ form on M of compact support, then

$$\int_{\partial M} \omega = \int_M d\omega.$$

PROOF. The model space is $\mathbb{H}^m$, and we first prove the theorem in this special case. The smooth $m-1$ form ω necessarily has an expansion

$$\omega = \sum_{r=1}^{m} F_r(x_1, \dots, x_r, \dots, x_m)\, dx_1 \wedge \cdots \wedge \widehat{dx_r} \wedge \cdots \wedge dx_m, \qquad (*)$$

the circumflex indicating a missing term. All the coefficient functions F_r are smooth and are equal to 0 off the compact support S of ω, and we have

$$\begin{aligned} d\omega &= \sum_{r=1}^{m} \sum_{s=1}^{m} \tfrac{\partial F_r}{\partial x_s}\, dx_s \wedge dx_1 \wedge \cdots \wedge \widehat{dx_r} \wedge \cdots \wedge dx_m \\ &= \sum_{r=1}^{m} (-1)^{r-1} \big(\tfrac{\partial F_r}{\partial x_r}\big)\, dx_1 \wedge \cdots \wedge dx_m. \qquad (**) \end{aligned}$$

The support of ω being compact, we choose real numbers a_j and b_j for $1 \leq j \leq m-1$ and a real number c such that all points $x = (x_1, \dots, x_m)$ of S have $a_j < x_j < b_j$ for $1 \leq j \leq m-1$ and $0 \leq x_m < c$.

On $\partial\mathbb{H}^m$, where x_m is identically 0 and dx_m is in effect 0, all the terms of $(*)$ drop out except for the term with $r = m$, and thus

$$\omega = F_m(x_1, \dots, x_{m-1}, 0)\, dx_1 \wedge \cdots \wedge dx_{m-1}.$$

We want to integrate ω over $\partial\mathbb{H}^m$, taking into account the orientation. Suppose for the moment that $m \geq 2$. Since $(-1)^m dx_1 \wedge \cdots \wedge dx_{m-1}$ is positively oriented on $\partial\mathbb{H}^m$ in the induced orientation, application of Theorem 1.29 gives

$$\begin{aligned} \int_{\partial\mathbb{H}^m} \omega &= (-1)^m \int_{\partial\mathbb{H}^m} F_m(x_1, \dots, x_{m-1}, 0)\, dx_1 \wedge \cdots \wedge dx_{m-1} \\ &= (-1)^m \int_{a_1}^{b_1} \cdots \int_{a_{m-1}}^{b_{m-1}} F_m(x_1, \dots, x_{m-1}, 0)\, dx_{m-1} \cdots dx_1. \qquad (\dagger) \end{aligned}$$

Special remarks are appropriate here when $m = 1$. Then $\partial\mathbb{H}^m$ reduces to a single point (0), and the differential form ω is the scalar $F_1(0)$ attached to that point. When it comes to integration, our convention about the induced orientation at the point 0 for this value of m, which was spelled out in the final paragraph of the previous section, is that we multiply $F_1(0)$ by -1. That is

$$\int_{\partial\mathbb{H}^1} \omega = -F_1(0),$$

and thus (†) still holds for $m = 1$. Therefore we take (†) as known for all $m \geq 1$.

Meanwhile, $d\omega$ is given on $\mathbb{H}^m$ for all $m \geq 1$ by $(**)$, and application of Theorem 1.29 and its Remark (2) yields

$$\begin{aligned}\int_{\mathbb{H}^m} d\omega &= \int_{\mathbb{H}^m} \sum_{r=1}^{m} (-1)^{r-1}\big(\tfrac{\partial F_r}{\partial x_r}\big)\, dx_1 \wedge \cdots \wedge dx_m \\ &= \sum_{r=1}^{m} (-1)^{r-1} \int_{\mathbb{H}^m} \big(\tfrac{\partial F_r}{\partial x_r}\big)\, dx_1 \cdots dx_m \qquad (\dagger\dagger)\end{aligned}$$

with the integral on the right side equal to an ordinary integral with respect to Lebesgue measure. On the right side of (††) in the r^{th} term, the integration is taking place in m variables, and we choose to do the integration in the variable x_r first. Since the set of integration is a product set, the inside integral in the case that $r < m$ is

$$\int_{a_r}^{b_r} \big(\tfrac{\partial F_r}{\partial x_r}\big)\, dx_r.$$

The function F_r in its dependence on x_r is smooth and compactly supported in the open interval $a_r < x_r < b_r$. By the Fundamental Theorem of Calculus, the integral in the variable x_r is 0. For $r = m$, the inside integral on the right side of (††) is

$$\int_0^c \big(\tfrac{\partial F_m}{\partial x_m}\big)\, dx_m = F_m(x_1, \ldots, x_{m-1}, c) - F_m(x_1, \ldots, x_{m-1}, 0)$$

with $F_m(x_1, \ldots, x_{m-1}, c) = 0$ by the support condition. Therefore the whole expression (††) boils down to

$$= (-1)^m \int_{a_1}^{b_1} \cdots \int_{a_{m-1}}^{b_{m-1}} F_m(x_1, \ldots, x_{m-1}, 0)\, dx_{m-1} \cdots dx_1,$$

which exactly equals (†). We conclude that

$$\int_{\partial\mathbb{H}^m} \omega = \int_{\mathbb{H}^m} d\omega. \qquad (\ddagger)$$

in the special case that $M = \partial\mathbb{H}^m$.

To handle the general case, we proceed in the same manner as in the proof of Theorem 2.1: About each point p in M of the compact support S of ω, we choose a *positive* compatible chart (M_α, α). Since the sets M_{α_j} form an open cover of the compact set S, we can choose a finite subcover $\{M_{\alpha_1}, \dots, M_{\alpha_k}\}$. By Proposition 2.5 (instead of Theorem 1.25), let $\{\psi_1, \dots, \psi_k\}$ be a smooth partition of unity of M subordinate to this finite open cover. For $1 \le i \le k$, the $m-1$ form $\psi_i\omega$ is compactly supported in M_{α_i}, and the $m-1$ form $(\alpha^{-1})^*(\psi_i\omega)$ is compactly supported in $\alpha_i(M_{\alpha_i}) \subseteq \mathbb{H}^m$. Let us extend it to all of $\mathbb{H}^m$ by setting it equal to 0 off $\alpha_i(M_{\alpha_i}) \subseteq \mathbb{H}^m$, leaving its name unchanged. Then

$$\begin{aligned}
\int_M d(\psi_i\omega) = \int_{M_{\alpha_i}} d(\psi_i\omega) &= \int_{\alpha_i(M_{\alpha_i})} (\alpha_i^{-1})^*(d(\psi_i\omega)) && \text{by Theorem 1.29} \\
&= \int_{\mathbb{H}^m} (\alpha_i^{-1})^*(d(\psi_i\omega)) && \text{after extension by 0} \\
&= \int_{\mathbb{H}^m} d((\alpha_i^{-1})^*(\psi_i\omega)) && \text{by Proposition 1.24} \\
&= \int_{\partial\mathbb{H}^m} (\alpha_i^{-1})^*(\psi_i\omega) && \text{by (‡)} \\
&= \int_{\partial M_{\alpha_i}} \psi_i\omega = \int_{\partial M} \psi_i\omega && \text{by Theorem 1.29.}
\end{aligned}$$

Summing over i from 1 to k and using the fact that $\sum_{i=1}^k \psi_i$ is identically 1, we obtain

$$\int_M d\omega = \sum_{i=1}^k \int_M d(\psi_i\omega) = \int_{\partial M} \Big(\sum_{i=1}^k \psi_i\omega\Big) = \int_{\partial M} \omega,$$

and the proof of the general case is complete. □

EXAMPLE. Suppose M is the closed bounded interval $[a, b]$ of $\mathbb{R}^1$. This manifold-with-boundary has $M_+ = (a, b)$ and $\partial M = \{a, b\}$. We can cover M with an atlas of two charts about boundary points, namely (M_α, α) and (M_β, β) with

$$\begin{aligned}
M_\alpha &= [a, b), \qquad & \alpha(x) &= x - a, \qquad & \alpha(M_\alpha) &= [0, b-a), \\
M_\beta &= (a, b], \qquad & \beta(x) &= b - x, \qquad & \beta(M_\beta) &= [0, b-a).
\end{aligned}$$

Proposition 2.6 does not modify this atlas before we restrict matters to ∂M. Fix a function ψ in $C^\infty_{\text{com}}([a, b))$ taking values in $[0, 1]$ and having $\psi(x) = 1$ near

$x = a$ and $\psi(x) = 0$ near $x = b$. Our partition of unity on $[a, b]$ can be taken as $\{\psi, 1 - \psi\}$.

The orientation on M_+ is given by the nowhere-vanishing 1 form $\eta = dx$; in other words, it is right to left as usual. The pullbacks of dx into the charts are given by $(\alpha^{-1})^*(dx) = dx$ and $(\beta^{-1})^*(dx) = -dx$. So the functions F_α and F_β in the proof of Proposition 2.6 are given by $F_\alpha = 1$ and $F_\beta = -1$. The induced orientation is obtained by multiplying these by -1 since m is odd. Thus we get a total contribution of -1 at the point a of ∂M and $+1$ at the point b.

Let ω be the 0 form $x \mapsto f(x)$, f being a C^∞ function on $[a, b]$. Then

$$\int_{\partial M} \omega = \int_{\partial M} \psi\omega + \int_{\partial M} (1 - \psi)\omega = -f(a) + f(b).$$

Meanwhile,

$$\int_M d\omega = \int_{(a,b)} \tfrac{d}{dx}(f(x))\, dx = \int_a^b f'(x)\, dx.$$

So the conclusion of the theorem, namely $\int_{\partial M} \omega = \int_M d\omega$, reduces to the Fundamental Theorem of Calculus, namely $f(b) - f(a) = \int_a^b f'(x)\, dx$.

5. Classical Vector Analysis

Vector analysis refers to the part of multivariable calculus in $\mathbb{R}^2$ and $\mathbb{R}^3$ that uses techniques of geometry and calculus to provide tools helpful in applications to science and engineering. These tools include

- vector notation for $\mathbb{R}^2$ and $\mathbb{R}^3$,
- dot product and vector product,
- various differentiation operators and notation for them,
- double and triple integrals,
- descriptions of curves and surfaces,
- tangent vectors and normal vectors,
- line integrals and surface integrals,
- arc length and surface area,
- Green's Theorem, the Divergence Theorem, and the Kelvin–Stokes Theorem.

Most of what is said in this section will be simply alternative notation for notions that are already known. Though the mathematics will not be new, it is important for good communication to be able to recognize this alternative notation and to be able to work with it.

The emphasis in this book has been and continues to be on the unified treatment of Green's Theorem, the Divergence Theorem, and the Kelvin–Stokes Theorem that was introduced by E. Cartan. The Cartan approach has led us to a certain amount of differential geometry (tangents vectors, tangent spaces, vector fields, differentials, derivatives, differential forms, pullbacks, integration of top-degree differential forms, and so on), and at the same time it has avoided making essential use of orthogonality anywhere. It avoided using orthogonality by being cast as a theory about general smooth manifolds with no additional structure. Some of the tools in vector analysis do make considerable use of orthogonality inherited from Euclidean space, and we shall touch on these tools only lightly.

In vector analysis one works with scalar-valued functions and vector-valued functions in two or three dimensions. To fix the ideas, let us work with dimension 3; dimension 2 van be handled by simply taking the third component to be 0. For a mathematician, $\mathbb{R}^3$ is often viewed as a space of column vectors of real numbers, such as $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Mathematicians allow themselves to write such a column vector horizontally with commas, as in (a, b, c), to save space, and the subject of vector analysis sometimes uses the same horizontal notation. Often in vector analysis, however, a different kind of abbreviation appears, in which one gives names to the three standard basis vectors, namely $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Then the vector (a, b, c) becomes $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Sometimes a vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = (a, b, c)$ is associated geometrically with an arrow that extends from the origin $(0, 0, 0)$ to the point (a, b, c). Vectors are often written with boldface symbols as in $\mathbf{v} = a\mathbf{i}+b\mathbf{j}+c\mathbf{k} = (a, b, c)$, or with symbols having arrows over them as in $\overrightarrow{v} = a\overrightarrow{i} + b\overrightarrow{j} + c\overrightarrow{k}$, but we shall usually not follow either of these conventions.

Functions into $\mathbb{R}^3$ with domain in $\mathbb{R}^1$ are called **curves** if they satisfy some additional properties, functions with domain in $\mathbb{R}^2$ are called **surfaces** if they satisfy some additional properties, and functions with domain in $\mathbb{R}^3$ are called "vector fields" in this language. The case of values in $\mathbb{R}^3$ requires some special comments. Such a function F, carrying part of $\mathbb{R}^3$ into $\mathbb{R}^3$, can be viewed conveniently as a system of arrows in $\mathbb{R}^3$, one such arrow having its tail is at the point (a, b, c) of the domain and having its tip is at $(a, b, c) + F(a, b, c)$. The arrows show how each point (a, b, c) moves under the function. Just as with vectors themselves, vector-valued functions are sometimes denoted by boldface symbols or by symbols with arrows over them, but we shall often not follow this convention.

Dot product in $\mathbb{R}^3$ is familiar to the reader from elementary linear algebra. The dot product of vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ is written $u \cdot v$ in the language of vector analysis, and its value is $u \cdot v = u_1v_1 + u_2v_2 + u_3v_3$. which is a scalar. As a function from $\mathbb{R}^3 \times \mathbb{R}^3$ into $\mathbb{R}$, dot product is linear in each variable, and it satisfies $u \cdot u \geq 0$ with equality if and only if $u = 0$. The length of a vector $u = (u_1, u_2, u_3)$, written $|u|$, is given by $|u| = \sqrt{u \cdot u} = \sqrt{u_1^2 + u_2^2 + u_3^2}$. Dot product has the geometric interpretation that $u \cdot v = |u||v| \cos\theta$, where θ is the angle that u and v make with the origin.

The cross product, also known as the vector product, of two vectors may be less well known. Cross product is defined reasonably only in $\mathbb{R}^3$ and does not generalize well to other dimensions. The **cross product** or **vector product** of vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ is the *vector* in $\mathbb{R}^3$ given by

$$u \times v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

Fortunately there is a mnemonic for this definition, the formal expression being either

$$u \times v = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} \quad \text{or} \quad u \times v = \det\begin{pmatrix} \mathbf{i} & u_1 & v_1 \\ \mathbf{j} & u_2 & v_2 \\ \mathbf{k} & u_3 & v_3 \end{pmatrix},$$

whichever is more convenient. As a function from $\mathbb{R}^3 \times \mathbb{R}^3$ into $\mathbb{R}^3$, vector product is linear in each variable. It is 0 if and only if u and v are collinear.

Let $w = (w_1, w_2, w_3)$ be a third vector. The triple product $w \cdot (u \times v)$ is given by substituting into the mnemonic the coordinates of w for $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. From this fact it is clear that $u \times v$ is orthogonal to u and v. Moreover, a little computation shows that $|u \times v|^2 + (u \cdot v)^2 = |u|^2|v|^2$. Therefore $|u \times v| = |u||v||\sin\theta|$, where θ is again the angle that u and v make with the origin. Consequently we know the magnitude of $u \times v$ (namely $|u||v||\sin\theta|$) and the direction up to sign, namely orthogonal to both u and v; that final sign can be determined from easy geometric considerations.[8] Finally if u, v, and w are three vectors, then $\det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$, up to sign, is the volume of the parallelepiped with sides u, v, and w. (See Problem 2 at the end of the chapter.)

Vector analysis makes use of three differential operators on functions on $\mathbb{R}^3$, known as gradient, divergence, and curl. They are defined by

$$\begin{aligned} \operatorname{grad} f &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} && \text{for } f \text{ scalar-valued,} \\ \operatorname{div} F &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} && \text{for } F = (F_1, F_2, F_3) \text{ vector-valued,} \end{aligned}$$

and

$$\operatorname{curl} F = \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\mathbf{i} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\mathbf{j} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\mathbf{k}$$

for $F = (F_1, F_2, F_3)$ vector-valued. Observe that $\operatorname{grad} f$ and $\operatorname{curl} F$ are vector-valued, but $\operatorname{div} F$ is scalar-vaued. The symbolic vector $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$,

[8] That sign is determined by the right-hand rule or the left-hand rule, whichever applies to the valid formula $\mathbf{i} \times \mathbf{j} = \mathbf{k}$,

pronounced "nabla," allows us to write these definitions more economically as follows:

$$\operatorname{grad} f = \nabla f, \quad \operatorname{div} F = \nabla \cdot F, \quad \text{and} \quad \operatorname{curl} F = \nabla \times F.$$

These operators may be interpreted as special cases of the exterior derivative operator d, as was shown in Example 2 in Section I.4.

Double and triple integrals are familiar from Chapter III of *Basic Real Analysis*, and it is not necessary to say any more about them now except that $dx\,dy$ is sometimes abbreviated as dA in dimension 2 and $dx\,dy\,dz$ is sometimes abbreviated dV is dimension 3.

Curves and surfaces were discussed somewhat in problems at the end of Chapter I, and the reader may wish to refer to that material. In the present chapter we are interested only in smooth curves and smooth surfaces, which are often given either "parametrically" or "implicitly." Parametric curves are usually given by a function of 1 parameter into $\mathbb{R}^3$, while parametric surfaces are usually given by a function of 2 parameters into $\mathbb{R}^3$. In both cases the defining function is assumed to satisfy a certain nondegeneracy condition so that the Inverse Function Theorem can be applied. In the implicit case, curves and surfaces are usually given as the set of simultaneous solutions of some (nonlinear) equations, usually $n-1$ equations in n variables in the case of a curve or $n-2$ equations in n variables in the case of a surface. In addition, the defining equations are assumed to satisfy a certain nondegeneracy condition so that the Implicit Function Theorem can be applied. Nothing more needs to be added to these remarks at this time.

One way that a curve can arise in physics and engineering is as the trajectory of a particle in space. The position is often written as $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, in which case the velocity is $\mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$. The velocity vector is always tangent to the curve. The nondegeneracy condition that was mentioned in the previous paragraph is that the velocity vector is nowhere the 0 vector. This condition ensures that the curve is locally a smooth manifold of dimension 1.

A surface can arise, for example, as a membrane through which fluid is flowing, or as the two dimensional boundary of an open subset of $\mathbb{R}^3$. Say that the surface is given in terms of two parameters s and t by three functions $x(s, t)$, $y(s, t)$, $z(s, t)$. We write $\mathbf{r}(s, t) = x(s, t)\mathbf{i} + y(s, t)\mathbf{j} + z(s, t)\mathbf{k}$. The nondegeneracy condition that was mentioned above is that the surface has two linearly independent tangent vectors at each point, hence a genuine tangent plane at each point that varies nicely with the point. A vector orthogonal to this tangent plane is called a **normal vector**, and such a vector of length 1 is often denoted by $\mathbf{n}$. A unit normal vector at a particular point is determined up to sign. A smoothly embedded surface need not have a continuously varying unit normal vector; the Möbius band does not. A surface in $\mathbb{R}^3$ has a continuously varying unit normal vector if and only if it is orientable. Orientability is often disposed of quickly in physics and engineering applications, often by a phrase such as "with the region on the left" or "with an outward pointing unit normal vector." We shall see examples in the next section.

In the language of differential forms, line integrals are integrals of 1 forms over oriented manifolds of dimension 1, and surface integrals are integrals of 2 forms over oriented manifolds of dimension 2. In a sense, that fact remains true in the notation used in physics and engineering, but the differential forms are somewhat concealed. In the notation of physics and engineering, a line integral is an expression like

$$\int_C \mathbf{F} \cdot d\mathbf{s}.$$

where C is a curve. We are to think of $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ as a force field, assigning a quantity of force to a particle at any particular point of C or perhaps at any point of an open set containing C. Also we are to think of $d\mathbf{s}$ as $\mathbf{i}dx+\mathbf{j}dy+\mathbf{k}dz$. The integral represents the limit of a sum of infinitesimal displacements multiplied by values of $\mathbf{F}$, each summand of force times displacement representing a quantity of work (energy). A rigorous definition using a limiting process appears in Chapter III of *Basic Real Analysis*, but we need not be concerned with that point at present. Operationally we evaluate $\int_C \mathbf{F} \cdot d\mathbf{s}$ the same way we evaluate the integral of the 1 form $F_1\,dx + F_2\,dy + F_3\,dz$, namely by parametrizing the oriented curve with a parameter t, substituting for dx, dy, and dz in terms of dt, and evaluating an ordinary Riemann integral.

Similarly in the notation of physics and engineering, a surface integral is an expression like

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where

$$d\mathbf{S} = \begin{pmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{pmatrix}.$$

The integral $\int_S \mathbf{F} \cdot d\mathbf{S}$ is evaluated in the same way as the integral of the 2 form $F_1\,dy \wedge dz + F_2\,dz \wedge dx + F_3\,dx \wedge dy$. The interpretation of the integral is of the total "flux" crossing the surface, with $\mathbf{F}$ telling how much flux per unit area is crossing the surface at each point and with $d\mathbf{S}$ representing infinitesimal area of the surface. "Flux" is a term in physics whose exact meaning depends on the particular application. In hydrodynamics it is a quantity of fluid. The term is used also in electromagnetic theory. Since we know how to work with the integral of a smooth 2 form, we need not be concerned with incorporating a rigorous passage to the limit into our definition of surface integral.

Let us turn to arc length and surface area. Arc length was defined rigorously in Chapter III of *Basic Real Analysis* by a passage to the limit. For surface area, however, we found that an approach by taking a limit of areas of inscribed surfaces does not work, and consequently the surface area of a manifold of dimension 2 requires extra structure for a meaningful definition. It would be enough to have the surface smoothly embedded in $\mathbb{R}^3$, and then unit normal vectors to the surface are available. This concept takes us beyond the mathematics needed for Stokes's Theorem, and we shall not pursue it after this paragraph except to say that a unit normal vector to the surface in $\mathbb{R}^3$ defined parametrically by $\mathbf{r}(s, t)$ is

$$\mathbf{n} = \Big|\frac{\partial \mathbf{r}(s,t)}{\partial s} \times \frac{\partial \mathbf{r}(s,t)}{\partial t}\Big|^{-1}\Big(\frac{\partial \mathbf{r}(s,t)}{\partial s} \times \frac{\partial \mathbf{r}(s,t)}{\partial t}\Big)$$

and that the total surface area of a surface is given by integration of a scalar quantity dS over the surface, dS being related to the vector quantity $d\mathbf{S}$ by the formulas

$$d\mathbf{S} = \frac{\partial \mathbf{r}(s,t)}{\partial s} \times \frac{\partial \mathbf{r}(s,t)}{\partial t}\, ds\, dt$$
$$dS = \Big|\frac{\partial \mathbf{r}(s,t)}{\partial s} \times \frac{\partial \mathbf{r}(s,t)}{\partial t}\Big|\, ds\, dt$$

Finally we want to see in some detail how Green's Theorem, the Divergence Theorem, and the Kelvin–Stokes Theorem arise as special cases in dimensions 2 and 3 of Stokes's Theorem. That is the topic for the next section.

6. Low Dimensional Cases of Stokes's Theorem

Let us examine the meaning of Stokes's Theorem for compact manifolds-with-boundary that are subsets of Euclidean spaces in dimensions 2 and 3. In every case we need to pay particular attention to orientations. In each case we shall state the classical result that comes from Theorem 2.7, explain the choices that are being made, and give a simple example. More complicated examples appear in the problems at the end of the chapter.

We have already handled the case of a closed interval in $\mathbb{R}^1$ as an example at the end of the previous section; it yields the Fundamental Theorem of Calculus on a closed bounded interval of $\mathbb{R}^1$. In dimensions 2 and 3, the situations to examine are those of Green's Theorem in $\mathbb{R}^2$, the Divergence Theorem in $\mathbb{R}^2$ and $\mathbb{R}^3$, the Kelvin–Stokes Theorem in $\mathbb{R}^3$, and integration of a differential along a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Before coming to the analysis of cases, let us summarize what we know about orientations from the previous section and Section I.6. Orientation of a smooth manifold M of dimension m amounts to a parity condition and is constant on connected components. One way of expressing it is as the sign of a nowhere-vanishing form of the top degree m. The standard orientation on $\mathbb{R}^m$ corresponds to the m form $dx_1 \wedge \cdots \wedge dx_m$. Permuting the variables corresponds to permuting an ordered basis, and the effect on orientation is given by the sign of the determinant of the change-of-basis matrix, which is the same as the sign of the permutation. An orientation on a smooth manifold-with-boundary is given by an orientation on the set of manifold points, and this induces an orientation on the boundary points. This process of inducing an orientation may or may not seem natural; primarily it is designed to make the signs come out right in Stokes's Theorem. One way of describing the process is the following: One works with the tangent space to M, starts with an outward pointing vector from the boundary, and extends that one-element set of vectors to an ordered basis of the tangent space by adjoining tangent vectors to the boundary. Then one takes that basis into account in parametrizing the boundary.

a. Green's Theorem. Rather than try to make the above general description more precise all at once, let us see how it is to work in increasingly complex examples. We begin with Green's Theorem, whose statement in the current setting is as follows.

Theorem 2.8 (GREEN'S THEOREM). Let M be a compact oriented smooth manifold-with-boundary of dimension 2 within $\mathbb{R}^2$. If P and Q are smooth functions on an open subset of $\mathbb{R}^2$ containing M, then

$$\int_{\partial M} P\,dx + Q\,dy = \int_M \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dx\,dy,$$

provided ∂M is given the induced orientation.

Here Theorem 2.7 is being applied on M to the 1 form $\omega = P\,dx + Q\,dy$. According to Example 1 in Section I.4, $d\omega$ equals $\big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\big)\,dx \wedge dy$. The manifold M_+ is understood to be given the standard orientation from $\mathbb{R}^2$, which is determined by the 2 form $dx \wedge dy$. Evaluation of the integral $\int_M d\omega$ can be done by using Theorem 1.29; since $dx \wedge dy$ corresponds to the standard orientation of $\mathbb{R}^2$, $dx \wedge dy$ is to be replaced by $dx\,dy$ in a double integral.

According to Theorem 2.8, ∂M is to be given the induced orientation. This means informally that a parametrization of the boundary curve ∂M is to trace out the curve "with the region on the left." More formally let the parametrization be $t \mapsto \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. The derivative is $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$; it is assumed that $x'(t)$ and $y'(t)$ never vanish simultaneously, so that at each point the Inverse Function Theorem applies either in x or in y and shows that locally one of the variables x and y is a smooth function of the other. At the point of the curve where $t = t_0$, the tangent space in $\mathbb{R}^2$ has an ordered basis $\Big(v, \begin{pmatrix} x'(t_0) \\ y'(t_0) \end{pmatrix}\Big)$, where v is a vector pointing outward from the boundary. This basis can be transformed into the standard basis of $\mathbb{R}^2$ by a linear map of nonzero determinant. In terms of orientations, the determinant is positive if and only if the parametrization of the curve at t_0 is consistent with the induced orientation of the boundary,[9] and it remains consistent for all t while the parametrization is in force.

EXAMPLE. Let M be the closed annulus $\{(x, y) \in \mathbb{R}^2 \mid 1 \le x^2 + y^2 \le 4\}$. The boundary consists of two circles, the outer circle being traversed counterclockwise (so that M_+ is on the immediate left) and the inner circle being traversed clockwise (so that M_+ is on the immediate left). Let $\omega = P\,dx + Q\,dy = y\,dx$. Then $d\omega = -dx \wedge dy$. So $\int d\omega = \int_M -1\,dx\,dy = -\text{Area}(M_+) = -3\pi$. We can parametrize the boundary by two circles, one being $t \mapsto (2\cos t, 2\sin t)$ for $0 \le t \le 2\pi$ and the other being $t \mapsto (\cos t, -\sin t)$ for $0 \le t \le 2\pi$. Then

$$\begin{aligned}\textstyle\int_{\partial M} x\,dx &= \textstyle\int_0^{2\pi} 2\sin t\,d(2\cos t) - \sin t\,d(\cos t)\\ &= \textstyle\int_0^{2\pi} (-4\sin^2 t + \sin^2 t)\,dt = -3\pi.\end{aligned}$$

b. Divergence Theorem. The Divergence Theorem works for manifolds-with-boundary in any number of dimensions $m \ge 2$. Let us begin with dimension 3. We state the theorem in that case, remark about orientation, and give an example. Then we make remarks about the case of general dimension m and say how the result in dimension 2 compares with Green's Theorem. Finally we restate the Divergence Theorem in dimension 3 in the notation of the previous section that is often used in physics and engineering.

Theorem 2.9 (DIVERGENCE THEOREM). Let M be a compact oriented smooth manifold-with-boundary of dimension 3 within $\mathbb{R}^3$. If F_1, F_2, F_3 are smooth real-valued functions on an open subset of $\mathbb{R}^3$ containing M, then

$$\int_{\partial M} F_1\,dy \wedge dz + F_2\,dz \wedge dx + F_3\,dx \wedge dy = \int_M \Big(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\Big)\,dx\,dy\,dz,$$

provided ∂M is given the induced orientation.

Theorem 2.9 is the special case of Theorem 2.7 applied to the 2 form

$$F_1\,dy \wedge dz + F_2\,dz \wedge dx + F_3\,dx \wedge dy.$$

According to Example 2 in Section I.4, $d\omega = \big(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial x}\big)\,dx \wedge dy \wedge dz$. M is oriented by the standard orientation of $\mathbb{R}^3$, the one determined by $dx \wedge dy \wedge dz$. To sort out the meaning of the induced orientation, we start with a

[9] Positive determinant means that the tangent vector to the curve points to the left of the outward-pointing vector v; thus the inward-pointing vector $-v$ points to the left of the tangent vector, and the region is on the left of the curve.

parametrization of the surface ∂M, say $(s,t) \mapsto \begin{pmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{pmatrix}$. This parametrization does not have to work everywhere on M at once. Consistent local parametrizations will be good enough because of the assumed orientability. The derivative matrix is the 3-by-2 matrix $\begin{pmatrix} \partial x/\partial s & \partial x/\partial t \\ \partial y/\partial s & \partial y\partial t \\ \partial z/\partial s & \partial z/\partial t \end{pmatrix}$. The assumption on the parametrization to make it locally invertible is that this derivative matrix has rank 2 everywhere. Then about every point of the surface, one can in principle solve for one of the variables x, y, z in terms of the other two, according to the Inverse Function Theorem. At the point of the surface where $(s,t) = (s_0, t_0)$, the tangent space in $\mathbb{R}^3$ has an ordered basis $\left(v, \begin{pmatrix} \partial x/\partial s \\ \partial y/\partial s \\ \partial z/\partial s \end{pmatrix}_{(s_0,t_0)}, \begin{pmatrix} \partial x/\partial t \\ \partial y/\partial t \\ \partial z/\partial t \end{pmatrix}_{(s_0,t_0)}\right)$, where v is a vector pointing outward from the boundary. This basis can be transformed into the standard basis of $\mathbb{R}^3$ by a linear map of nonzero determinant. If the determinant is positive, then the parametrization of the surface near (s_0, t_0) is consistent with the induced orientation of the boundary. Conversely if the determinant is negative, then the parametrization of the surface is consistent with the opposite of the induced orientation.

EXAMPLE. Let M be the closed unit ball

$$M = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \le 1\}.$$

The boundary is the unit sphere $\partial M = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$, and it is to be given the induced orientation. Let $\omega = z\,dx \wedge dy$. Then $d\omega = dx \wedge dy \wedge dz$. So $\int_M d\omega = \int_M dx\,dy\,dz = \text{Volume}(M_+) = 4\pi/3$.

To evaluate $\int_{\partial M} z\,dx \wedge dy$ directly and check Theorem 2.7 in this case, we need to parametrize the sphere. We can use ordinary spherical coordinates (φ, θ) near most points for this purpose:

$$\begin{pmatrix} x(\varphi,\theta) \\ y(\varphi,\theta) \\ z(\varphi,\theta) \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \end{pmatrix}$$

for $0 < \varphi < \pi$ and $-\pi < \theta < \pi$. The derivative matrix is

$$\begin{pmatrix} -\sin\varphi & 0 \\ \cos\varphi\cos\theta & -\sin\varphi\sin\theta \\ \cos\varphi\sin\theta & \sin\varphi\cos\theta \end{pmatrix}.$$

This derivative matrix is convenient to examine at $(\varphi,\theta) = (\pi/2, 0)$, which corresponds to $(x,y,z) = (0,1,0)$. At this point, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an outward pointing vector from the closed unit ball. The derivative matrix at this point is

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

If the outward pointing vector is adjoined to this matrix as its first column, the determinant of the resulting 3-by-3 matrix is $+1$, positive. Thus our parametrization gives us the induced orientation, not its opposite. To evaluate $\int_{\partial M} z\,dx\wedge dy$, we compute

$$\begin{aligned} dx\wedge dy &= \left(\tfrac{\partial(x,y)}{\partial(\varphi,\theta)}\right) d\varphi\wedge d\theta \\ &= \det\begin{pmatrix} -\sin\varphi & 0 \\ \cos\varphi\cos\theta & -\sin\varphi\sin\theta\end{pmatrix} d\varphi\wedge d\theta = \sin^2\varphi\sin\theta\,d\varphi\wedge d\theta. \end{aligned}$$

Then

$$\int_{\partial M} z\,dx\wedge dy = \int_{\varphi=0}^{\pi}\int_{\theta=-\pi}^{\pi} \sin\varphi\sin\theta\sin^2\varphi\sin\theta\,d\theta\,d\varphi = \pi\int_{\varphi=0}^{\pi}\sin^3\varphi\,d\varphi.$$

One readily checks that the φ integral equals $4/3$, and thus the surface integral equals $4\pi/3$, in agreement with the statement of the Divergence Theorem.

REMARK. It may at first appear that Theorem 2.9 applies to many familiar regions of $\mathbb{R}^3$. But one has to remember that the hypotheses require the closure of the region to be a smooth manifold-with-boundary. In particular the boundary has to be smooth. A solid hemisphere does not fit the hypotheses. Often the region between two smooth surfaces suffers the same drawback, having "corners" where the two surfaces meet. The relevant setting to handle this situation is that of a "smooth manifold-with-corners." Such objects will be discussed in Chapter III.

In general dimension m, Theorem 2.7 gives

$$\int_{\partial M}\sum_{i=1}^{m}(-1)^{i-1}F_i\,dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_m = \int_M\sum_{i=1}^{m}\frac{\partial F_i}{\partial x_i}\,dx_1\cdots dx_m,$$

where the circumflex indicates a missing factor. In dimension 2, the formula reduces to

$$\int_{\partial M} F_1 dy - F_2\,dx = \int_M\Big(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\Big)\,dx\,dy.$$

This matches the formula of Green's Theorem if we put $F_2 = -P$ and $F_1 = Q$.

In the notation of the previous section that often arises in physics and engineering, the integral formula in the Divergence Theorem can be written more briefly as

$$\int_{\partial M} F\cdot d\mathbf{S} = \int_M (\operatorname{div} F)\,dV$$

or as $\int_{\partial M} F\cdot d\mathbf{S} = \int_M(\nabla\cdot F)\,dV$, where dV is shorthand for the volume element. The manifold-with-boundary M has dimension 3 and is assumed to lie in $\mathbb{R}^3$. It follows that its set M_+ of manifold points is an open subset of $\mathbb{R}^3$. Then M_+ inherits the standard orientation from $\mathbb{R}^3$, and it is understood that ∂M gets the induced orientation. Thus nothing explicit needs to be said about orientations or normal vectors.

c. Kelvin–Stokes Theorem. The classical form of Stokes's Theorem, also known as the Kelvin–Stokes Theorem, applies to a manifold-with-boundary of dimension 2 realized in $\mathbb{R}^3$. First we state the theorem and relate it to Theorem 2.7 for differential forms, and we make a few general comments about orientations in the Kelvin–Stokes Theorem. Second we work through a simple example, paying particular attention to orientations. Third we look at the example in the light of the notation in the previous section.

Theorem 2.10 (KELVIN–STOKES THEOREM). Let M be a compact oriented smooth manifold-with-boundary of dimension 2 within $\mathbb{R}^3$. If P, Q, R are smooth real-valued functions on an open subset of $\mathbb{R}^3$ containing M, then

$$\iint_S \Big(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\Big)\, dy \wedge dz + \Big(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\Big)\, dz \wedge dx + \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\, dx \wedge dy$$
$$= \int_\gamma P\, dx + Q\, dy + R\, dz.$$

provided ∂M is given the induced orientation.

Here Theorem 2.7 is being applied to the 1 form

$$\omega = P\, dx + Q\, dy + R\, dz.$$

According to Example 2 in Section I.4,

$$d\omega = \big(\tfrac{\partial R}{\partial y} - \tfrac{\partial Q}{\partial z}\big)\, dy \wedge dz + \big(\tfrac{\partial P}{\partial z} - \tfrac{\partial R}{\partial x}\big)\, dz \wedge dx + \big(\tfrac{\partial Q}{\partial x} - \tfrac{\partial P}{\partial y}\big)\, dx \wedge dy.$$

About orientations for this setting, M_+ is not an open subset of the Euclidean space $\mathbb{R}^3$ is which it lives; thus it does not automatically inherit an orientation from $\mathbb{R}^3$. By assumption, M_+ is orientable, and we must actually choose an orientation. One way to do so is to make use of a local parametrization, since a local parametrization allows us to identify part of M_+ with an open subset of the Euclidean space of parameters and transfer the standard orientation from that Euclidean space to M_+. Once that step is done, then the orientation of M_+ can be pieced together, ∂M acquires the induced orientation, and we can proceed. Observe that an orientation of $\mathbb{R}^3$ plays no role in this construction.

EXAMPLE. Let M be the cylinder in Figure 2.3 given by

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ \ z^2 \le 1\}.$$

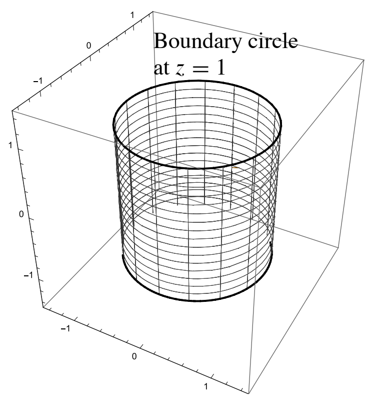

FIGURE 2.3. M and ∂M in an example for the Kelvin–Stokes Theorem.

The boundary ∂M consists of two circles:

$$\partial M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 \text{ and } z = \pm 1\},$$

and it is to be given the induced orientation from M_+, whatever the orientation of M_+ might mean. To get at such an orientation locally, we can parametrize the cylinder locally with a pair (r, θ) of parameters, r for the z component and θ for the angle made with (x, y). One parametrization of M_+ is

$$\alpha(r, \theta) = \begin{pmatrix} x(r,\theta) \\ y(r,\theta) \\ z(r,\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ r \end{pmatrix},$$

valid for $-1 < r < 1$ and $-\pi < \theta < \pi$, let us say. The space of all parameters (r, θ), being an open subset of $\mathbb{R}^2$, contains a standard orientation given by $dr \wedge d\theta$, and we move this over to M_+ by the pullback of α^{-1}. Thus we obtain a nowhere-vanishing 2 form on an open set of M_+. We can argue similarly with different parameters for a second open subset of M_+, and the two open sets together cover M_+. The assumption that M_+ is orientable implies that these nowhere-vanishing 2 forms can be chosen consistently from the one open set to the other, and then we have realized our orientation of M_+ more or less concretely. To use this information, we form the derivative matrix of α, which is

$$D\alpha(r, \theta) = \begin{pmatrix} 0 & -\sin\theta \\ 0 & \cos\theta \\ 1 & 0 \end{pmatrix}.$$

Its columns span the tangent space of M_+ at the point of M corresponding to (r, θ).

We can parametrize the boundary circles one at a time, the one at $z = 1$ being given by

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ 1 \end{pmatrix}, \quad \text{with derivative} \quad \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix}.$$

To orient this component of ∂M, we seek a tangent vector to M that points outward from ∂M. Consider a single point of ∂M, say $(1, 0, 1)$, which arises when $r = 1$ and $\theta = 0$. The tangent space at this point has basis $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$. An example of an outward pointing tangent vector at this point is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, since the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is in the span of the two columns of the derivative matrix and is not a multiple of the second vector in the basis. Working with the induced orientation of ∂M means that when ∂M is parametrized, the derivative vector of the parametrization points in a direction that is a positive multiple of the second column. In other words the above parametrization of the circle at $z = 1$ is consistent with the induced orientation on ∂M.

The candidates for such a vector are $\pm$ any vector that is in the span of the two columns of the full derivative matrix but is not a multiple of the second column, and $(0, 0, 1)$ will do fine. To have the vector point outward, we can use $(0, 0, 1)$ at $z = 1$. In our ordering of basis vectors yielding an orientation for M, this vector is to precede a tangent vector to ∂M, and that situation is already the case with the columns of the derivative matrix as is. Thus the above parametrization of the circle at $z = 1$ is consistent with the induced orientation.

Now let us orient the boundary circle[10] at $z = -1$. We select a single point of this part of the boundary to examine. The point $(0, 1, -1)$, which arises when $\theta = \pi/2$, will do fine. An outward pointing tangent vector can be taken to be $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$. If we write v for the second vector in a basis purporting to give the induced orientation on ∂M, then the linear map that carries the tangent space to itself, sends $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and sends v to $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ must have positive determinant. This means that v is a negative multiple of $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. In other words the parametrization of ∂M as $\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ -1 \end{pmatrix}$ is inconsistent with the induced orientation on ∂M. So we

[10] An observant reader will say that we are merely ensuring that the circle with $z = -1$ is traversed with the region on its left, just as the circle with $z = 1$ was. Resorting to familiar geometric intuition is all very well in this case, but the method being discussed here works even in higher dimensional cases when ∂M need not be 1 dimensional.

should use its opposite as in Figure 2.4, parametrizing the circle by

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \\ -1 \end{pmatrix} \quad \text{with derivative} \quad \begin{pmatrix} -\sin t \\ -\cos t \\ 0 \end{pmatrix}.$$

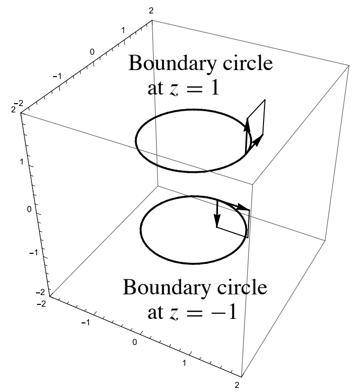

FIGURE 2.4. Tangent planes at points on the boundary circles in the example. The indicated planes are the respective tangents at $(1, 0, 1)$ and $(0, 1, -1)$.

Now let $\omega = yz\,dx$. Having parametrized both components of ∂M consistently with the induced orientation, we shall evaluate $\int_{\partial M} \omega = \int_{\partial M} yz\,dx$ in the two ways that Theorem 2.10 says should give the same answer. The signs will be crucial. One way is directly as the sum of two line integrals, namely as

$$\begin{aligned} &= \textstyle\int_{z=1} yz\,dx + \int_{z=-1} yz\,dx \\ &= \textstyle\int_{-\pi}^{\pi} (\sin t)(+1)(-\sin t)\,dt + \int_{-\pi}^{\pi} (\sin t)(-1)(\sin t)\,dt = -2\pi. \end{aligned}$$

The other way is as

$$\begin{aligned} \textstyle\int_M d\omega &= \textstyle\int_M d(yz) \wedge dx \\ &= \textstyle\int_M z\,dy \wedge dx + \int_M y\,dz \wedge dx \\ &= \textstyle -\int_M z\,dx \wedge dy + \int_M y\,dz \wedge dx. \end{aligned}$$

Referring to the derivative matrix $D\alpha(r, \theta)$, we have

$$dx \wedge dy = \tfrac{\partial(x,y)}{\partial(r,\theta)} = \det \begin{pmatrix} 0 & -\sin\theta \\ 0 & \cos\theta \end{pmatrix} = 0$$

and

$$dz \wedge dx = \tfrac{\partial(z,x)}{\partial(r,\theta)} = \det \begin{pmatrix} 1 & 0 \\ 0 & -\sin\theta \end{pmatrix} = -\sin\theta.$$

Therefore

$$\int_M d\omega = \int_M y\,dz \wedge dx = \int_0^{2\pi} \int_{-1}^{1} \sin\theta(-\sin\theta)\,dr\,d\theta$$
$$= -2\int_0^{2\pi} \sin^2\theta\,d\theta = -2\pi.$$

Thus indeed the two computations give the same answer.

Finally let us review the example in the light of the other systems of notation. The vector-valued function that we have been using is $F = (yz, 0, 0)$ or $F = yz\mathbf{i}$ or $F = \begin{pmatrix} yz \\ 0 \\ 0 \end{pmatrix}$, and

$$\operatorname{curl} F = \det \begin{pmatrix} \mathbf{i} & \frac{\partial}{\partial x} & yz \\ \mathbf{j} & \frac{\partial}{\partial y} & 0 \\ \mathbf{k} & \frac{\partial}{\partial z} & 0 \end{pmatrix} = -\mathbf{k}\big(\tfrac{\partial(yz)}{\partial y}\big) + \mathbf{j}\big(\tfrac{\partial(yz)}{\partial z}\big) = y\mathbf{j} - z\mathbf{k}.$$

Then

$$\operatorname{curl} F \cdot d\mathbf{S} = \begin{pmatrix} 0 \\ y \\ -z \end{pmatrix} \cdot \begin{pmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{pmatrix} = y\,dz \wedge dx - z\,dx \wedge dy.$$

d. Integration of a differential along a curve. In many applications of Stokes's Theorem, we are given $\int_{\partial M} \omega$ and we want to compute $\int_M d\omega$. Occasionally an application arises in which one wants to go in the other direction. In this case we are evaluating an integral $\int_M \eta$ for some m form η on M, where m is the dimension of M, and we recognize η as d of something, say $\eta = d\omega$. Then we can use the equality $\int_M \eta = \int_M d\omega = \int_{\partial M} \omega$.

This is what happens in the last low dimensional instance of Stokes's Theorem mentioned at the beginning os this section, namely the integration of a differential along a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$. We are to compute a line integral $\int_C \eta$, where η is a 1 form and C is a smooth curve with endpoints A and B in $\mathbb{R}^2$ or $\mathbb{R}^3$. A smooth curve with endpoints present is an example of a 1 dimensional manifold-with-boundary, and the above theory can apply. The only case in which Stokes's Theorem applies in straightforward fashion, however, is the case that the 1 form η is d of something, specifically d of a smooth function f. Thus suppose that the 1 form η that we are integrating is equal to a differential df. Then we have

$$\int_C \eta = \int_C df = f(B) - f(A).$$

This formula is an instance of Stokes's Theorem, but it is really easier than that. If the curve C is parametrized as $\gamma(t)$ for $a \le t \le b$ with $\gamma(a) = A$ and $\gamma(b) = B$, then an application of the Fundamental Theorem of Calculus to the composition $f \circ \gamma$ immediately gives

$$\int_C df = \int_a^b f(\gamma(t))\,dt = f(\gamma(b)) - f(\gamma(a)) = f(B) - f(A).$$

e. Final remarks. In many authors' formulations of versions of Stokes's Theorem, inner products and normal vectors play a role in the statements of the theorems and in the proofs. This is so in the formulations of the classical theorems of the Introduction, for example. In the text we have systemically avoided this extra layer of structure. Stokes's Theorem is really something about the exterior derivative and integration of differential forms, not about orthogonality, and the text has sought to emphasize this point. The cost has been small. We have had to work with "outward pointing tangent vectors" from the boundary of a manifold-with-boundary rather than outward "normal vectors." The inner product focuses attention on one good choice of an outward vector, but it does not help otherwise in the theory.

7. Problems

1. In $\mathbb{R}^3$, show that $|u \times v|^2 + (u \cdot v)^2 = |u|^2|v|^2$.
2. If $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, and $w = (w_1, w_2, w_3)$ are vectors in $\mathbb{R}^3$, show that $\det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$, up to sign, is the volume of the parallelepiped with sides u, v, and w.
3. If $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, and $w = (w_1, w_2, w_3)$ are vectors in $\mathbb{R}^3$, which of the six expressions $u \cdot (v \times w)$, $u \cdot (w \times v)$, $v \cdot (u \times w)$, $v \cdot (w \times u)$, $w \cdot (u \times v)$, and $w \cdot (v \times u)$ are equal to the first one. What is the relationship of the first one to the others?
4. (a) Compute div F and curl F for $F = x^2y\mathbf{i} - (z^3 - 3x)\mathbf{j} + 4y^2\mathbf{k}$.
   (b) Compute div F and curl F for $F = (3x + 2z^2)\mathbf{i} + x^3y^2\mathbf{j} - (z - 7x)\mathbf{k}$.
5. Let M be a smooth compact orientable manifold without boundary of dimension m. Proposition 1.30 showed that M has a nowhere-vanishing smooth m form ω. Use Stokes's Theorem to show that ω cannot be obtained as $d\eta$ for a smooth $m - 1$ form η.
6. (a) Exhibit a smooth differential 2 form ω on $\mathbb{R}^4$ such that $\omega \wedge \omega \neq 0$.
   (b) Suppose that M is a compact orientable smooth manifold of dimension $2n$ without boundary. Suppose that α is a smooth differential 1 form on M, so that $d\alpha$ is a 2 form. Can the n fold wedge product $\omega = d\alpha \wedge \cdots \wedge d\alpha$ be nowhere vanishing? If so, exhibit such an ω for some M. If not, prove that such an ω can never be nowhere vanishing.

7. (a) Show that
$$\omega = \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$
has $d\omega = 0$ in $\mathbb{R}^3 - \{0\}$.
(b) Let T be the torus in $\mathbb{R}^3$ given by rotating the unit circle in the x-z plane about the line where $x = 2$ and $y = 0$. It is the locus where
$$\Big(\sqrt{(x-2)^2 + y^2} - 2\Big)^2 + z^2 = 1.$$
Evaluate the integral $\int_T \omega$, where ω is as in (a) and where T is oriented so that the unbounded component of $\mathbb{R}^3 - T$ is "outside" the torus.

8. Let M be the subset of $\mathbb{R}^3$ lying between the sphere S_1 of radius 1 and the sphere S_a of positive radius a with $a < \frac{1}{2}$. Regard M as a manifold-with-boundary that inherits its orientation from the standard orientation of $\mathbb{R}^3$, and give its boundary $S = S_1 \cup S_a$ the induced orientation. Let F be the vector-valued function $F(x) = |x|^{-3}x$.
(a) Show that $\operatorname{div} F = 0$ on M.
(b) Why is $\int_{S_1} F \cdot d\mathbf{S} = \int_{S_a} F \cdot d\mathbf{S}$?

9. Generalize the formula in (a) of the Problem 7a by finding a smooth $n-1$ form $\omega = f(x_1, \dots, x_n)^{-1}\eta$ on $\mathbb{R}^n - \{0\}$ such that $d\eta = dx_1 \wedge \cdots \wedge dx_n$ and $d\omega = 0$.

10. By examining the example of $\mathbb{H}^m$ in Sections 3 and 4, show for every $m \geq 1$ that if $\partial\mathbb{H}^m$ is made to correspond to $x_1 = 0$ and if $\partial\mathbb{H}^m$ gets its orientation from $dx_2 \wedge \cdots \wedge dx_m$, then one is led to Stokes formula for $\mathbb{H}^m$ with a single minus sign (rather than $(-1)^m$) on one of the two sides of the formula.

Problems 11–15 concern surface integrals and the Kelvin–Stokes Theorem in $\mathbb{R}^3$.

11. Evaluate the surface integral $\int_S x\mathbf{i} \cdot d\mathbf{S}$, where S is the surface in $\mathbb{R}^3$ given by $z = x^2 + y^2$ for $z \leq 4$ and S is oriented by an outward/downward pointing normal vector.

12. Use the Kelvin–Stokes Theorem to compute $\int_S \operatorname{curl} F \cdot d\mathbf{S}$, where $F(x, y, z) = yz\mathbf{i} + xy\mathbf{k}$ and S is the part of sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the x-y plane. The surface is oriented by an outward pointing normal vector.

13. Let F be the vector-valued function $F = (-yz, 4y+1, xy+e^z)$, and let C be the oriented curve $\mathbf{s}(t) = (3\cos t, 4, 3\sin t)$. This is the circle of radius 3 given by $x^2 + y^2 = 9$ and $y = 4$. With the help of the Kelvin–Stokes Theorem, evaluate the line integral $\int_C F \cdot d\mathbf{s}$.

14. Use the Kelvin–Stokes Theorem to evaluate $\int_S \operatorname{curl} F \cdot d\mathbf{S}$ if $F = (y, -x, yx^3)$ and S is the portion of the sphere of radius 4 about the origin having $z \geq 0$ and the upward orientation.

15. Evaluate $\int_C F \cdot d\mathbf{s}$, where $F(x, y, z) = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$, and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. The curve C is to be oriented counterclockwise when the x-y plane is viewed as horizontal and the curve is viewed from above. Do this in two ways, as follows:
    (a) directly by parametrizing the curve by the angle θ in the x-y plane,
    (b) by using the Kelvin–Stokes Theorem, taking C to be the boundary of the filled ellipse in the plane where $y + z = 2$.

Problems 16–20 establish the Brouwer Fixed-Point Theorem, which says that whenever a continuous function f carries the closed unit ball $\overline{B} = \{x \in \mathbb{R}^n \mid |x \leq 1\}$ of $\mathbb{R}^n$ into itself, then there is some x in the ball with $f(x) = x$. Let

$$B = \{x \in \mathbb{R}^n \mid |x < 1\} \quad \text{and} \quad \partial B = \{x \in \mathbb{R}^n \mid |x = 1\}.$$

A **retraction** of $\overline{B}$ into ∂B is a continuous function $r : \overline{B} \to \partial B$ such that r is the identity on ∂B. The line of proof will be to show that there is no smooth retraction, that the fixed-point theorem follows in the smooth case from the nonexistence of a smooth retraction, and that the fixed-point theorem in the smooth case implies the fixed-point theorem in the general case.

16. This problem and the next show that there is no smooth retraction of $\overline{B}$ onto ∂B. In fact, suppose that a smooth retraction $r : \overline{B} \to \partial B$ exists. Let ω be a nowhere-vanishing $n-1$ form on ∂B; this has to exist on ∂B by Proposition 1.30 because Problem 15 at the end of Chapter I showed that all spheres are oriented. Justify the following steps in a computation for the smooth manifold-with-boundary $\overline{B}$:
$$0 < \int_{\partial B} \omega = \int_{\partial B} r^*(\omega) = \int_B dr^*(\omega) = \int_B r^*(d\omega).$$

17. Explain why the right side is 0 in the displayed line of the previous problem and why the retraction r cannot exist.

18. Show that if $f : \overline{B} \to \overline{B}$ is a smooth function such that $f(x) \neq x$ for all x in $\overline{B}$, then one can construct from f a smooth retraction r of $\overline{B}$ onto ∂B. Since the previous two problems have shown that there is no such smooth retraction, every smooth $f : \overline{B} \to \overline{B}$ has a fixed point.

19. If $f : \overline{B} \to \overline{B}$ is a continuous function, show that there exists a sequence $\{f_k\}$ of smooth functions from $\overline{B}$ into $\overline{B}$ that converges uniformly to f on $\overline{B}$.

20. If $f : \overline{B} \to \overline{B}$ is a continuous function, choose by the previous problem a sequence $\{f_k\}$ of smooth functions carrying $\overline{B} \to \overline{B}$ and converging uniformly to f on $\overline{B}$. Using Problem 18, let x_k be a point in $\overline{B}$ with $f_k(x_k) = x_k$. If x_0 is a limit point of $\{x_k\}$ in $\overline{B}$, show that $f(x_0) = x_0$. Consequently f has a fixed point in $\overline{B}$.