

II. THE RESOLVENT

13. We shall follow, with some material deviations, König's exposition of Kronecker's method of solving equations by means of the resolvent. The equations are in general supposed to be non-homogeneous; and homogeneous equations are regarded as a particular case. Thus a homogeneous equation in n variables represents a cone of $n-1$ dimensions with its vertex at the origin. *Homogeneous coordinates are excluded.*

The problem is to find all the solutions of any given system of equations $F_1 = F_2 = \dots = F_k = 0$ in n unknowns x_1, x_2, \dots, x_n . The unknowns are supposed if necessary to have been subjected to a homogeneous linear substitution beforehand, the object being to make the equations and their solutions of a general character, and to prevent any inconvenient result happening (such as an equation or polynomial being irregular* in any of the variables) which could have been avoided by a linear substitution at the beginning. In theoretical reasoning *this preliminary homogeneous substitution is always to be understood*; but is seldom necessary in dealing with a particular example.

The solutions we shall seek are (i) those, if any, which exist for x_1 when x_2, x_3, \dots, x_n have arbitrary values; (ii) those which exist for x_1, x_2 , not included in (i), when x_3, \dots, x_n have arbitrary values; (iii) those which exist for x_1, x_2, x_3 , not included in (i) or (ii), when x_4, \dots, x_n have arbitrary values; and so on. A set of solutions for x_1, x_2, \dots, x_r when x_{r+1}, \dots, x_n have arbitrary values is said to be of *rank* r , and the *spread* of the points whose coordinates are the solutions is of *rank* r and *dimensions* $n-r$. If there are solutions of rank r and no solutions of rank $< r$ the system of equations $F_1 = F_2 = \dots = F_k = 0$ and the module (F_1, F_2, \dots, F_k) are both said to be of rank r .

14. The polynomials F_1, F_2, \dots, F_k , and also all their factors are regular in x_1 . Hence their common factor D can be found by the ordinary process of finding the H.C.F. of F_1, F_2, \dots, F_k treated as polynomials in a single variable x_1 . If D does not involve the variables we take it to be 1. If it does involve the variables the solutions of $D=0$ treated as an equation for x_1 give the first set of solutions of the equations $F_1 = F_2 = \dots = F_k = 0$ mentioned above.

* A polynomial of degree l is said to be regular or irregular in x_1 according as the term x_1^l is present in it or not.

In the algebraic theory of modules we regard any algebraic equation in one unknown, whether the coefficients involve parameters or not, as completely soluble, i.e. we regard any given non-linear polynomial in one variable as *reducible*. A polynomial in two or more variables is called reducible if it is the product of two polynomials both of which involve the variables. A polynomial which is not reducible is called (absolutely) *irreducible*. Any given polynomial is either irreducible or uniquely expressible as a product of irreducible factors, leaving factors of degree zero out of account. It is assumed that the irreducible factors of any given polynomial are known. Thus the polynomial D above may be supposed to be expressed in its irreducible factors in x_1, x_2, \dots, x_n , and to each irreducible factor corresponds an irreducible or non-degenerate spread.

Put $F_i = D\phi_i$ ($i = 1, 2, \dots, k$). Then $\phi_1, \phi_2, \dots, \phi_k$ have no common factor involving the variables, and the same is true of the two polynomials

$$\lambda_1\phi_1 + \lambda_2\phi_2 + \dots + \lambda_k\phi_k \quad \text{and} \quad \mu_1\phi_1 + \mu_2\phi_2 + \dots + \mu_k\phi_k,$$

where the λ 's and μ 's are arbitrary quantities. Regarding them as two polynomials in a single variable x_1 we calculate their resultant, and arrange it in the form

$$\rho_1 F_1^{(1)} + \rho_2 F_2^{(1)} + \dots + \rho_{k_1} F_{k_1}^{(1)},$$

where $\rho_1, \rho_2, \dots, \rho_{k_1}$ are different power products of the λ 's and μ 's, and $F_1^{(1)}, F_2^{(1)}, \dots, F_{k_1}^{(1)}$ are polynomials in x_2, x_3, \dots, x_n not involving the λ 's and μ 's. Each $F_i^{(1)}$ is regular in x_2 ; for any homogeneous linear substitution beforehand of x_2, x_3, \dots, x_n among themselves only would be carried through to the $F_i^{(1)}$.

Find the H.C.F. $D^{(1)}$ of $F_1^{(1)}, F_2^{(1)}, \dots, F_{k_1}^{(1)}$ treated as polynomials in a single variable x_2 , and put $F_i^{(1)} = D^{(1)}\phi_i^{(1)}$ ($i = 1, 2, \dots, k_1$). Then find the resultant of

$$\lambda_1\phi_1^{(1)} + \lambda_2\phi_2^{(1)} + \dots + \lambda_{k_1}\phi_{k_1}^{(1)} \quad \text{and} \quad \mu_1\phi_1^{(1)} + \mu_2\phi_2^{(1)} + \dots + \mu_{k_1}\phi_{k_1}^{(1)}$$

and arrange it in the form

$$\rho_1 F_1^{(2)} + \rho_2 F_2^{(2)} + \dots + \rho_{k_2} F_{k_2}^{(2)}$$

as before, where $F_1^{(2)}, F_2^{(2)}, \dots, F_{k_2}^{(2)}$ are polynomials in x_3, x_4, \dots, x_n , which may be assumed regular in x_3 , and whose H.C.F. $D^{(2)}$ can be found. We thus get the following series in succession :

$$\begin{aligned} &F_1, F_2, \dots, F_k, \quad \text{with H.C.F. } D, \\ &\phi_1, \phi_2, \dots, \phi_k, \\ &F_1^{(1)}, F_2^{(1)}, \dots, F_{k_1}^{(1)}, \quad \text{with H.C.F. } D^{(1)}, \\ &\phi_1^{(1)}, \phi_2^{(1)}, \dots, \phi_{k_1}^{(1)}, \\ &F_1^{(2)}, F_2^{(2)}, \dots, F_{k_2}^{(2)}, \quad \text{with H.C.F. } D^{(2)}, \\ &\phi_1^{(2)}, \phi_2^{(2)}, \dots, \phi_{k_2}^{(2)}, \quad \text{and so on.} \end{aligned}$$

Now any solution of $F_1 = F_2 = \dots = F_k = 0$ is a solution of $D = 0$ or of $\phi_1 = \phi_2 = \dots = \phi_k = 0$. And any solution of $\phi_1 = \phi_2 = \dots = \phi_k = 0$ is a solution of $F_1^{(1)} = F_2^{(1)} = \dots = F_{k_1}^{(1)} = 0$, since $\Sigma \rho_i F_i^{(1)} \equiv 0 \pmod{(\Sigma \lambda_i \phi_i, \Sigma \mu_i \phi_i)}$, and therefore a solution of $D^{(1)} = 0$ or of $\phi_1^{(1)} = \phi_2^{(1)} = \dots = \phi_{k_1}^{(1)} = 0$. Hence any solution of $F_1 = F_2 = \dots = F_k = 0$ is a solution of $D = 0$ or of $D^{(1)} = 0$ or of $\phi_1^{(1)} = \phi_2^{(1)} = \dots = \phi_{k_1}^{(1)} = 0$. Proceeding in a similar way we find that any solution of $F_1 = \dots = F_k = 0$ is a solution of $DD^{(1)} \dots D^{(n-1)} = 0$, since $\phi_1^{(n-1)}, \phi_2^{(n-1)}, \dots, \phi_{k_{n-1}}^{(n-1)}$ are polynomials in a single variable x_n at most and have no common factor.

Conversely if ξ_3, x_4, \dots, x_n is any solution of $D^{(2)} = 0$ the resultant of $\Sigma \lambda_i \phi_i^{(1)}$ and $\Sigma \mu_i \phi_i^{(1)}$ with respect to x_2 vanishes when $x_3 = \xi_3$, and $\Sigma \lambda_i \phi_i^{(1)} = \Sigma \mu_i \phi_i^{(1)} = 0$ have a solution $x_2 = \xi_2$ when $x_3 = \xi_3$; i.e. the equations $\phi_1^{(1)} = \dots = \phi_{k_1}^{(1)} = 0$, and therefore also the equations $F_1^{(1)} = \dots = F_{k_1}^{(1)} = 0$, have a solution $\xi_2, \xi_3, x_4, \dots, x_n$; and, by the same reasoning, the equations $F_1 = F_2 = \dots = F_k = 0$ have a solution $\xi_1, \xi_2, \xi_3, x_4, \dots, x_n$. Similarly to any solution of $DD^{(1)} \dots D^{(n-1)} = 0$, say a solution $\xi_i, x_{i+1}, \dots, x_n$ of $D^{(i-1)} = 0$, there corresponds a solution $\xi_1, \xi_2, \dots, \xi_i, x_{i+1}, \dots, x_n$ of the equations $F_1 = F_2 = \dots = F_k = 0$. Hence from the solutions of the single equation $DD^{(1)} \dots D^{(n-1)} = 0$ we can get all the solutions of the system $F_1 = F_2 = \dots = F_k = 0$, since all the solutions of the latter satisfy the former.

Definitions. $DD^{(1)} \dots D^{(n-1)}$ is called the *complete (total) resolvent* of the equations $F_1 = F_2 = \dots = F_k = 0$ and of the module (F_1, F_2, \dots, F_k) . $D^{(i-1)}$ is called the *complete partial resolvent of rank i*, and any whole factor of $D^{(i-1)}$ is called a *partial resolvent of rank i*.

15. *The complete resolvent is a member of the module (F_1, F_2, \dots, F_k) .*

For $\Sigma \rho_i F_i^{(1)} \equiv 0 \pmod{(\Sigma \lambda_i \phi_i, \Sigma \mu_i \phi_i)} = A \Sigma \lambda_i \phi_i + B \Sigma \mu_i \phi_i$,

where A, B are whole functions of $x_1, x_2, \dots, x_n, \lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_k$. Hence by equating coefficients of the power products ρ_i on both sides, we have

$$F_i^{(1)} \equiv 0 \pmod{(\phi_1, \phi_2, \dots, \phi_k)},$$

and $DF_i^{(1)} = 0 \pmod{(F_1, F_2, \dots, F_k)}$,*

or $DD^{(1)} \phi_i^{(1)} = 0 \pmod{(F_1, F_2, \dots, F_k)}$.

Similarly $DD^{(1)} \dots D^{(n-1)} \phi_i^{(n-1)} = 0 \pmod{(F_1, F_2, \dots, F_k)}$;

and since the $\phi_i^{(n-1)}$ include one variable only (or none at all) and have

* Not $DF_i^{(1)} \equiv 0 \pmod{(F_1, F_2, \dots, F_k)}$ because any common factor of F_1, F_2, \dots, F_k not involving the variables is not included in D and is left out of account.

no common factor, we can choose polynomials a_i in the single variable so that $\sum a_i \phi_i^{(n-1)} = 1$. Hence

$$D D^{(1)} \dots D^{(n-1)} = 0 \pmod{(F_1, F_2, \dots, F_k)}.$$

If the equations $F_1 = F_2 = \dots = F_k = 0$ have no finite solution the complete resolvent is equal to 1; consequently 1 is a member of (F_1, F_2, \dots, F_k) , and every polynomial is a member.

16. We have seen that to every solution $x_i = \xi_i$ of $D^{(i-1)} = 0$ there corresponds a solution $\xi_1, \xi_2, \dots, \xi_i, x_{i+1}, \dots, x_n$ of the equations $F_1 = F_2 = \dots = F_k = 0$. It may happen that there is an earlier complete partial resolvent $D^{(j-1)}$ which vanishes when $x_j = \xi_j, \dots, x_i = \xi_i$. In such a case the solution $\xi_1, \dots, \xi_i, x_{i+1}, \dots, x_n$ of $F_1 = \dots = F_k = 0$ corresponding to a solution of $D^{(i-1)} = 0$ is included in the solutions corresponding to $D^{(j-1)} = 0$, and may be neglected if we are seeking merely the complete solution of $F_1 = F_2 = \dots = F_k = 0$. Such a solution is called an *imbedded* solution. All solutions corresponding to an irreducible factor of $D^{(i-1)}$ will be imbedded if one of them is imbedded.

17. Examples on the Resolvent. Geometrically the resolvent enables us to resolve the whole spread represented by any given set of algebraic equations into definite irreducible spreads (§ 21). It has been supposed that the complete resolvent also supplies a definite answer to certain other questions. The following examples disprove this to some extent.

Example i. Find the resolvent of n homogeneous equations $F_1 = F_2 = \dots = F_n = 0$ of the same degree l and having no proper solution.

Since there are no solutions of rank $< n$ the complete resolvent is $D^{(n-1)}$. The first derived set of polynomials $F_1^{(1)}, F_2^{(1)}, \dots, F_{k_1}^{(1)}$ are homogeneous and of degree l^2 , the 2nd set $F_1^{(2)}, F_2^{(2)}, \dots$ are homogeneous and of degree l^4 , and the $(n-1)$ th set $F_1^{(n-1)}, F_2^{(n-1)}, \dots$ are homogeneous and of degree $l^{2^{n-1}}$. This last set involve only one variable x_n , and therefore have the common factor $x_n^{l^{2^{n-1}}}$, which is therefore the required complete resolvent.

We should arrive at a similar result if we changed x_i to $x_i + a_i$ ($i = 1, 2, \dots, n$) beforehand, thus making the polynomials non-homogeneous. The complete resolvent would then be $(x_n + a_n)^{l^{2^{n-1}}}$. The resultant would be $(x_n + a_n)^{l^n}$. The difference in the two results is explained by the fact that the resultant is obtained by a process

applying uniformly to all the variables, and the resolvent by a process applied to the variables in succession.

Example ii. König (K, p. 219) defines a module or system of equations as being *simple* or *mixed* according as only one or more than one of the complete partial resolvents $D, D^{(1)}, \dots, D^{(n-1)}$ differs from unity. Kronecker (Kr, p. 31) says that the system of equations $F_1 = F_2 = \dots = F_n = 0$ is irreducible in this case; and the *Ency. des Sc. Math.* (W₂, p. 352) repeats König's definition. We give two examples to show that this definition is a valueless one.

If u, v, w are three linear functions of three or more variables, any polynomial which contains the spread of $u = v = 0$ is of the form $Au + Bv$; if it also contains the spread of $u = w = 0$, B must vanish when $u = w = 0$, hence B must be of the form $Cu + Dw$, and $Au + Bv$ of the form $A'u + B'vw$; if it also contains the spread of $v = w = 0$, A' must be of the form $C'v + D'w$, and $A'u + B'vw$ of the form $C'uv + D'uw + B'vw$. Hence a polynomial which contains all three spreads is a member of the module (vw, wu, uv) , and also any member of the module contains the three spreads. This module, although composite, is not mixed in any proper sense of the word.

Besides having partial resolvents of rank 2 corresponding to the three spreads the module has a partial resolvent of rank 3 corresponding to its singular spread $u = v = w = 0$. This last partial resolvent does not correspond to any property of the module which is not included in the properties corresponding to its partial resolvents of rank 2; in other words the partial resolvent of rank 3 is purely redundant.

The resolvent $D^{(1)}D^{(2)}$ can be found as follows: Suppose

$$u = a_0 + a_1x_1 + a_2x_2 + \dots, \quad v = b_0 + b_1x_1 + b_2x_2 + \dots, \quad w = c_0 + c_1x_1 + c_2x_2 + \dots$$

Then the resultant of $\lambda_1vw + \lambda_2wu + \lambda_3uv$ and $\mu_1vw + \mu_2wu + \mu_3uv$ with respect to x_1 , apart from a constant factor, is

$$(c_1v - b_1w)(a_1w - c_1u)(b_1u - a_1v) \times \left\{ \frac{c_1v - b_1w}{\lambda_2\mu_3 - \lambda_3\mu_2} + \frac{a_1w - c_1u}{\lambda_3\mu_1 - \lambda_1\mu_3} + \frac{b_1u - a_1v}{\lambda_1\mu_2 - \lambda_2\mu_1} \right\},$$

its four irreducible factors corresponding to the spreads

$$v = w = 0, \quad w = u = 0, \quad u = v = 0,$$

$$(\lambda_2\mu_3 - \lambda_3\mu_2)u = (\lambda_3\mu_1 - \lambda_1\mu_3)v = (\lambda_1\mu_2 - \lambda_2\mu_1)w.$$

Hence $D^{(1)} = (c_1v - b_1w)(a_1w - c_1u)(b_1u - a_1v)$;

and $\phi_1^{(1)} = (c_1v - b_1w)$, $\phi_2^{(1)} = (a_1w - c_1u)$, $\phi_3^{(1)} = (b_1u - a_1v)$,

from which we obtain

$$D^{(2)} = (b_1c_2 - b_2c_1)u + (c_1a_2 - c_2a_1)v + (a_1b_2 - a_2b_1)w.$$

Example iii. Compare and find the resolvents of the two modules

$$M = (x_1^3, x_2^3, x_1^2 + x_2^2 + x_1x_2x_3),$$

$$M' = (x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1^2 + x_2^2 + x_1x_2x_3).$$

The resolvent of M' will be found by obtaining the resultant with respect to x_1 of the two equations

$$\lambda_1x_1^3 + \lambda_2x_1^2x_2 + \lambda_3x_1x_2^2 + \lambda_4x_2^3 + \lambda_5(x_1^2 + x_2^2 + x_1x_2x_3) = 0,$$

and
$$\mu_1x_1^3 + \mu_2x_1^2x_2 + \mu_3x_1x_2^2 + \mu_4x_2^3 + \mu_5(x_1^2 + x_2^2 + x_1x_2x_3) = 0.$$

This resultant is the same as that of the first equation and

$$(\lambda_1\mu_5)x_1^3 + (\lambda_2\mu_5)x_1^2x_2 + (\lambda_3\mu_5)x_1x_2^2 + (\lambda_4\mu_5)x_2^3 = 0$$

except for a factor λ_5^3 . The roots of the last equation are a_1x_2, a_2x_2, a_3x_2 . Hence the resultant, apart from a constant factor, is

$$\Pi \{(\lambda_1\alpha^3 + \lambda_2\alpha^2 + \lambda_3\alpha + \lambda_4)x_2^3 + \lambda_5(\alpha^2 + 1 + \alpha x_3)x_2^2\}, \quad (\alpha = a_1, a_2, a_3)$$

or
$$x_2^6 \Pi \{(\lambda_1\alpha^3 + \lambda_2\alpha^2 + \lambda_3\alpha + \lambda_4)x_2 + \lambda_5(\alpha^2 + 1 + \alpha x_3)\}.$$

Hence the complete resolvent is x_2^6 , since no values of x_2, x_3 independent of the λ 's and μ 's will make the remaining product of factors of the above resultant vanish.

The complete resolvent of M , worked in the same way, is also x_2^6 ; i.e. M and M' have the same complete resolvent, although they are not the same module. M , but not M' , contains the two modules

$$M'' = (x_3 - 1, x_1^2 + x_1x_2 + x_2^2, x_1^2x_2 + x_1x_2^2),$$

$$M''' = (x_3 + 1, x_1^2 - x_1x_2 + x_2^2, x_1^2x_2 - x_1x_2^2),$$

i.e. every member of M is a member of M'' and of M''' . Thus

$$x_1^3 = x_1(x_1^2 + x_1x_2 + x_2^2) - (x_1^2x_2 + x_1x_2^2),$$

$$x_2^3 = x_2(x_1^2 + x_1x_2 + x_2^2) - (x_1^2x_2 + x_1x_2^2),$$

$$x_1^2 + x_2^2 + x_1x_2x_3 = (x_1^2 + x_1x_2 + x_2^2) + x_1x_2(x_3 - 1).$$

The module M is what is called the L.C.M. of M', M'', M''' . The two modules M'', M''' have $x_1 = x_2 = x_3 - 1 = 0$ and $x_1 = x_2 = x_3 + 1 = 0$ for their spreads, which are imbedded in the spread $x_1 = x_2 = 0$ of the first component of M , viz. M' .

M is then properly speaking a mixed module although this is not indicated by its complete resolvent x_2^6 . It has two imbedded spreads, the points $(0, 0, \pm 1)$. The complete resolvent should have the factors

$x_3 \pm 1$ to indicate these, but it has no such factors. The complete resolvent may indicate imbedded modules which do not exist as in Ex. ii, or it may give no indication of them when they do exist as in Ex. iii.

Example iv. It is stated in the *Encyk. der Math. Wiss.* (W_1 , p. 305) and repeated in (W_2 , p. 354) that if only one complete partial resolvent $D^{(v)}$ differs from 1, and $D^{(v)}$ has no repeated factor, the module is the product of the prime modules corresponding to the irreducible factors of $D^{(v)}$. The absurdity of this statement is shown by applying it to the module (u, vw) , where u, v, w are the same as in Ex. ii. The complete resolvent is $D^{(v)} = (b_1u - a_1v)(c_1u - a_1w)$, and the product of the prime modules $(u, v), (u, w)$ corresponding to its two factors is $(u^2, uv, uv, vw) \neq (u, vw)$.

18. The u -resolvent. The solutions of $F_1 = F_2 = \dots = F_k = 0$ are obtained in the most useful way by introducing a general unknown x standing for $u_1x_1 + u_2x_2 + \dots + u_nx_n$, where u_1, u_2, \dots, u_n are undetermined coefficients. This is done by putting

$$x_1 = \frac{x - u_2x_2 - \dots - u_nx_n}{u_1}$$

in the system of equations $F_1 = F_2 = \dots = F_k = 0$. We thus get a new system $f_1 = f_2 = \dots = f_k = 0$ in x, x_2, x_3, \dots, x_n , where

$$f_i = u_1^{l_i} F_i \left(\frac{x - u_2x_2 - \dots - u_nx_n}{u_1}, x_2, \dots, x_n \right) \quad (i = 1, 2, \dots, k),$$

the multiplier $u_1^{l_i}$ being introduced to make f_i integral in u_1 . There is evidently a one-one correspondence between the solutions of the two systems, viz. to the solution $\xi_1, \xi_2, \dots, \xi_n$ of $F_1 = F_2 = \dots = F_k = 0$ there corresponds the solution ξ, ξ_2, \dots, ξ_n of $f_1 = f_2 = \dots = f_k = 0$, and *vice versa*, where $\xi = u_1\xi_1 + u_2\xi_2 + \dots + u_n\xi_n$.

Definition. The complete resolvent $D_u D_u^{(1)} \dots D_u^{(n-1)} (= F_u)$ of (f_1, f_2, \dots, f_k) obtained by eliminating x_2, x_3, \dots, x_n in succession is called the *complete u -resolvent* of (F_1, F_2, \dots, F_k) .

Since $F_u = 0 \pmod{(f_1, f_2, \dots, f_k)}$, by § 15, we have

$$(F_u)_{x=u_1x_1+\dots+u_nx_n} = 0 \pmod{(F_1, F_2, \dots, F_k)}.$$

F_u is a whole function of $x, x_2, \dots, x_n, u_1, u_2, \dots, u_n$ which resolves into linear factors when regarded as a function of x only. The linear factors of rank r , that is, the linear factors of $D_u^{(r-1)}$, are of the type

$$x - u_1\xi_1 - \dots - u_r\xi_r - u_{r+1}x_{r+1} - \dots - u_nx_n$$

where $\xi_1, \dots, \xi_r, x_{r+1}, \dots, x_n$ is a solution of $F_1 = F_2 = \dots = F_k = 0$. For if $x - \xi$ is any linear factor of $D_u^{(r-1)}$ then ξ is a root of $D_u^{(r-1)} = 0$ to which corresponds a solution $\xi, \xi_2, \dots, \xi_r, x_{r+1}, \dots, x_n$ of $f_1 = f_2 = \dots = f_k = 0$ (§ 14) and a solution $\xi_1, \xi_2, \dots, \xi_r, x_{r+1}, \dots, x_n$ of $F_1 = F_2 = \dots = F_k = 0$, where $\xi = u_1 \xi_1 + \dots + u_r \xi_r + u_{r+1} x_{r+1} + \dots + u_n x_n$.

The linear factors of F_u expressed in the above form supply all the solutions of $f_1 = f_2 = \dots = f_k = 0$, viz. $\xi, \xi_2, \dots, \xi_r, x_{r+1}, \dots, x_n$, and all the solutions of $F_1 = F_2 = \dots = F_k = 0$, viz. $\xi_1, \xi_2, \dots, \xi_r, x_{r+1}, \dots, x_n$, of the several ranks $r = 1, 2, \dots, n$; but it is only when $\xi_1, \xi_2, \dots, \xi_r$ are independent of u_1, u_2, \dots, u_n that we know the solution from merely knowing the factor.

19. A linear factor of F_u of rank r such as the above will be called a *true* linear factor if $\xi_1, \xi_2, \dots, \xi_r$ are independent of u_1, u_2, \dots, u_n , that is, if it is linear in x, u_1, u_2, \dots, u_n .

If a linear factor of F_u is not a true linear factor the solution supplied by it is an imbedded one.

Let $x - \xi$ or $x - u_1 \xi_1 - \dots - u_s \xi_s - u_{s+1} x_{s+1} - \dots - u_n x_n$ be a non-true linear factor of F_u , so that $\xi_1, \xi_2, \dots, \xi_s$ depend on u_1, u_2, \dots, u_n . Then $\xi_1, \xi_2, \dots, \xi_s, x_{s+1}, \dots, x_n$ is a solution of $F_1 = F_2 = \dots = F_k = 0$, and so also is $\eta_1, \eta_2, \dots, \eta_s, x_{s+1}, \dots, x_n$ where $\eta_1, \eta_2, \dots, \eta_s$ are obtained from $\xi_1, \xi_2, \dots, \xi_s$ by changing u_1, u_2, \dots, u_n to v_1, v_2, \dots, v_n . Hence $\eta, \eta_2, \dots, \eta_s, x_{s+1}, \dots, x_n$ (where $\eta = u_1 \eta_1 + \dots + u_s \eta_s + u_{s+1} x_{s+1} + \dots + u_n x_n$) is a solution of $f_1 = f_2 = \dots = f_k = 0$, and therefore makes F_u vanish. But it does not make $D_u^{(s-1)} \dots D_u^{(n-1)}$ vanish since this does not involve x_2, \dots, x_s , and cannot have a factor $x - \eta$, where η involves v_1, v_2, \dots, v_n . Hence it makes some factor $D_u^{(r-1)}$ of F_u of rank $r < s$ vanish. Then $D_u^{(r-1)}$ vanishes when x, x_{r+1}, \dots, x_s are put equal to $\eta, \eta_{r+1}, \dots, \eta_s$; and by putting v_1, v_2, \dots, v_n (of which $D_u^{(r-1)}$ is independent) equal to u_1, u_2, \dots, u_n it follows that $D_u^{(r-1)}$ vanishes when x, x_{r+1}, \dots, x_s are put equal to $\xi, \xi_{r+1}, \dots, \xi_s$. Hence the solution $\xi, \xi_2, \dots, \xi_s, x_{s+1}, \dots, x_n$ is an imbedded one (§ 16).

It follows that all the solutions of $F_1 = F_2 = \dots = F_k = 0$ are obtainable from true linear factors of F_u ; and that all the linear factors of the first complete partial u -resolvent (different from 1) are true linear factors.

It also follows that if there is a spread of rank s which is not imbedded there must be true linear factors of F_u of rank s corresponding to the spread.

We have not proved that all linear factors of F_u are true linear factors*, and whether this is so or not must be considered doubtful.

20. *If an irreducible factor R_u of F_u considered as a whole function of all the quantities $x, x_2, \dots, x_n, u_1, u_2, \dots, u_n$ has a true linear factor all its linear factors are true linear factors.*

Let R_u be of rank r . Then R_u is independent of x_1, x_2, \dots, x_r and there is a one-one correspondence between its true linear factors and the sets of values $\xi_1, \xi_2, \dots, \xi_r$ of x_1, x_2, \dots, x_r (not involving u_1, u_2, \dots, u_n) for which $(R_u)_{x=u_1x_1+\dots+u_nx_n}$ vanishes. Let

$$(R_u)_{x=u_1x_1+\dots+u_nx_n} = \rho_1 R_1 + \rho_2 R_2 + \dots + \rho_\mu R_\mu,$$

where $\rho_1, \rho_2, \dots, \rho_\mu$ are different power products of u_1, u_2, \dots, u_n and R_1, R_2, \dots, R_μ are whole functions of x_1, x_2, \dots, x_n independent of u_1, u_2, \dots, u_n . Then the sets of values $\xi_1, \xi_2, \dots, \xi_r$ required are the solutions of $R_1 = R_2 = \dots = R_\mu = 0$ regarded as equations for x_1, x_2, \dots, x_r . These come from the solutions $\xi_1, \xi_2, \dots, \xi_r, x_{r+1}, \dots, x_n$ of rank r of the same equations in x_1, x_2, \dots, x_n . Now there is at least one solution of rank r ; since R_u has a true linear factor; and only a finite number of such solutions altogether, since R_u has only a finite number of such factors. Hence the first complete partial u -resolvent (different from 1) of the equations $R_1 = R_2 = \dots = R_\mu = 0$ is of rank r , and resolves completely into true linear factors (§ 19)

$$x - u_1 \xi_1 - \dots - u_r \xi_r - u_{r+1} x_{r+1} - \dots - u_n x_n.$$

This complete partial u -resolvent of rank r is therefore R_u itself (or else a power of R_u), which proves the theorem.

If F_u is resolved into factors of the R_u type (irreducible with respect to $x, x_2, \dots, x_n, u_1, u_2, \dots, u_n$), and these into irreducible factors as regards x, x_2, \dots, x_n only, F_u will be resolved into all its irreducible factors. Hence every irreducible factor of F_u is a factor of a factor of the R_u type, and has all or none of its linear factors true linear factors.

It follows that any factor of F_u irreducible with respect to x, x_2, \dots, x_n , and having a true linear factor, has all its linear factors true linear factors, and is a whole function of u_1, u_2, \dots, u_n .

* Kronecker states this as a fact without proving it. König's proof contains an error (K, p. 210). It is not correct to say as he does that $\bar{E}_t^{(h)} \bar{X}_t^{(h)}$ vanishes when $x = \xi_t$, but only when $x, \xi_1, \xi_2, \dots, \xi_h$ are put equal to $\xi_t, \xi_1', \xi_2', \dots, \xi_h'$.

21. The irreducible spreads of a module. Let R_u be any irreducible factor of F_u of rank r having a true linear factor. We know that

$$R_u = A \prod_{i=1}^{i=d} (x - u_1 x_{1i} - \dots - u_r x_{ri} - u_{r+1} x_{r+1i} - \dots - u_n x_n).$$

Hence $(R_u)_{x=u_1 x_1 + \dots + u_n x_n} = A \prod_{i=1}^{i=d} \{u_1 (x_1 - x_{1i}) + \dots + u_r (x_r - x_{ri})\}.$

To R_u corresponds what is called an *irreducible spread*, viz. the spread of all points $x_{1i}, \dots, x_{ri}, x_{r+1i}, \dots, x_{ni}$ in which x_{r+1i}, \dots, x_{ni} take all finite values, and x_{1i}, \dots, x_{ri} the d sets of values supplied by the linear factors of R_u , which vary as x_{r+1i}, \dots, x_{ni} vary.

The degree d of R_u is called the *order* of the irreducible spread.

From the two identities above several useful results can be deduced. It must be remembered that R_u is a known polynomial in $x, x_{r+1}, \dots, x_n, u_1, u_2, \dots, u_n$. No linear factor of R_u can be repeated, unless x_{r+1}, \dots, x_n are given special values; for otherwise R_u and $\frac{\partial R_u}{\partial x}$ would have an H.C.F. involving x , and R_u would be the product of two factors. Whatever set of values x_{r+1}, \dots, x_n have, whether general or special, the d sets of corresponding values of x_1, x_2, \dots, x_r , viz. $x_{1i}, x_{2i}, \dots, x_{ri}$ are definite and finite, because R_u is regular in x .

From the second identity it is seen that $(R_u)_{x=u_1 x_1 + \dots + u_n x_n}$ is independent of u_{r+1}, \dots, u_n , and vanishes identically (i.e. irrespective of u_1, u_2, \dots, u_n) at every point of the spread and no other point. Hence the whole coefficients* of the power products of u_1, u_2, \dots, u_r in $(R_u)_{x=u_1 x_1 + \dots + u_n x_n}$ all vanish at every point of the spread and do not all vanish at any other point. These coefficients equated to zero give a system of equations for the spread; but it is not necessary to take them all, and some are simpler than others. The coefficient of u_r^d gives an equation $\phi(x_r, x_{r+1}, \dots, x_n) = A \prod (x_r - x_{ri}) = 0$ for x_r , whose roots are the d values of x_r corresponding to given arbitrary values of x_{r+1}, \dots, x_n . The coefficient of $u_1 u_r^{d-1}$ gives an equation

$$x_1 \phi' - \phi_1 = \phi \sum \frac{x_1 - x_{1i}}{x_r - x_{ri}} = 0,$$

where ϕ' is $\frac{\partial \phi}{\partial x_r}$ and ϕ_1 , or $\phi \sum \frac{x_{1i}}{x_r - x_{ri}}$, is a polynomial in x_r, x_{r+1}, \dots, x_n .

* Also these coefficients are members of (F_1, F_2, \dots, F_k) if $(R_u)_{x=u_1 x_1 + \dots + u_n x_n}$ is a member of (F_1, F_2, \dots, F_k) , as it will be proved to be when (F_1, F_2, \dots, F_k) is a prime module (§ 31).

Similarly we have $x_2\phi' - \phi_2 = 0, \dots, x_{r-1}\phi' - \phi_{r-1} = 0$. The equations

$$\phi = 0, \quad x_1 = \frac{\phi_1}{\phi'}, \quad x_2 = \frac{\phi_2}{\phi'}, \quad \dots, \quad x_{r-1} = \frac{\phi_{r-1}}{\phi'}$$

are called more particularly the equations of the spread, the first giving the different values of x_r as functions of x_{r+1}, \dots, x_n , and the others giving x_1, x_2, \dots, x_{r-1} as rational functions of x_r, x_{r+1}, \dots, x_n . If x_r, x_{r+1}, \dots, x_n have such values that $\phi = \phi' = 0$ then $\phi_1, \phi_2, \dots, \phi_{r-1}$ all vanish and the expressions above for x_1, x_2, \dots, x_{r-1} become indeterminate. In such a case the values of x_1, x_2, \dots, x_{r-1} may be found by taking other equations from $(R_u)_{x=u, x_1+\dots+x_n}$ for them.

22. Geometrical property of an irreducible spread.

An algebraic spread in general is one which is determined by any finite system of algebraic equations, and consists of all points whose coordinates satisfy the equations and no other points. Such a spread has already been shown to consist of a finite number of irreducible spreads each of which is determined by a finite system of equations. The characteristic property of an irreducible spread is that any algebraic spread which contains a part of it, of the same dimensions as the irreducible spread, contains the whole of it.

Let $F_1 = F_2 = \dots = F_k = 0$ be the equations determining any algebraic spread, and $F'_1 = F'_2 = \dots = F'_{k'} = 0$ the equations determining an irreducible spread. The spread they have in common is determined by the combined system of equations $F_1 = F_2 = \dots = F_k = F'_1 = \dots = F'_{k'} = 0$, and is contained in the irreducible spread and has the same or less dimensions. If it is of the same dimensions as the irreducible spread the complete u -resolvent of $F_1 = \dots = F_k = F'_1 = \dots = F'_{k'} = 0$ will have an irreducible factor R_u'' of the same rank as the irreducible factor R_u' of the complete u -resolvent of $F'_1 = F'_2 = \dots = F'_{k'} = 0$ corresponding to the spread of the same. Also all the roots of $R_u'' = 0$ regarded as an equation for x are roots of $R_u' = 0$. Hence R_u' is divisible by R_u'' , and since they are both irreducible they must be identical. Hence the spread of $F_1 = \dots = F_k = F'_1 = \dots = F'_{k'} = 0$ contains the whole of the spread of $F'_1 = F'_2 = \dots = F'_{k'} = 0$, and the spread of $F_1 = F_2 = \dots = F_k = 0$ contains the same. This proves the property stated above.