Chapter 18

GEOMETRIC 2-MANIFOLDS

The concept "two-dimensional manifold" or "surface" will not be associated with points in three-dimensional space; rather it will be a much more general abstract idea. — Hermann Weyl (1913)

There is clearly a large variety of different surfaces around in our experiential world. The study of the geometry of general surfaces is the subject of differential geometry. In this chapter we will study ***geometric 2-manifolds***, that is, a connected space that locally is isometric to either the (Euclidean) plane, a sphere, or a hyperbolic plane. The surface of a cylinder (no top or bottom and indefinitely long) and a cone (with the cone point removed) are examples of geometric 2-manifolds. We study these because their geometry is simpler and closely related to the geometry we have been studying of the plane, spheres, and hyperbolic planes.

There are no prerequisites for this chapter from after Chapter 7, but some of the ideas may be difficult the first time around. Problems **18.1**, **18.3**, and **18.6** are the minimum that is needed from this chapter before you study Chapter 24 (3-Manifolds — Shape of Space); the other problems can be skipped.

We use the term "manifold" here instead of "surface" because we usually think of surfaces as sitting extrinsically in 3-space. Here we want to study only the intrinsic geometry; and thus, any particular extrinsic embedding does not matter. Moreover, we will study some geometric 2-manifolds (for example, the flat torus) that cannot be (isometrically) embedded in 3-space. We ask: what is a two-dimensional bug's intrinsic geometric experience on geometric 2-manifolds? How will the bug view geodesics (intrinsically straight lines) and triangles? How can a bug on a geometric 2-manifold discover the global shape of its universe? These questions will help us as we think about how we as human beings perceive our 3-dimensional physical universe, where *we* are the (3-dimensional) bugs.

This chapter will only be an introduction to these ideas. For a geometric introduction to differential geometry, see [**DG**: Henderson]. For more details about geometric 2-manifolds, see [**DG**: Weeks] and Chapter 1 of [**DG**: Thurston]. For the classification of (triangulated) 2-manifolds, see [**TP**: Francis & Weeks], which contains an accessible proof due to John H. Conway.

In Problem **4.1**, we already studied two examples of geometric 2-manifolds — cylinders and cones (without the cone point). Because these surfaces are locally isometric to the Euclidean plane, these types of geometric manifolds are called ***flat*** (or ***Euclidean***) ***2-manifolds***. It would be good at this point for you to review what you know from Chapter 4 about cylinder and cones.

PROBLEM 18.1 FLAT TORUS AND FLAT KLEIN BOTTLE

FLAT TORUS

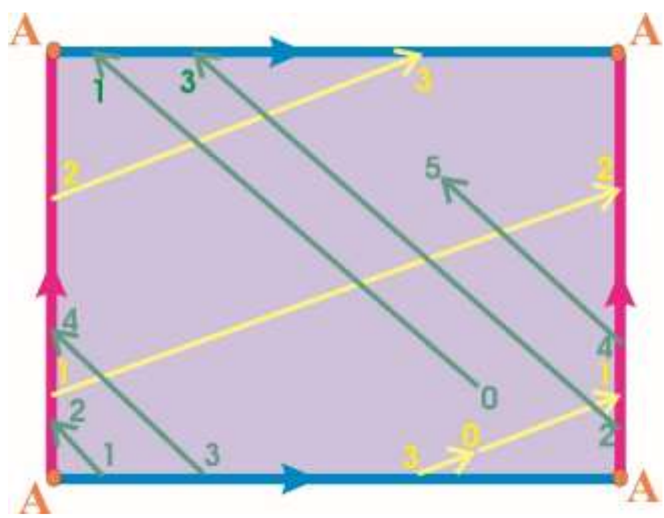


Figure 18.1 1-sheeted covering of flat torus

Another example of a flat (Euclidean) 2-manifold was one the first video games popular a while ago. A blip on the video screen representing a ball travels in a straight line until it hits an edge of the screen. Then the blip reappears traveling parallel to its original direction from a point at the same position on the opposite edge. Is this a representation of some surface? If so, what surface? First, imagine rolling the screen into a tube where the top and bottom edges are glued (Figure 18.1). This is a representation of the screen as a 1-sheeted covering of the cylinder. A blip on the screen that goes off the top edge and reappears on the bottom is the lift of a point on the cylinder that travels around the cylinder, crossing the line that corresponds to the joining of the top and bottom of the screen.

Now let us further imagine that the cylinder can be stretched and bent so that we can glue the two ends to make a torus. Now the screen represents a 1-sheeted covering of the torus. If the blip goes off on one side and comes back on the other at the same height, this represents the lift of a point moving around the torus and crossing the circle that corresponds to the place where the two ends of the cylinder are joined. The possible motions of a point on the torus are represented by the motions on the video screen! You can play some of torus games on Jeff Week's website
(http://geometrygames.org/TorusGames/index.html)

You can't make a model in 3-space of a flat torus from a flat piece of paper without distorting it. Such a torus is called a ***flat torus***. It is best not to call this a "surface," because there is no way to realize it isometrically in 3-space and it is not the surface of anything. But the question of whether or not you can make an isometric model in 3-space is not important — the point is that the gluings in Figure 18.1 intrinsically define a flat 2-manifold.

If you distort the cylinder in Figure 18.1 in 3-space, you can get the torus pictured in Figure 18.2. This is not a geometric 2-manifold because the original flat (Euclidean) geometry has been distorted and it is also not exactly either spherical or hyperbolic.

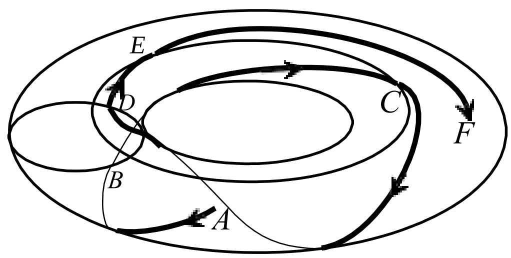


Figure 18.2 Non-flat torus

a. *Show that the flat torus is locally isometric to the plane and thus, is a geometric 2-manifold, in particular, a flat (Euclidean) 2-manifold.*

Note that each point on the interior of an edge of the screen is the lift of a point that has another lift on the opposite edge. Thus, a lift of a neighbor- hood of that is in two pieces (one near each of the two opposite edges). What happens at the four corners of the computer screen (which are lifts of the same point)?

The torus in Figure 18.2 and the flat torus are related in that there is a continuous one-to-one mapping from either to the other. We say that they are ***homeomorphic***, or ***topologically equivalent***. We can further express this situation by saying that the torus in Figure 18.2 and the flat torus are both ***topological tori***.

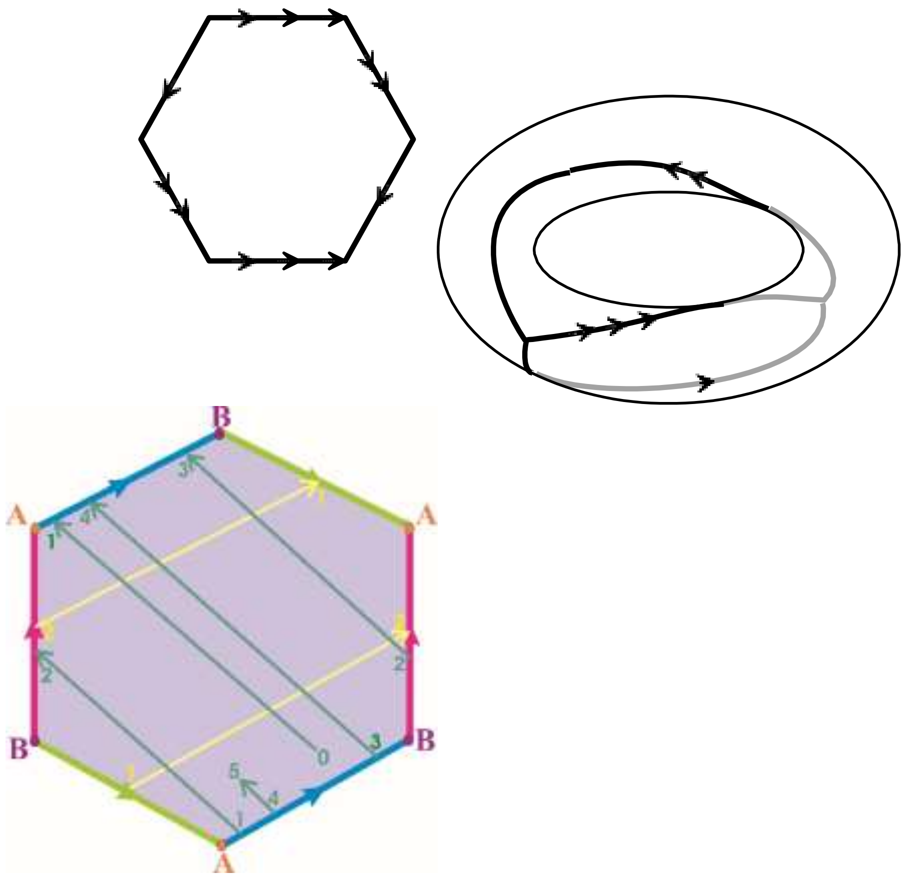


Figure 18.3 Flat torus from a hexagon

There is another representation of the flat torus based on a hexagon. Start with a regular hexagon in the plane and glue opposite sides as indicated in Figure 18.3.

> **b.** *Show that gluing the edges of the hexagon as in* Figure 18.3 *forms a flat 2-manifold (called the* ***hexagonal torus****).*

In order for a flat 2-manifold to be formed from a regular polygon, the interior angle of the polygon must be an integral factor of 360°. (*Do you see why?*) Thus, it is not possible to use other regular polygons to create flat 2-manifolds; see Problem **11.7**. However, we can see from Problem **11.7** that it may be possible for spherical and/or hyperbolic 2-manifolds to be created from regular polygons; see Problems **18.3** and **18.4**.

FLAT KLEIN BOTTLE

Now we will describe a related geometric 2-manifold, traditionally called a ***flat Klein bottle***, named after Felix Klein (1849–1925, German). It may have been originally named the *Kleinsche Fläche* ("Klein surface") and then misinterpreted as *Kleinsche Flasche* ("Klein bottle"), which ultimately may have led to the adoption of this term in the

German language as well. Imagine the same video screen, again with a traveling blip representing a ball that travels in a straight line until it hits an edge of the screen. When it hits the top edge then the blip proceeds exactly the same as for the flat torus (traveling parallel to its original direction from a point at the same position on the opposite edge). However, when the blip hits a vertical edge of the screen it reappears on the opposite edge but in the *diametrically opposite* position and travels in a direction with slope that is the negative of the original slope. (See Figure 18.4.) As before, imagine rolling the screen into a tube where the top and bottom edges are joined. This is again a representation of the screen as a 1-sheeted covering of the cylinder.

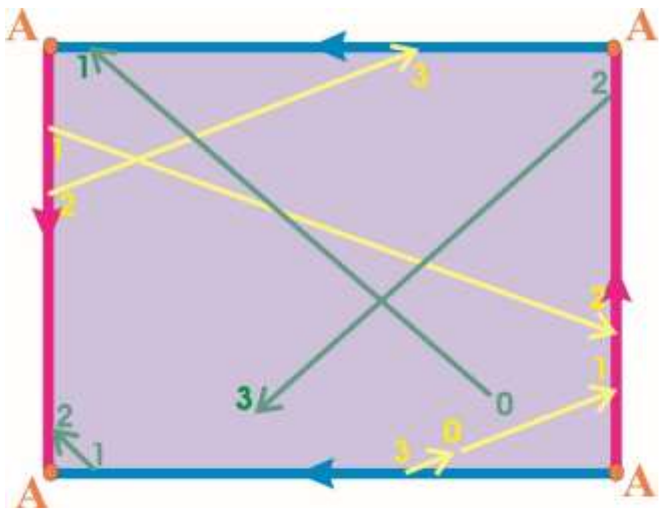


Figure 18.4 1-sheeted covering of a flat Klein bottle

A blip on the screen that goes off the top edge and reappears on the bottom is the lift of a point on the cylinder that travels around the cylinder crossing the line that corresponds to the joining of the top and bottom of the screen. Now the screen represents a 1-sheeted covering of the Klein bottle. The possible motions of a point on the Klein bottle are represented by the motions on the video screen.

Now, let us further imagine that the cylinder can be stretched and bent so that we can join the two ends to make a topological Klein bottle; see Figure 18.5. The figure in Figure 18.5 is not a geometric 2-manifold, because most points do not have neighborhoods that are isomorphic to either the plane, a sphere, or a hyperbolic plane. (see some animations here: https://plus.maths.org/content/introducing-klein-bottle)

You also can't make an isometric model in 3-space of a flat Klein bottle without distorting it and having self-intersections. But the gluings in Figure 18.4 define intrinsically a flat 2-manifold.

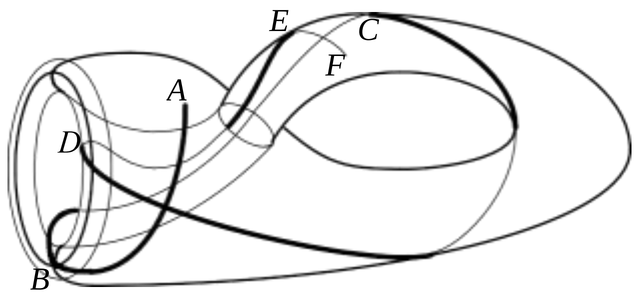


Figure 18.5 Topological Klein bottle

c. *Show that the flat Klein bottle is locally isometric to the plane and thus is a geometric 2-manifold, in particular, a flat (Euclidean) 2-manifold.*

Note that the four corners of the video screen are lifts of the same point and that a neighborhood of this point has 360° — that is, 90° from each of the four corners.

It can be shown that

THEOREM 18.1. *Flat tori and flat Klein bottles are the only flat* (*Euclidean*) 2-*manifolds that are finite and geodesically complete* (*every geodesic can be extended indefinitely*). See the last section in Chapter 4, Relations to Differential Geometry.

For a detailed discussion, see [**DG**: Thurston], pages 25–28. For a more elementary discussion, see [**DG**: Weeks], Chapters 4 and 11.

Note that a finite cylinder is not geodesically complete; and if it is extended indefinitely, then it is geodesically complete but not finite. A cone with the cone point is not a flat manifold at the cone point; with the cone point removed the cone is not geodesically complete.

Note that we get a flat torus for each size rectangle in the plane. These flat tori are different geometrically because there are different distances around the tori. However, topologically they are all the same as (homeomorphic to) the surface of a doughnut.

Note that if you move a right-hand glove (which we stylize by) around the flat torus, it will always stay right handed; however, if you move it around the flat Klein bottle horizontally, it will become left handed. See Figure 18.6. We describe these phenomena by saying that the flat torus is ***orientable*** and a Klein bottle is ***non-orientable***.

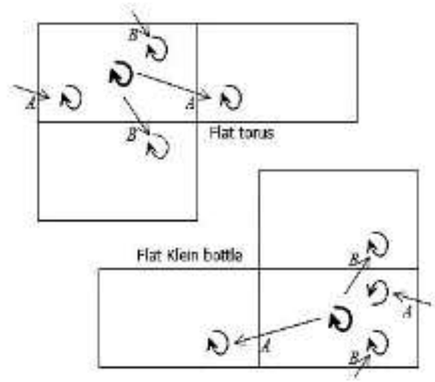


Figure 18.6 Orientable and non-orientable

PROBLEM 18.2 UNIVERSAL COVERING OF FLAT 2-MANIFOLDS

Problem **4.2** contains an introduction to some of the ideas in this problem and thus we urge the reader to look at Problem **4.2** before proceeding with this chapter.

a. *On a flat torus or flat Klein bottle, how do we determine the different geodesics connecting two points? How many are there? How can we justify our conjectures?*

Look at straight lines in the universal coverings introduced below.

b. *Show that some geodesics on the flat torus or flat Klein bottle are closed curves (in the sense that they come back and continue along themselves like great circles), though possibly self- intersecting. How can you find them?*

Look in the universal coverings introduced below.

c. *Show that there are geodesics on the flat torus and flat Klein bottle that never come back and continue along themselves.*

Look at the slopes of the geodesics found in part **b**.

The geodesics found in part **c** can be shown to come arbitrarily close to *every* point on the manifold. Such curves are said to be ***dense*** in the manifold.

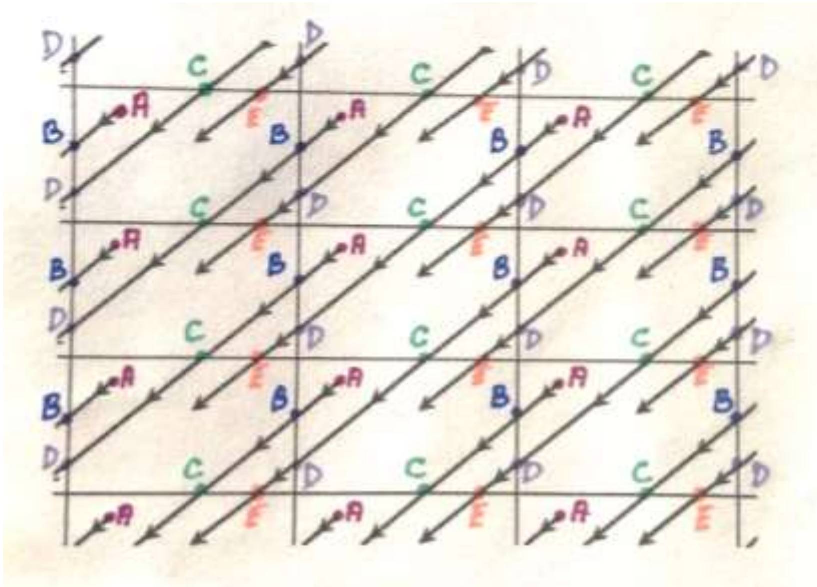


Figure 18.7 Universal covering of a flat torus

SUGGESTIONS

We suggest that you use coverings just as you did for cones and cylinders. The difference is that in this case the sheets of the coverings extend in two directions. See Figure 18.7 for a covering of the flat torus. If this covering is continued indefinitely in all directions, then the whole plane covers the flat torus with each point in the torus having infinitely many lifts. When a covering is the whole of either the (Euclidean) plane, a sphere, or a hyperbolic plane, it is called ***the universal covering***. See Figure 18.8 for a universal covering of a flat Klein bottle. These coverings are called "universal" because there are no coverings of the plane, spheres, or hyperbolic planes that have more than one sheet; see the next section for a discussion of this for a sphere.

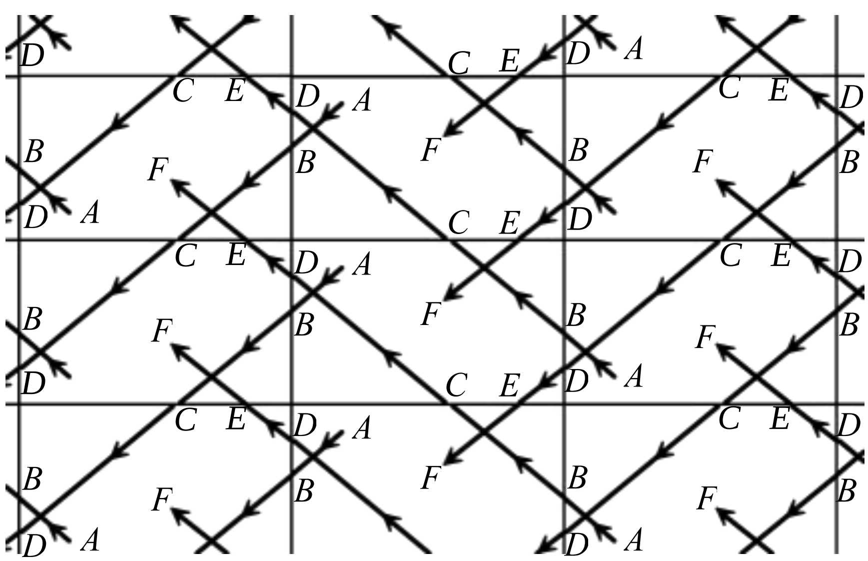


Figure 18.8 Universal covering of the flat Klein bottle

PROBLEM 18.3 SPHERICAL 2-MANIFOLDS

Start by considering another version of the video screen as depicted in Figure 18.9.

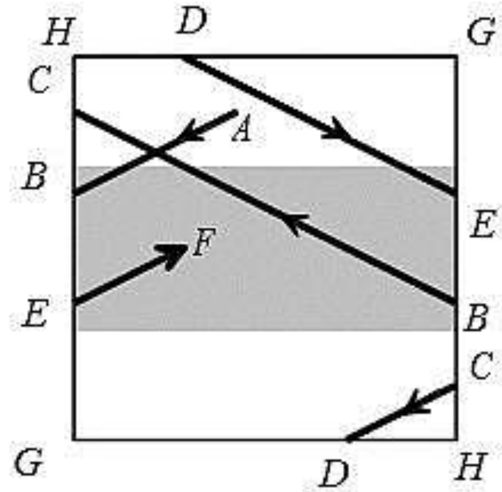


Figure 18.9 This is *not* a geometric 2-manifold

Imagine the same video screen, again with a traveling blip representing a ball that travels in a straight line until it hits an edge of the screen. When the blip reaches *any* edge of the screen, it reappears on the opposite edge but in the *diametrically opposite* position and travels in a direction with slope that is the negative of the original slope. (See Figure 18.9.)

a. *Show that the situation in Figure 18.9 does not represent a geometric 2-manifold because the corners represent two cone points with cone angles 180°.*

Cut around the corners marked *G* and tape together the edges as indicated.

If you restrict yourself to the shaded strip in Figure 18.9 and identify the left and right edges as indicated, then you obtain a ***Möbius strip***, named after August Möbius (German, 1790–1868). You may have seen this surface before — if not, you should be sure to construct one from a (preferably long) strip of paper. To have some fun with a Möbius strip, take a pencil and draw a line in a middle of your Möbius strip parallel to the edges. What happens? Cut your strip in half, following this line in a middle. What do you get? Make another Möbius strip and cut it starting about 1/3 from the edge of a strip and cut parallel to edges. What happens now? The Möbius strip fails to be a (flat) geometric manifold only because it has an edge (note that there is only one edge!); but it is an example of what is called a "flat geometric manifold with boundary."

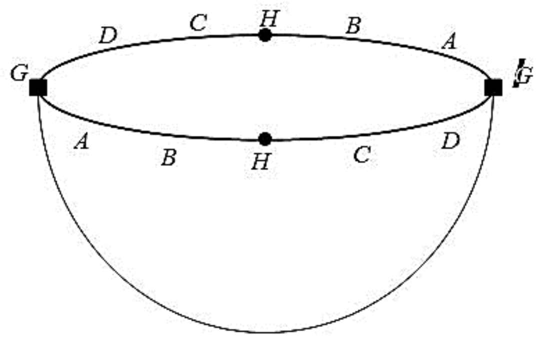


Figure 18.10 Gluings on a hemisphere producing a projective plane

The gluings (on all four edges) indicated in Figure 18.9 fail to produce a geometric 2-manifold because the interior angles are only 90°. It seems that we might get a geometric manifold if the interior angles were 180°. Thus, we need a quadrilateral with equal opposite sides and 180° interior angles. There is no such quadrilateral in the plane; however, on the sphere there *is* such a quadrilateral! See Figure 18.10.

What we have in Figure 18.10 is a hemisphere with each point of the bounding equator being glued to its antipode. In this way we get what is called the (real) projective plane, often denoted $\mathbf{RP}^2$.

b. *Show that a projective plane is a spherical 2-manifold.*

Examine the neighborhood of a point of the projective plane that comes from the bounding equator. Note that there is a "strip" connecting *B-H-C* to itself along the hemisphere that is very similar to the Möbius strip except that it is a *spherical* manifold with boundary.

c. *What are the geodesics on a projective plane?*

It is clear that the geodesics come from half great circles in the hemisphere, but what happens as one of these half great circles is crossing the equator that is glued?

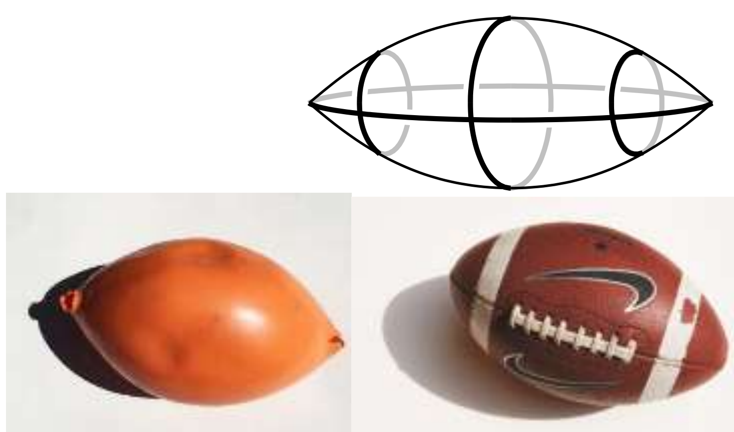


Figure 18.11 A spherical cone

If we cut out and remove a lune from a sphere and then join the two edges of the lune, we have what can reasonably be called a spherical cone. See Figure 18.11. Note that at least some of these spherical cones have a shape similar to an American football.

By pasting together several (equal radius) spheres with the same lune removed, you can get multiple-sheeted branched coverings of a spherical cone. A spherical cone with the two cone points removed is a finite spherical 2-manifold, but (as with ordinary cones) it is not geodesically complete.

d. *Show that the spherical cones as described above are spherical* 2*-manifolds if you remove the two cone points.*

e. *Identify the geodesics on a spherical cone with cone angle 180° (that is, you remove from the sphere a lune with angle 360° – 180° = 180°). What happens with other cone angles?*

Look at the great circles in the sphere minus the lune before its edges are joined to produce the spherical cone.

It can be shown that

THEOREM 18.3. *Spheres and projective planes are the only spherical* 2*-manifolds that are finite and geodesically complete* (*every geodesic can be extended indefinitely*).

For a detailed discussion, see [**DG**: Thurston], pages 25–28. For a more elementary discussion, see [**DG**: Weeks], Chapters 4 and 11.

COVERINGS OF A SPHERE

There is no way to construct a covering of a sphere that has more than one sheet unless the covering has some "branch points." A ***branch point*** on a covering is a point such that every neighborhood (no matter how small) surrounding the point contains at least two lifts of some point. In any covering of a cone with more than one sheet, the lift of the cone point is a branch point, as you can see in Figure 18.12.

Notice that the coverings of a cylinder and a flat torus have no branch points. For a sphere the matter is very different — any covering of a sphere will have a branch point. You can see this if you try to construct a cover by slitting two spheres, as depicted in Figure 18.13 and then sticking the two together along the slit. The ends of the slit would become branch points. This topic may be explored further in textbooks on geometric or algebraic topology.

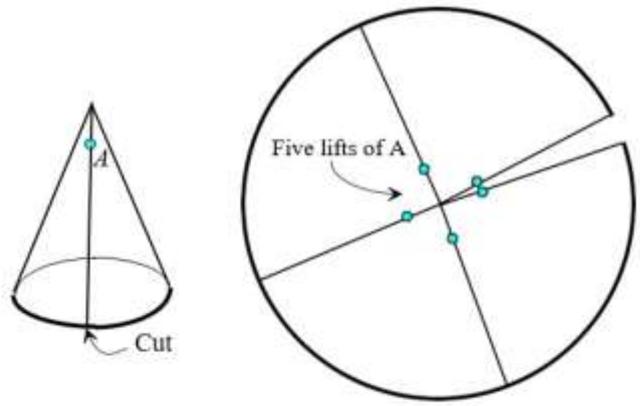


Figure 18.12 Covering space of a cone has branch points

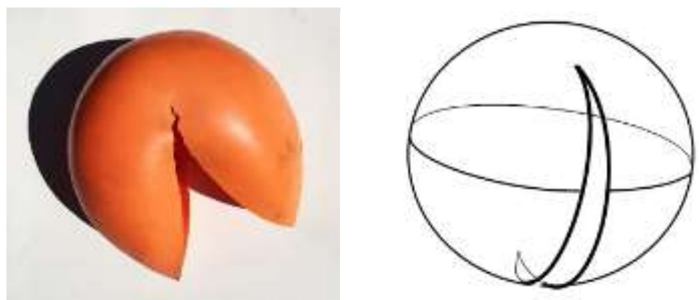


Figure 18.13 Covering space of a sphere has branch points

In fact, any surface that has no (non-branched) coverings and that is bounded and without an edge can be continuously deformed (without tearing) into a round sphere. The surfaces of closed boxes and of footballs are two examples. A torus is bounded and without an edge, but it cannot be deformed into a sphere. A cylinder also cannot be deformed into a sphere, and a cylinder either has an edge or (if we imagine it as extending indefinitely) it is unbounded.

A 3-dimensional analog of this situation arises from a famous problem called the *Poincaré conjecture*. The analog of a surface is called a 3-*dimensional manifold*, a space locally like Euclidean 3-space (in the same sense that a surface is locally like the plane). The 3-sphere, which we will study in Chapter 22, is a 3-dimensional manifold. Henri Poincaré (1854–1912, French) conjectured that any 3-dimensional manifold that has no (non-branched) coverings and that is bounded and without boundary must be homeomorphic to a 3-dimensional sphere $\mathbf{S}^3$. For some 80 years, numerous mathematicians have tried to decide whether Poincaré's conjecture is true. On May 24, 2000, the Clay Mathematics Institute announced (www.claymath.org) that it was offering a $1,000,000 prize for a solution of the Poincaré conjecture! In 2002, Grigori Perelman of St. Petersburg, Russia, announced that he proved the Poincaré conjecture. Perelman has settled completely the question of which topological 3-manifolds are geometric 3-manifolds. For discussion of Perelman's ideas and their relationships to the Poincaré conjecture, see articles by J. Milnor (*Notices of the AMS*, Nov. 2003, pp. 1226–1233) and M. T. Anderson (*Notices of the AMS*, Feb. 2004, pp. 184–193). See Chapter 22 and [**EG**: Hilbert] and [**DG**: Weeks] for more discussion of 3-dimensional manifolds and the 3-dimensional sphere. You can read about this fascinating story Donal O'Shea's book *The Poincare Conjecture:In Search of the Shape of Universe*, Walker Bokks, 2008 and more about Grigori Perelman and his quest for the proof in Masha Gessen's book *Perfect Rigor: A Genius and a Mathematical Breakthrough of the Century*, Houghton Mifflin, 2009.

PROBLEM 18.4 HYPERBOLIC MANIFOLDS

Is it possible to make a two-holed torus (sometimes called an anchor ring, or the surface of a two-holed donut) into a geometric 2-manifold? See Figure 18.14.

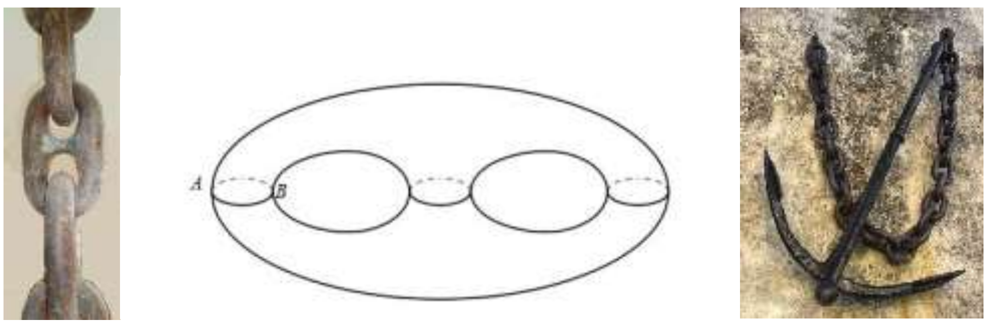


Figure 18.14 Two-holed torus — not a geometric manifold

Note that as it is pictured in Figure 18.14 the two-holed torus is definitely **not** a geometric 2-manifold because the intrinsic geometry is not the same at every point — for example, points *A* and *B*. But can we distort the geometry so that the surface is a geometric 2-manifold?

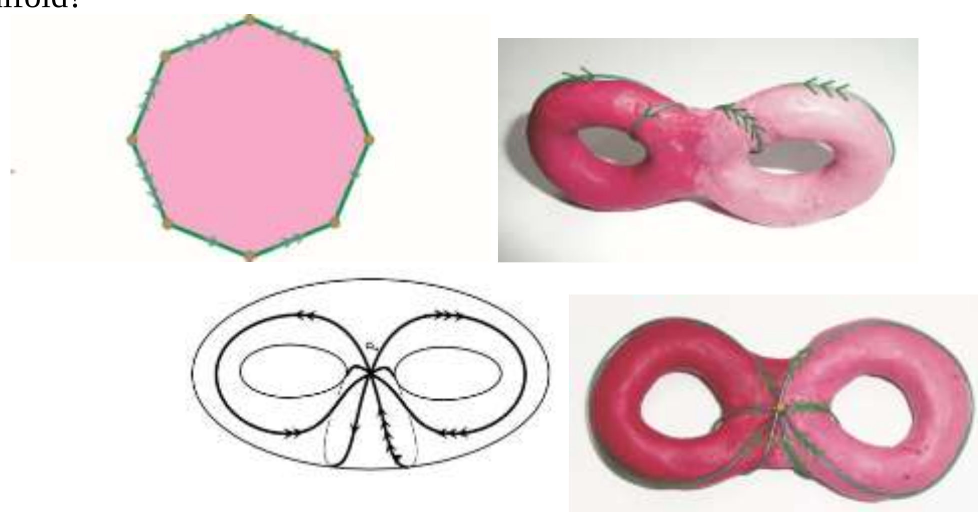


Figure 18.15 Cutting a two-holed torus into an octagon

Imagine cutting the two-holed torus along the four loops emanating from the point *P*, as indicated in Figure 18.15. You will get a distorted octagon with 45° (= 360°/8) interior angles at each vertex. This distorted octagon is topologically equivalent to a regular planar octagon. Walking around the point P, we find the gluings as indicated in Figure 18.15. Be sure you understand how the gluings on the octagon were determined from the loops on the two-holed torus. If you glue the edges of the regular planar octagon together as indicated in Figure 18.15, you will get a version of the two-holed torus that is locally isometric to the plane except at the (one) vertex. (*Why*? *What will a neighborhood of the vertex look like*?)

In the plane all octagons have the same interior angle sum. But in the hyperbolic plane, regular octagons have different angles. In fact, we can find such an octagon in the hyperbolic plane with 45° interior angles. (See Figure 18.16.) If we glue the edges of this octagon as indicated, then we will get a hyperbolic 2-manifold. An intermediate step in this process (when every other side is identified) is the so-called "pair of pants."

Figure 18.16 Hyperbolic octagon with 45° angles and "pair of pants"

To see that there is such an octagon, imagine placing a small (regular) octagon on the hyperbolic plane. Because the octagon is small, its interior angles must be very close to the interior angles of an octagon in the (Euclidean) plane. Because the exterior angle of a planar octagon must be 360°/8, the interior angle must be 180° – (360°/8) = 135°. Now let the small octagon grow, keeping it always regular. From Problem **7.2** (remembering that an octagon can be divided into triangles as in Figure 18.16), we conclude that the interior angles of the octagon will decrease in size until, if we let the vertices go to infinity, the angles would decrease to zero. Somewhere between 135° and 0° the interior angles will be the desired 45°.

a. *What is the area of the hyperbolic octagon with 45° interior angles?*

b. *Why is the two-holed torus obtained from a hyperbolic octagon with 45° interior angles a hyperbolic 2-manifold? What is its area?*

It is not easy to determine the geodesics on a hyperbolic 2-manifold. Some discussion about these issues is contained in William Thurston's *Three-Dimensional Geometry and Topology*, Volume 1 [**DG**: Thurston].

There are other ways of making a two-holed torus into a geometric 2-manifold, but it is always a *hyperbolic* 2-manifold. However, there are many different hyperbolic structures for a two-holed torus; for example, look in Figure 18.17 for a different way to represent the two-holed torus — this time as the gluing of the boundary of a dodecagon with 90° interior angles.

c. *Follow the steps above to check that Figure 18.17 leads to the representation of a two-holed torus as a dodecagon (with 90° interior angles) with gluing on the boundary and thus to a hyperbolic 2-manifold. What is its area?*

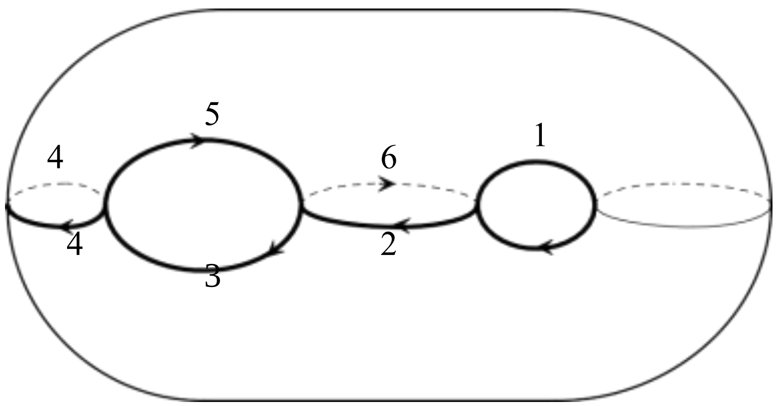


Figure 18.17 Cutting two-holed torus into a dodecagon

You should have found the same area in parts **b** and **c**. In fact, we (you!) will show in the next problem that *any hyperbolic geometric manifold structure on a two-holed torus has the same area.*

All the above discussion can be extended in straightforward ways to tori with 3, 4, and more holes.

Do you see how a two-holed torus can be made from four rectangular hexagons?
How many possibilities are there?

PROBLEM 18.5 AREA AND EULER NUMBER

Now is a good time to go back and review the material on area and holonomy in Problems **7.1–7.5a**. Recall that you showed that the area of any polygon is

$$K\,Area(\Gamma) = [\,\Sigma\beta_i - (n-2)\pi\,],$$

where K is the Gaussian curvature equal to $1/\rho^2$ for a sphere and $-1/\rho^2$ for a hyperbolic plane of radius ρ, $\Sigma\beta_i$ is the sum of the interior angles, and n is the number of edges.

We will now use those results to study the area of geometric manifolds. All of the geometric manifolds that we have described above have cell divisions. A **0-*cell*** is a point we usually call a *vertex*. A **1-*cell*** is a straight-line segment we usually call an *edge* (the edge need not be straight, but in most of our applications it will be). A **2-*cell*** is a polygon that is usually called a *face*. We say that a geometric manifold has a ***cell division*** if it is divided into cells so that every edge has its boundary consisting of vertices and every face has its boundary divided into edges and vertices and two cells only intersect on their boundaries. We call the cell division a ***geodesic cell division*** if all the edges are geodesic segments, and thus the faces are polygons. For example, in Figure 18.17 we have a two-holed torus divided into one face, six edges, and three vertices. Note that some edges have only one vertex and thus form a loop (circle). If we make this two-holed torus into a hyperbolic manifold (by using a regular hyperbolic dodecagon with 90° angles), then the cell division is a geodesic cell division.

Suppose we have a geodesic cell division of a geometric manifold M into f faces (2-cells), where the j^{th} face has n_j edges. We can calculate the area of M as follows:

K area(M) =
$= K\{\text{area}(1^{\text{st}}\text{ face})\} + K\{\text{area}(2^{\text{nd}}\text{ face})\} + \ldots + K\{\text{area}(f^{\text{ th}}\text{ face})\}$
$= \{\Sigma\beta_i(1^{\text{st}}\text{ face}) - (n_1 - 2)\pi\} + \ldots + \{\Sigma\beta_i(f^{\text{ th}}\text{ face}) - (n_f - 2)\pi\}$
$= \{\Sigma\beta_i(1^{\text{st}}\text{ face}) + \ldots + \Sigma\beta_i(f^{\text{ th}}\text{ face})\} - \{(n_1 - 2)\pi + \ldots + (n_f - 2)\pi\}$

$= \{\text{sum of all the angles}\} - (n_1 + n_2 + \ldots + n_f)\pi + (2 + 2 + \ldots + 2)\pi.$

Fill in the steps to prove the following:

a. *If a geometric manifold M has a geodesic cell division, then*

$$K\text{ area}(M) = 2\pi(v - e + f),$$

where the cell division has v vertices and e edges and f faces. Check that this agrees with your results in Problems **18.1**, **18.3**, *and* **18.4**.

The quantity ($v - e + f$) is called the ***Euler number*** (or sometimes the ***Euler characteristic***). This quantity is named after the mathematician Leonhard Euler (1707–1783), who was born and educated in Basel, Switzerland but worked in St. Petersburg, Russia, and Berlin, Germany. It follows directly from part **a** and what we know about area and curvature that

b. *The Euler number of any geodesic cell division of a sphere must be 2. The Euler number of any geodesic cell division of a projective plane must be 1. The Euler number of any geodesic cell division of a flat (Euclidean) 2-manifold must be equal to 0. The Euler number of any geodesic cell division of a hyperbolic 2-manifold must be negative.*

We see from part **b** that in the cases of a sphere, a projective plane, or a flat 2-manifold, the Euler number does not depend on the specific cell division (with geodesic edges). So, we can talk about the ***Euler number of the sphere*** (= 2) and the ***Euler number of the torus*** or ***Klein bottle*** (= 0). We also saw above that the two different cell divisions that were given of the two-holed torus have the same area and thus the same Euler number. Can we prove that for every hyperbolic 2-manifold the Euler number (and therefore the area) depends only on the topology and not on the particular cell division? In fact,

THEOREM 18.5a. *The Euler number of any cell division of a* 2*-manifold depends only on the topology of the manifold and not on the specific cell division. Furthermore, two* 2*-manifolds are homeomorphic if and only if they have the same Euler number and are either both orientable or both non-orientable.*

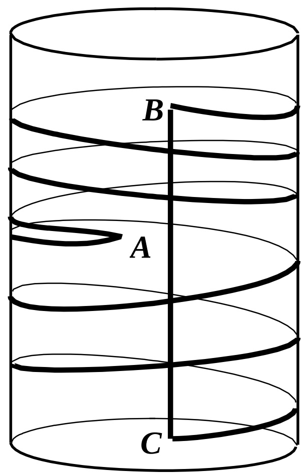
Proofs of this result are somewhat fussy and involve much of the foundational results of topology that were only developed in the 20th century. See Imre Lakatos's *Proofs and Refutations* [**PH**: Lakatos] for an accessible and interesting account of the long and complicated history and philosophy of the Euler number. Lakatos describes an imaginary class discussion about the Euler number in which the tortuous route students take toward a proof mirrors the actual route that mathematicians took. Other proofs are with different additional assumptions. For example, [**DG**: Thurston] (Propositions 1.3.10 and 1.3.12) gives an accessible proof assuming that the reader has some familiarity with vector fields on differentiable manifolds. In Sections 2.4 and 2.5 of [**TP**: Blackett], there is a combinatorial-based proof that assumes the topological 2-manifold has some cell division.

TRIANGLES ON GEOMETRIC MANIFOLDS

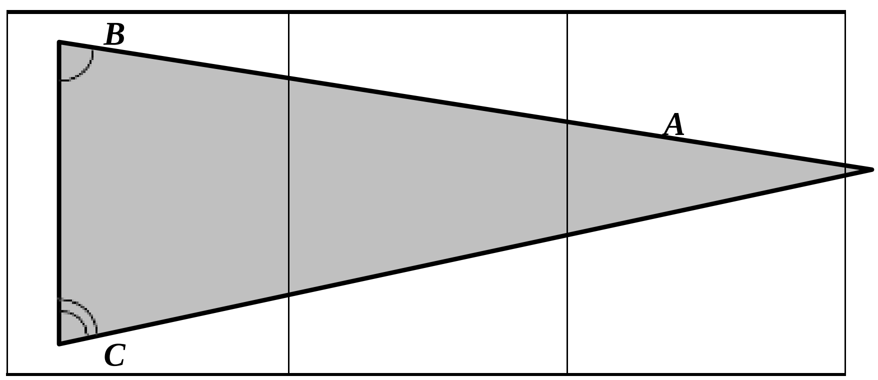
Clearly, if on a flat (Euclidean) 2-manifold a triangle is contained in a region that is isometric to the plane, then the triangle is a planar triangle and has all the properties of a triangle in the plane. The same can be said about triangles in spherical and hyperbolic 2-manifolds. In fact, it can be shown (see any topology text that deals with covering spaces) that

THEOREM 18.5b. *If* Δ *is a triangle in a Euclidean* [*spherical, hyperbolic*] 2*-manifold, M, such that* Δ *can be shrunk to a point in the interior of* Δ*, then* Δ (*and its interior*) *can be lifted to the plane* [*sphere, hyperbolic plane*] *that is the universal covering space of M; and thus* Δ *has all the same properties of a triangle in the plane* [*sphere, hyperbolic plane*].

It is natural to be uncomfortable using covering spaces, but covering spaces are a helpful tool for thinking intrinsically. Some triangles, even though they look strange extrinsically, will look like reasonable triangles for the bug. In Figure 18.18 we give an example of an extrinsically strange triangle that intersects itself but that can be considered a normal triangle from an intrinsic point of view. In fact, it is a planar triangle. Such triangles have all the properties of plane triangles, including SAS and ASA.



Unroll to a 3-sheeted cover, and ...



This is a triangle!

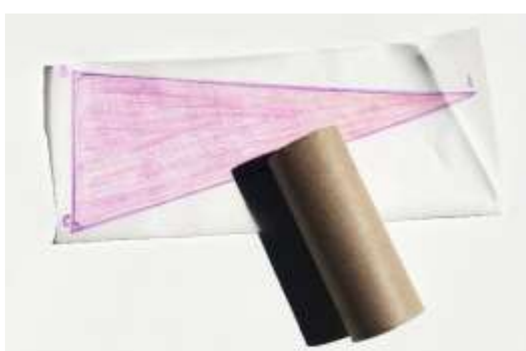


Figure 18.18 Think intrinsically

PROBLEM 18.6 CAN THE BUG TELL WHICH MANIFOLD?

Our physical universe is apparently a geometric 3-manifold. In Chapter 24 we will explore ways in which we (human beings) may be able to determine the global shape and size of our physical universe. But first we look at the situation of a 2-dimensional (2-D) bug on a geometric 2-manifold in order to get some help for our 3-dimensional question.

a. *Suppose a 2-D bug lives on a geometric 2-manifold M and suppose that M is the bug's whole universe. How can the bug determine intrinsically what the local geometry of its universe is?*

b. *How can the bug in part **a** tell what the global shape of its universe is? For example, how can the bug tell (intrinsically) the difference between being on a flat torus and being on a hexagonal torus? Again, for this part you may imagine that the bug can crawl over the whole manifold and leave markers and make measurements.*

c. *Suppose that the bug in part **a** can only travel in a very small region of the manifold (so small that all triangles in the region are indistinguishable from planar triangles), but the bug can see for very long distances. Can the bug still determine which geometric 2-manifold is its universe?*

You must assume that light travels on geodesics in the bug's universe (which is the case of light in our physical 3-dimensional universe). Or forget about light and think about vibrations that travel along the surface. Our 2-D bugs have antennae with which they can receive these vibrations and through them "see" the surface. Imagine that there are bright stars on the 2-manifold that the bug can see; but remember that these stars must be on the 2-manifold — we are taking an intrinsic view of 2-manifolds. You may also assume that there are many stars and they are distributed (roughly) uniformly.

This is different from the situation on the earth, where we have the extrinsic observation of stars (and sun and moon) to help us. The sun was used to measure the radius of the earth by Eratosthenes of Cyrene (Egyptian, 276–194 B.C.) and others. Mariners have used the sun, stars, and eclipses (and clocks) since early times for navigating on the oceans. For an interesting and accessible historical account of the problem of determining one's longitude on the earth, see *Longitude* by Dava Sobel [**CE**: Sobel].