I. THE RESULTANT

1. The Resultant is defined in the first instance with respect to \( n \) homogeneous polynomials \( F_1, F_2, \ldots, F_n \) in \( n \) variables, of degrees \( l_1, l_2, \ldots, l_n \), each polynomial being complete in all its terms with literal coefficients, all different. The resultant of any \( n \) given homogeneous polynomials in \( n \) variables is the value which the resultant in the general case assumes for the given case. The resultant of \( n \) given non-homogeneous polynomials in \( n - 1 \) variables is the resultant of the corresponding homogeneous polynomials of the same degrees obtained by introducing a variable \( x_0 \) of homogeneity.

Definitions. An elementary member of the module \((F_1, F_2, \ldots, F_n)\) is any member of the type \( \omega_i F_i \) \((i = 1, 2, \ldots, n)\), where \( \omega \) is any power product of \( x_1, x_2, \ldots, x_n \). What is and what is not an elementary member depends on the basis chosen for the module.

The total number of elementary members of an assigned degree is evidently finite.

The diagram below represents the array of the coefficients of all elementary members of \((F_1, F_2, \ldots, F_n)\) of degree \( t \), arranged under the power products \( \omega_1(t), \omega_2(t), \ldots, \omega_{\mu}(t) \) of degree \( t \left( \mu = \frac{t + n - 1}{t + n - 1} \right) \):

\[
\begin{array}{cccc}
\omega_1(t) & \omega_2(t) & \ldots & \omega_{\mu}(t) \\
\lambda_1 & a_1 & b_1 & k_1 \\
\lambda_2 & a_2 & b_2 & k_2 \\
\vdots & \vdots & \vdots & \vdots \\
\lambda_\mu & a_\mu & b_\mu & k_\mu
\end{array}
\]

Each row of the array, in association with \( \omega_1(t), \omega_2(t), \ldots, \omega_{\mu}(t) \), represents an elementary member of degree \( t \); and the rows of the array corresponding to \( F_i \) all consist of the same elements (the coefficients of \( F_i \) and zeros) but in different columns.

Any member \( F = X_1 F_1 + X_2 F_2 + \ldots + X_n F_n \) of degree \( t \) is evidently a linear combination \( \lambda_1 \omega_1 F_1 + \lambda_2 \omega_2 F_1 + \ldots + \lambda_\mu \omega_{\mu} F_i + \ldots + \lambda_\mu \omega_{\mu} F_n \) of elementary members of degree \( t \), and is represented by the above array when bordered by \( \lambda_1, \lambda_2, \ldots, \lambda_\mu \) on the left, where \( \lambda_1, \lambda_2, \ldots, \lambda_\mu \) are the coefficients of \( X_1, X_2, \ldots, X_n \), some of which may be zeros.

This bordered array also shows in a convenient way the whole coefficients of the terms of \( F_i \), viz. \( \Sigma \lambda a, \Sigma \lambda b, \ldots, \Sigma \lambda k \).
These remarks and definitions are equally applicable to any module $(F_1, F_2, \ldots, F_k)$ of homogeneous or non-homogeneous polynomials; but the following definition applies only to the particular module $(F_1, F_2, \ldots, F_n)$.

The resultant $R$ of $F_1, F_2, \ldots, F_n$ is the h.c.f. of the determinants of the above array for degree $t = l + 1$, where $l = l_1 + l_2 + \ldots + l_n - n$. It will be shown (§ 7) that $R$ is homogeneous and of degree $l_1 l_2 \ldots l_n / l_i$ in the coefficients of $F_i$ ($i = 1, 2, \ldots, n$).

2. Resultant of two homogeneous polynomials in two variables.

Let

\[ F_1 = a_1 x_1^{l_1} + b_1 x_1^{l_1 - 1} x_2 + \ldots + k_1 x_2^{l_1}, \]
\[ F_2 = k_2 x_1^{l_2} + \ldots + a_2 x_2^{l_2}, \]
\[ l = l_1 + l_2 - 2. \]

The array of the coefficients of all elementary members of $(F_1, F_2)$ of degree $l + 1$, viz. $x_1^{l_1 - 1} F_1, x_1^{l_1 - 2} x_2 F_1, \ldots, x_2^{l_2 - 1} F_2, x_1^{l_2 - 1} F_2, \ldots, x_2^{l_2 - 1} F_2$, has $l_2$ rows corresponding to $F_1$ and $l_1$ rows corresponding to $F_2$, and the same number $l_1 + l_2$ of rows in all as columns. The resultant $R$ is therefore the determinant of this array. The array is

\[
\begin{array}{cccccc}
\lambda_1 & a_1 & b_1 & \cdots & k_1 & \cdots \\
\lambda_2 & a_1 & b_1 & \cdots & k_1 & \cdots \\
\lambda_{l_1} & a_1 & b_1 & \cdots & k_1 & \cdots \\
\lambda_{l_1 + 1} & \cdots & a_1 & b_1 & \cdots & k_1 \\
\lambda_{l_2} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\lambda_{l_2 + 1} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

\[
\begin{array}{c}
= x_1^{l_2 - 1} F_1 \\
= x_1^{l_2 - 2} x_2 F_1 \\
= x_1^{l_1 - 1} F_2 \\
= x_1^{l_1 - 2} x_2 F_2 \\
= x_2^{l_1 - 1} F_2 \\
= x_2^{l_1 - 2} x_2 F_2 \\
\end{array}
\]

On the right are written the elementary members which the rows represent. Thus, neglecting the left hand border, we may regard the diagram as a set of $l + 2$ identical equations for

\[
\begin{array}{c}
\omega_1^{(l+1)} \quad \omega_2^{(l+1)} \quad \cdots \quad \omega_{l+1}^{(l+1)} \\
\omega_{l+1}^{(l+1)} \quad \omega_{l+2}^{(l+1)} \\
\end{array}
\]

Solving them we have

\[ R \omega_1^{(l+1)} = A_{i1} F_1 + A_{i2} F_2 \quad (i = 1, 2, \ldots, l + 2), \]

where $A_{i1}, A_{i2}$ are polynomials whose coefficients are whole functions of the coefficients of $F_1, F_2$. Hence

\[ R \omega^{(l+1)} \equiv 0 \mod (F_1, F_2), \]
where \( w^{(l+1)} \) is any power product of \( x_1, x_2 \) of degree \( l + 1 \). This expresses the first important property of \( R \).

3. Irreducibility of \( R \). The general expression for the resultant \( R \) is irreducible in the sense that it cannot be resolved into two factors each of which is a whole function of the coefficients of \( F_1, F_2 \). When this has been proved it follows that any whole function of the coefficients of \( F_1, F_2 \) which vanishes as a consequence of \( R \) vanishing must be divisible by \( R \).

\( R \) has a term \( a_1^{l_2} a_2^{l_1} \) obtained from the diagonal of the determinant, and this is the only term of \( R \) containing \( a_2^{l_1} \). Also, when \( a_1 = 0 \), \( R \) has a term \((-1)^{l_2} k_2 b_1 a_2^{l_1-1} \), and this is the only term of \( R \) containing \( a_2^{l_1-1} \) when \( a_1 = 0 \). Hence, when \( R \) is expanded in powers of \( a_2 \) to two terms, we have

\[
R = a_1^{l_2} a_2^{l_1} + b a_2^{l_1-1} + \ldots,
\]

where

\[
b = (-1)^{l_2} k_2 b_1 \text{mod } a_1.
\]

Hence if \( R \) can be written as a product of two factors, we have

\[
R = (a_1^{p_1} a_2^{q_1} + \ldots) (a_1^{p_2} a_2^{q_2} + \ldots),
\]

where \( p_1 + q_1 = l_2 \) and \( p_2 + q_2 = l_1 \), and either \( p_1 \) or \( q_1 \) is zero; for otherwise the coefficient \( b \) of \( a_2^{l_1-1} \) would be zero or divisible by \( a_1 \), which is not the case. Hence one of the factors of \( R \) is independent of the coefficients of \( F_1 \), since both factors must be homogeneous in the coefficients of \( F_1 \). Similarly one of the factors must be independent of the coefficients of \( F_2 \), i.e.

\[
R = (a_1^{l_1} + \ldots) (a_2^{l_1} + \ldots) = a_1^{l_2} a_2^{l_1},
\]

since the whole coefficient of \( a_1^{l_2} \) in \( R \) is \( a_2^{l_1} \), and of \( a_2^{l_1} \) is \( a_1^{l_2} \). This is not true; hence \( R \) is irreducible.

4. The necessary and sufficient condition that the equations \( F_1 = F_2 = 0 \) may have a proper solution (i.e. a solution other than \( x_1 = x_2 = 0 \)) is the vanishing of \( R \).

This is the fundamental property of the resultant. If the equations \( F_1 = F_2 = 0 \) have a solution other than \( x_1 = x_2 = 0 \) it follows from

\[
R x_1^{l+1} \equiv 0 \text{ mod } (F_1, F_2), \quad R x_2^{l+1} \equiv 0 \text{ mod } (F_1, F_2),
\]

that \( R = 0 \), by giving to \( x_1, x_2 \) the values (not both zero) which satisfy the equations \( F_1 = F_2 = 0 \).

Conversely if \( R = 0 \) we can choose \( \lambda_1, \lambda_2, \ldots, \lambda_{l+2} \) so that the sum of their products with the elements in each column of the
determinant \( R \) vanishes. Multiplying each sum by the power product corresponding to its column, and adding by rows, we have

\[(\lambda_1 a_1^{l_1-1} + \lambda_2 a_2^{l_2-2} x_2 + \ldots + \lambda_{l_2} a_2^{l_2-1}) F_1 + (\lambda_{l_2+1} a_1^{l_1-1} + \ldots + \lambda_{l_2+2} a_2^{l_2-1}) F_2 = 0,\]

where \( \lambda_1, \lambda_2, \ldots \lambda_{l_2+2} \) do not all vanish. Hence, since \( \lambda_1 a_1^{l_2-1} + \ldots \) is of less degree than \( F_2 \), \( F_1 \) must have a factor in common with \( F_2 \), and the equations \( F_1 = F_2 = 0 \) have a proper solution.

In the following article another proof is given which can be extended more easily to any number of variables.

5. When \( R \neq 0 \) there are \( l+2 \) linearly independent members of \((F_1, F_2)\) of degree \( l+1 \), and \( l \) of degree \( l \). When \( R = 0 \) there are only \( l+1 \) linearly independent members of degree \( l+1 \) and still \( l \) of degree \( l \), i.e. in each case 1 less than the number of terms in a polynomial of degree \( l+1 \) and \( l \) respectively. Hence there will be one and only one identical linear relation between the coefficients of the general member of \((F_1, F_2)\) whether of degree \( l+1 \) or \( l \). Let this identical relation for degree \( l+1 \) be

\[c_{i+1,0} z_{i+1,0} + c_{i,1} z_{i,1} + \ldots + c_{0,i+1} z_{0,i+1} = 0,\]

where \( z_{i,j} \) denotes the coefficient of \( x_1^i a_2^j \) in the general member of \((F_1, F_2)\) of degree \( i+j \), and the \( c_{i,j} \) are constants. Then, if \( F \) is the general member

\[z_{i,0} x_1^i + z_{i-1,1} a_1 x_1^{l-1} x_2 + \ldots + z_{0,l} a_2^l \]

of \((F_1, F_2)\) of degree \( l \), \( x_1 F \) is a member of degree \( l+1 \) whose coefficients must satisfy the relation above. Hence

\[c_{i+1,0} z_{i,0} + c_{i,1} z_{i-1,1} + \ldots + c_{1,l} z_{0,l} = 0.\]

Similarly

\[c_{i,1} z_{i,0} + c_{i-1,2} z_{i-1,1} + \ldots + c_{0,i+1} z_{0,l} = 0,\]

since \( x_2 F \) is a member of \((F_1, F_2)\) of degree \( l+1 \). These two relations must be equivalent to one only, since only one identical relation exists for degree \( l \). Hence we have

\[\frac{c_{i+1,0}}{c_{i,1}} = \frac{c_{i,1}}{c_{i-1,2}} = \ldots = \frac{c_{1,l}}{c_{0,i+1}} = \frac{a_1}{a_2} \text{ (say)},\]

i.e. \( c_{i+1,0}, c_{i,1}, \ldots, c_{0,i+1} \) are proportional to \( a_2^{i+1}, a_1 a_2, \ldots, a_2^{i+1} \). Hence the original identical relation may be written

\[z_{i+1,0} a_2^{i+1} + z_{i,1} a_1 a_2 + \ldots + z_{0,i+1} a_2^{i+1} = 0,\]

showing that the general member \( z_{i+1,0} a_2^{i+1} + \ldots \) of \((F_1, F_2)\) of degree \( l+1 \) vanishes when \( x_1 = a_1, \ x_2 = a_2, \) and that the equations \( F_1 = F_2 = 0 \) have the proper solution \((a_1, a_2)\). The theorem being thus proved true in general is assumed to be true in particular.
6. Resultant of \( n \) homogeneous polynomials in \( n \) variables.

The general theory of the resultant to be now given is exactly parallel to that already given for two variables, although it involves points of much greater difficulty as might be expected. Another method of exposition depending on a different definition of the resultant is given in (K, p. 260 ff.).

Let \( F_1, F_2, \ldots, F_n \) be \( n \) homogeneous polynomials of degrees \( l_1, l_2, \ldots, l_n \) of which all the coefficients are different letters. In particular, let \( a_1, a_2, \ldots, a_n \) be the coefficients of \( x_1^{l_1}, x_2^{l_2}, \ldots, x_n^{l_n} \) in \( F_1, F_2, \ldots, F_n \) respectively, and \( c_1, c_2, \ldots, c_n \) the constant terms of \( F_1, F_2, \ldots, F_n \) when \( x_n \) is put equal to 1, so that \( c_n = a_n \). Let

\[
l = l_1 + l_2 + \ldots + l_n - n, \quad L = l_1 l_2 \ldots l_n, \quad L_1 = L/l_1, \quad L_2 = L/l_2, \ldots, L_n = L/l_n.
\]

The resultant \( R \) of \( F_1, F_2, \ldots, F_n \) has already been defined (§1) as the H.C.F. of the determinants of the array of the coefficients of all elementary members of \( (F_1, F_2, \ldots, F_n) \) of degree \( l + 1 \).

We shall first consider a particular determinant \( D \) of the array, viz. that representing (§1) the polynomial

\[
X^{(0)} F_1 + X^{(1)} F_2 + \ldots + X^{(n-1)} F_n \text{ of degree } l + 1,
\]

where \( X^{(j)} \) denotes a polynomial in which all terms divisible by \( x_1^{l_1} \) or \( x_2^{l_2} \) or \( x_i^{l_i} \) are absent, which may be expressed by saying that \( X^{(j)} \) is reduced in \( x_1, x_2, \ldots, x_i \). The polynomial

\[
X^{(0)} F_1 + X^{(1)} F_2 + \ldots + X^{(n-1)} F_n
\]

is represented by the bordered array

\[
\begin{array}{cccc}
\omega_1^{(l+1)} & \omega_2^{(l+1)} & \ldots & \omega_{\mu}^{(l+1)} \\
\lambda_1 & a_1 & b_1 & \ldots & k_1 \\
\lambda_2 & a_2 & b_2 & \ldots & k_2 \\
& \vdots & \vdots & \ddots & \vdots \\
\lambda_{\mu} & a_\mu & b_\mu & \ldots & k_\mu \\
\end{array}
\]

where \( \omega_1^{(l+1)}, \omega_2^{(l+1)}, \ldots, \omega_{\mu}^{(l+1)} \) are all the power products of \( x_1, x_2, \ldots, x_n \) of degree \( l + 1 \), and \( \lambda_1, \lambda_2, \ldots, \lambda_\mu \) are the coefficients of \( X^{(0)}, X^{(1)}, \ldots, X^{(n-1)} \). That this array has the same number \( \mu \) of rows as columns is seen from the fact that one and only one of the elements \( a_1, a_2, \ldots, a_n \) (the coefficients of \( x_1^{l_1}, x_2^{l_2}, \ldots, x_n^{l_n} \) in \( F_1, F_2, \ldots, F_n \)) occurs in each row and each column. This is evident as regards the rows. To prove

* These are not the same as the \( a_1, a_2, \ldots, a_n \) in the first column of the array. The latter should be represented by some other symbols.
that the same is true of the columns, we notice firstly that there
is no power product \( \omega^{l+1} \) of degree \( l+1 \) reduced in all the variables,
for the highest power product of this kind is \( x_1^{l_1-1} x_2^{l_2-1} \ldots x_n^{l_n-1} \)
which is of degree \( l<l+1 \); and secondly, if we put every coefficient
of \( F_1, F_2, \ldots, F_n \), except only \( a_1, a_2, \ldots, a_n \), equal to zero, the
diagram will represent the polynomial

\[
X^{(0)} a_1 x_1^{l_1} + X^{(1)} a_2 x_2^{l_2} + \ldots + X^{(n-1)} a_n x_n^{l_n},
\]
in which each power product \( \omega^{l+1} \) occurs once and once only, so
that one and only one element \( a_1, a_2, \ldots, a_n \) occurs in each column
of \( D \).

Thus \( D \) when expanded has a term \( \pm a_1^{\mu_1} a_2^{\mu_2} \ldots a_n^{\mu_n} \), where \( \mu_i \) is
the number of terms in \( X^{(l-1)} \), and by saying that the coefficient
of this term in \( D \) is to be \(+1\) we remove any ambiguity as to the
sign of \( D \). Also it is to be noted that \( D \) vanishes when \( c_1, c_2, \ldots, c_n \)
all vanish, for the column of \( D \) corresponding to \( x_n^{l+1} \) contains no
elements other than \( c_1, c_2, \ldots, c_n \) and zeros.

Regarding the diagram as giving \( \mu \) identical equations for
\( \omega_1^{l+1}, \omega_2^{l+1}, \ldots, \omega_\mu^{l+1} \),
and solving, we have

\[
D \omega^{l+1} \equiv 0 \mod (F_1, F_2, \ldots, F_n),
\]
where \( \omega^{l+1} \) is any power product of \( x_1, x_2, \ldots, x_n \) of degree \( l+1 \).
It can be proved that the factors of \( D \) other than \( R \) can be divided
out of this congruence equation, so that

\[
R \omega^{l+1} \equiv 0 \mod (F_1, F_2, \ldots, F_n);
\]
but this will not be assumed in what follows.*

7. The number of rows in \( D \) corresponding to \( F_n \) is the number
of terms in \( X^{(n-1)} \). But \( X^{(n-1)} \) is of degree \( l+1-l_n \) or

\( (l_1-1) + (l_2-1) + \ldots + (l_{n-1}-1) \),
and its terms consist of all the power products in

\[
(1 + x_1 + \ldots + x_1^{l_1-1}) \ldots (1 + x_{n-1} + \ldots + x_{n-1}^{l_{n-1}-1})
\]
each multiplied by a power of \( x_n \); hence the number of the terms is
\( l_1 l_2 \ldots l_{n-1} = L_n \). Thus \( D \) is homogeneous and of degree \( L_n \) in the

* No proof of this has been published so far as I know. It can be proved that
if \( A \) is any whole function of the coefficients of \( F_1, F_2, \ldots, F_n \) not divisible by \( R \),
and \( AF \equiv 0 \mod (F_1, F_2, \ldots, F_n) \), then \( F \equiv 0 \mod (F_1, F_2, \ldots, F_n) \). Hence from
\( D \omega^{l+1} \equiv 0 \mod (F_1, F_2, \ldots, F_n) \) we have \( R \omega^{l+1} \equiv 0 \mod (F_1, F_2, \ldots, F_n) \). The
condition that \( A \) is not divisible by \( R \) is not needed if \( F \) is of degree \( \leq l \).
coefficients of $F_n$, and homogeneous and of degree $> L_i$ in the coefficients of $F_i (i = 1, 2, ..., n - 1)$. It follows that $R$, which is a factor of $D$, is at most of degree $L_n$ in the coefficients of $F_n$. We shall prove that $R$ is of this degree, and consequently of degree $L_i$ in the coefficients of $F_i$.

Let $D'$ be any other non-vanishing determinant of the array, viz.

\[
\begin{vmatrix}
\omega_1^{(i+1)} & \omega_2^{(i+1)} & \cdots & \omega_{\mu}^{(i+1)} \\
a_1 & a_1' & b_1' & \cdots & k_1' \\
a_2 & a_2' & b_2' & \cdots & k_2' \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_\mu & a_\mu' & b_\mu' & \cdots & k_\mu'
\end{vmatrix}
\]

This represents the polynomial $A_1 F_1 + A_2 F_2 + \cdots + A_n F_n$, in which $a_1, a_2, \ldots, a_\mu$ are the (arbitrarily chosen) coefficients of $A_1, A_2, \ldots, A_n$ which are not zeros. Choose $\lambda_1, \lambda_2, \ldots, \lambda_\mu$ in the previous diagram so that we have identically

\[X^{(0)} F_1 + X^{(1)} F_2 + \cdots + X^{(n-1)} F_n = A_1 F_1 + A_2 F_2 + \cdots + A_n F_n.\]

This gives, by equating coefficients of power products on both sides,

\[\Sigma \lambda a = \Sigma a a', \quad \Sigma \lambda b = \Sigma a b', \ldots, \quad \Sigma \lambda k = \Sigma a k'\]

as equations for $\lambda_1, \lambda_2, \ldots, \lambda_\mu$; and they have a unique solution, since $D$ does not vanish.

Let $\left( \begin{array}{c} \lambda \\ a \end{array} \right)$ denote the determinant of the substitution corresponding to the solution of the above equations for $\lambda_1, \lambda_2, \ldots, \lambda_\mu$ as linear functions of $a_1, a_2, \ldots, a_\mu$. Then if we put

\[\Sigma \lambda a = \Sigma a a' = \lambda_1', \quad \Sigma \lambda b = \Sigma a b' = \lambda_2', \ldots, \quad \Sigma \lambda k = \Sigma a k' = \lambda_\mu',\]

we have

\[\left( \begin{array}{c} \lambda' \\ a' \end{array} \right) = D, \quad \left( \begin{array}{c} \lambda' \\ a \end{array} \right) = D', \quad \text{and} \quad \left( \begin{array}{c} \lambda' \\ a' \end{array} \right) \cdot \left( \begin{array}{c} \lambda \\ a \end{array} \right) = \left( \begin{array}{c} \lambda' \\ a \end{array} \right),\]

i.e. $D \left( \begin{array}{c} \lambda \\ a \end{array} \right) = D'$, by the rule of successive substitutions, or the rule for multiplying determinants. Hence

\[\frac{D'}{D} = \left( \begin{array}{c} \lambda \\ a \end{array} \right).\]

Now we can find the solution for $\lambda_1, \lambda_2, \ldots, \lambda_\mu$, or the solution of

\[X^{(0)} F_1 + X^{(1)} F_2 + \cdots + X^{(n-1)} F_n = A_1 F_1 + A_2 F_2 + \cdots + A_n F_n,\]

in the following way. First solve the equation

\[Y^{(0)} F_1 + Y^{(1)} F_2 + \cdots + Y^{(n-2)} F_{n-1} + X^{(n-1)} = A_n\]
for the unknowns $Y^{(0)}, Y^{(1)}, \ldots, Y^{(n-2)}, X^{(n-1)}$. This equation has a unique solution, since the more particular equation

$$Y^{(0)} x_1^{l_1} + Y^{(1)} x_2^{l_2} + \ldots + Y^{(n-2)} x_{n-1}^{l_{n-1}} + X^{(n-1)} = A_n$$

has a unique solution (for any given polynomial $A_n$ can be expressed in one and only one way in the form on the left) and shows that the number of the coefficients of $Y^{(0)}, Y^{(1)}, \ldots, Y^{(n-2)}, X^{(n-1)}$ is equal to the number of equations they have to satisfy.

Substituting the value thus found for $X^{(w_1)}$ in the equation

$$X^{(0)} F_1 + X^{(1)} F_2 + \ldots + X^{(n-1)} F_n = A_1 F_1 + A_2 F_2 + \ldots + A_n F_n,$$

it becomes

$$X^{(0)} F_1 + X^{(1)} F_2 + \ldots + X^{(n-2)} F_n = (A_1 + Y^{(0)} F_n) F_1 + \ldots + (A_{n-1} + Y^{(n-2)} F_n) F_{n-1},$$

where $Y^{(0)}, Y^{(1)}, \ldots, Y^{(n-2)}$ have been found. Next solve the equation

$$Z^{(0)} F_1 + Z^{(1)} F_2 + \ldots + Z^{(n-3)} F_{n-2} + X^{(n-2)} = A_{n-1} + Y^{(n-2)} F_n,$$

which has a unique solution for $Z^{(0)}, Z^{(1)}, \ldots, Z^{(n-3)}, X^{(n-2)}$. We can proceed in this way till $X^{(0)}, X^{(1)}, \ldots, X^{(n-1)}$, i.e. $\lambda_1, \lambda_2, \ldots, \lambda_\mu$, have all been found.

In this method of solving the unknowns on the left are associated with $F_1, F_2, \ldots, F_{n-1}$ only and not with $F_n$. Hence $\binom{\lambda}{a}$ is a rational function of the coefficients of $F_1, F_2, \ldots, F_n$ whose denominator is independent of the coefficients of $F_n$, and the same is therefore true of $\frac{D'}{D} = \binom{\lambda}{a}$. Hence every determinant $D'$ of the array has a factor in common with $D$ which is of degree $L_n$ in the coefficients of $F_n$. The resultant $R$, which is the H.C.F. of all the determinants $D'$, is therefore of degree $L_i$ in the coefficients of $F_i (i = 1, 2, \ldots, n)$.

If we put $D = AR$, $A$ is called the extraneous factor of $D$. We have proved that $A$ is independent of the coefficients of $F_n$; and it is proved at the end of § 8 that $A$ depends only on the coefficients of $(F_1, F_2, \ldots, F_{n-1})_{x_n=0}$.

### 8. Properties of the Resultant.

Since $D$ has a term $a_1^{\mu_1} \ldots a_n^{\mu_n}$ (§ 6) $R$ has a term $a_1^{L_1} a_2^{L_2} \ldots a_n^{L_n}$. This is called the leading term of $R$.

Since $D$ vanishes when $c_1, c_2, \ldots, c_n$ all vanish (§ 6) the same is true of $R$; for $D = AR$ and $A$ is independent of $c_1, c_2, \ldots, c_n$.

The extraneous factor $A$ of $D$ is a minor of $D$, viz. the minor obtained by omitting all the columns of $D$ corresponding to power
products reduced in \(n-1\) of the variables and the rows which contain the elements \(a_1, a_2, ..., a_n\) in the omitted columns \((M_1, p. 14)\). Thus \(D/A\), where \(A\) is this minor of \(D\), is an explicit expression for \(R\).

Each coefficient \(a\) of \(F_1, F_2, ..., F_n\) is said to have a certain numerical weight, equal to the index of the power of one particular variable (say \(x_n\)) in the term of which \(a\) is the coefficient. In the case of non-homogeneous polynomials the variable chosen is generally the variable \(x_0\) of homogeneity. Also the weight of \(a^p\) is defined as \(p\) times the weight of \(a\), and the weight of \(a^pb^q...\) as the sum of the weights of \(a^p, b^q, c^r, ...\). A whole function of the coefficients is said to be isobaric when all its terms are of the same weight.

The resultant is isobaric and of weight \(L\). Assign to \(x_1, x_2, ..., x_n\) the weights 0, 0, ..., 0, 1. Then the coefficients of \(F_1, F_2, ..., F_n\) have the same weights as the power products of which they are the coefficients. The \(i\)th row of the determinant \(D\) represents the polynomial \(\omega_1 F_i = a_i x_i^{(i+1)} + b_i x_i^{(i+1)} + ... + k_i \omega_i^{(i+1)}\). Thus the weights of \(a_i, b_i, ..., k_i\) are less than the weights of \(\omega_1, \omega_2, ..., \omega_i\) respectively by the same amount, viz. the weight of \(\omega_i\). Hence, on expanding \(D\), the weight of any term is less than the sum of the weights of \(\omega_1, \omega_2, ..., \omega_i\); i.e. \(D\) is isobaric. Again, if in \(D\) each letter \(a\) is changed to \(a^q\), where \(q\) is the weight of \(a\), \(D\) becomes \(D^{wq}\), where \(w\) is the weight of \(D\); and consequently if \(D\) be expressed as a product of whole factors each factor must be isobaric. Thus \(R\) is isobaric and its weight is that of its leading term \(a_1^{L_1} a_2^{L_2} ... a_n^{L_n}\), which is \(l_n L_n = L\). The weight of \(D\) is the weight of \(a_1^{L_1} a_2^{L_2} ... a_n^{L_n}\), which is also \(L\), since \(\mu_n = L_n\).

The whole coefficient of \(a_1^{L_1} a_2^{L_2} ... a_n^{L_n-1}\) in \(R\) is \(a_n^{L_n}\). For the coefficient must be a whole function of the coefficients of \(F_n\) only of degree \(L_n\) and weight \(l_n L_n\), and \(a_n\) is the only coefficient of \(F_n\) of weight \(l_n\).

A more general result (§ 9) is that the whole coefficient of \(a_n^{L_n}\) in \(R\) is \(R_n^{l_n}\) where \(R_n\) is the resultant of \((F_1, F_2, ..., F_{n-1})_{x_n=0}\). Hence also the whole coefficient of \(a_1^{L_1} a_2^{L_2} ... a_r^{L_r+1}\) is \(R_r^{l_r L_{r+1}}\), where \(R_r\) is the resultant of \((F_1, F_2, ..., F_{r-1})_{x_r=...=x_n=0}\).

Since the weights of \(D\) and \(R\) are the same the weight of the extraneous factor \(A\) of \(D\) is zero. This, taken in conjunction with the fact that \(A\) is independent of the coefficients of \(F_n\), shows that \(A\) is a whole function of the coefficients of \((F_1, F_2, ..., F_{n-1})_{x_n=0}\) only.
9. The resultant of $F_1, F_2, \ldots, F_n$ is irreducible and invariant. It has been proved that the resultant is irreducible when $n = 2$ (§ 3); and the proof can be extended to the general case by induction.

Let $R_n$ = the resultant of $(F_1, F_2, \ldots, F_{n-1})x_n = 0$;

$F_0$ = the resultant of the homogeneous polynomials $F_1^{(0)}, F_2^{(0)}, \ldots, F_{n-1}^{(0)}$ in $x_1, x_2, \ldots, x_{n-2}, x_0$ obtained from $F_1, F_2, \ldots, F_{n-1}$ by changing $x_n$ to $x_0$;

$F'_n = (F_n)x_n = \ldots = x_{n-2} = 0 = k_n x_{n-1}^n + \ldots + a_n x_0^n$;

$R'$ = the resultant of $F_1, F_2, \ldots, F_{n-1}, F'_n$;

$R_0$ = the resultant of $F_0, F'_n$, two polynomials in $x_n, x_0$;

$L_1 l_1 = L_2 l_2 = \ldots = L_{n-1} l_{n-1} = l_1 l_2 \ldots l_{n-1} = l_n$.

Finally let $a_1, a_2, \ldots, a_n, c_1, c_2, \ldots, c_{n-1}$ denote the same coefficients of $F_1, F_2, \ldots, F_n$ as in § 6. We assume $R_n$ irreducible and have to prove that $R$ is irreducible.

$F'_0$ is of weight $L_n$ in the coefficients of $F_1^{(0)}, F_2^{(0)}, \ldots, F_{n-1}^{(0)}$, and each coefficient is a homogeneous polynomial in $x_{n-1}, x_0$ of degree equal to its weight in reference to $x_0$. Hence

$$F'_0 = A x_{n-1}^{L_n} + B x_{n-2}^{L_1} x_n + \ldots,$$

where $A, B, \ldots$ are whole functions of the coefficients of $F_1, F_2, \ldots, F_{n-1}$ of the same dimensions as $R_n$. When $x_n = 0$, $F_0$ becomes the resultant of $(F_1, F_2, \ldots, F_{n-1})x_n = 0$, viz. $R_n x_{n-1}^{L_n}$; hence $A = R_n$. Also the whole coefficient of $a_1^{L_n} a_2^{L_2} \ldots a_{n-1}^{L_{n-1}}$ in $F'_0$ is $a'^{L_{n-1}}$, where $a'^{L_{n-1}}$ is the coefficient of $x_{n-1}^{L_{n-1}}$ in $F_{n-1}^{(0)}$ (§ 8), viz.

$$a'^{L_{n-1}} = a_{n-1} x_{n-1}^{L_{n-1}} + b x_{n-2}^{L_{n-1}} x_n + \ldots + c_{n-1} x_0^n a_{n-1}^{L_{n-1}}.$$

Hence $B$ has a term $L_{n-1} a_1^{L_n} a_2^{L_2} \ldots a_{n-2}^{L_{n-2}} a_{n-1}^{L_{n-1}} b$, and cannot be divisible by $R_n$, since $R_n$ does not involve $b$. Hence we find that

$$F'_0 = R_n x_{n-1}^{L_n} + B x_{n-2}^{L_1} x_n + \ldots,$$

where $B$ is neither zero nor divisible by $R_n$.

Now if $R'$ vanishes one of the solutions of $F'_n = 0$ for $x_{n-1} : x_n$ will be the same as in one of the solutions of $F_1 = \ldots = F_{n-1} = 0$ (§ 10), and will therefore be a solution of $F'_0 = 0$; i.e. $R' = 0$ requires $R_0 = 0$, and $R_0$ is divisible by each irreducible factor of $R'$. But (§ 3)

$$R_0 = R_n x_n^{L_n} + B' x_n^{L_{n-1}} + \ldots,$$

where $B' = (-1)^{L_n} k_n B^{L_n} \mod R_n$, so that $B'$ is neither zero nor divisible by $R_n$. Hence, as in § 3, $R_0$ has an irreducible factor of the form $R_n x_n^{L_n} a_n + \ldots$, and has no other
factor involving the coefficients of \( F_1, F_2, \ldots, F_{n-1} \). This must therefore be a factor of \( R' \).

Again \( R' \) is what \( R \) becomes when all the coefficients of \( F_n \) other than those of \( F_n' \) are put equal to zero. Hence \( R \) has an irreducible factor of the form \( R_n a_n^q + \ldots \), where \( q \geq p \). The remaining factor of \( R \) is independent of the coefficients of \( F_1, F_2, \ldots, F_{n-1} \) and therefore also of the coefficients of \( F_n \) when \( n > 2 \). Hence \( R \) is irreducible.

It easily follows that \( R \) is invariant for a homogeneous linear substitution whose determinant \( \begin{pmatrix} x \\ x' \end{pmatrix} \) does not vanish. Suppose that \( R = 0 \) and that this is the only relation existing between the coefficients of \( F_1, F_2, \ldots, F_n \). Then not more than one relation can exist between the coefficients of \( F_1', F_2', \ldots, F_n' \), the polynomials into which \( F_1, F_2, \ldots, F_n \) transform. Since \( R = 0 \) there are less than \( \mu \) linearly independent members of \( (F_1, F_2, \ldots, F_n) \) of degree \( l+1 \) and therefore less than \( \mu \) linearly independent members of \( (F_1', F_2', \ldots, F_n') \) of degree \( l+1 \), and the only single relation between the coefficients of \( F_1', F_2', \ldots, F_n' \) which will admit this is \( R' = 0 \). Hence \( R = 0 \) requires \( R' = 0 \), and \( R' \) is divisible by \( R \). The remaining factor of \( R' \) is independent of the coefficients of \( F_1, F_2, \ldots, F_n \), and can be shown to be \( \begin{pmatrix} x^L \\ x' \end{pmatrix} \). A proof that \( R \) is invariant without assuming it irreducible is given in (E, p. 17).

10. The necessary and sufficient condition that the equations \( F_1 = F_2 = \ldots = F_n = 0 \) may have a proper solution is the vanishing of \( R \).

In the general case, when the coefficients are letters, \[ ARx_n^{l+1} \equiv 0 \mod (F_1, F_2, \ldots, F_n). \]
Put \( x_n = 1 \) and change* \( c_i \) to \( c_i - F_i (i = 1, 2, \ldots, n) \); then \( A \) does not change, being independent of \( c_1, c_2, \ldots, c_n \) (§ 8); but \( R \) changes to \( R - A_1 F_1 - A_2 F_2 - \ldots - A_n F_n \), and this must vanish; hence \( R \equiv 0 \mod (F_1, F_2, \ldots, F_n)x_n = 1 \).

Hence \( R \) vanishes if the equations \( F_1 = F_2 = \ldots = F_n = 0 \) have a solution in which \( x_n = 1 \), i.e. if they have a proper solution.

To prove that \( R = 0 \) is a sufficient condition, we shall assume that \( R = 0 \) is the only relation existing between the coefficients of \( F_1, F_2, \ldots, F_n \). There are then less than \( \mu \) linearly independent members of \( (F_1, F_2, \ldots, F_n) \) of degree \( l+1 \). Hence the coefficients \( x_{p_1, p_2, \ldots, p_n} \) of

* Called the Kronecker substitution.
the general member of degree $l+1$ must satisfy an identical linear relation
\[ \sum c_{p_1, p_2, \ldots, p_n} z_{p_1, p_2, \ldots, p_n} = 0, \quad p_1 + p_2 + \ldots + p_n = l + 1. \]

The coefficients of the general member of degree $l$ also satisfy one and only one identical linear relation, whether $R$ vanishes or not. To prove this it has to be shown that the number $N$ of linearly independent members of $(F_1, F_2, \ldots, F_n)$ of degree $l$ is 1 less than the number $\rho$ of power products of degree $l$. If no relation exists between the coefficients of $F_1, F_2, \ldots, F_n$ the equation
\[ X^{(0)} F_1 + X^{(1)} F_2 + \ldots + X^{(n-1)} F_n = A_1 F_1 + A_2 F_2 + \ldots + A_n F_n \]
can always be solved by the method of § 7, where $A_1, A_2, \ldots, A_n$ are arbitrary given polynomials. Hence $N$ is not greater than the number of coefficients in $X^{(0)}, X^{(1)}, \ldots, X^{(n-1)}$, or in
\[ X^{(0)} x_1^{l_1} + X^{(1)} x_2^{l_2} + \ldots + X^{(n-1)} x_n^{l_n}, \]
viz. $\rho - 1$, since, when this expression is of degree $l$, every power product except $x_1^{l_1-1} x_2^{l_2-1} \ldots x_n^{l_n-1}$ occurs once and only once in it. Hence $N \leq \rho - 1$.

Any particularity in $F_1, F_2, \ldots, F_n$ can only affect the value of $N$ by diminishing it. Hence for the remainder of the proof it will be sufficient to show that $N = \rho - 1$ in a particular example in which $R = 0$. Let
\[ F_1 = (x_1 - x_2) x_1^{l_1-1}, \quad F_2 = (x_2 - x_3) x_2^{l_2-1}, \ldots, \quad F_n = (x_n - x_1) x_n^{l_n-1}. \]
Then $R = 0$ since the equations $F_1 = F_2 = \ldots = F_n = 0$ have the proper solution $x_1 = x_2 = \ldots = x_n = 1$. Let $x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}$ be any power product of degree $l$. If $p_1 \geq l_1$ change $x_1^{p_1} x_2^{p_2}$ to $x_1^{l_1-1} x_2^{q_2}$ where $p_1 + p_2 = l_1 - 1 + q_2$; this is equivalent to changing $x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}$ to $x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n} + A_1 F_1$. Again if $q_2 \geq l_2$ change $x_2^{q_2} x_3^{p_3}$ to $x_2^{l_2-1} x_3^{q_3}$; and if $q_2 < l_2$ proceed to the first $p_r \geq l_r$ and change $x_r^{p_r} x_{r+1}^{p_{r+1}}$ to $x_r^{l_r-1} x_{r+1}^{q_{r+1}}$. If we continue this process, going round the cycle $x_1, x_2, \ldots, x_n$ as many times as is necessary, the power product $x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}$ will eventually become changed to $x_1^{l_1-1} x_2^{l_2-1} \ldots x_n^{l_n-1}$. Hence these two power products are congruent mod $(F_1, F_2, \ldots, F_n)$, while neither of them is a member of $(F_1, F_2, \ldots, F_n)$, since they do not vanish when $x_1 = \ldots = x_n = 1$. Hence $N = \rho - 1$.

Let $F = \sum z_{q_1, q_2, \ldots, q_n} x_1^{q_1} x_2^{q_2} \ldots x_n^{q_n}$ be the general member of $(F_1, F_2, \ldots, F_n)$ of degree $l$; then $x_i F$ is a member of degree $l + 1$ in which the coefficient of $x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}$ is $z_{p_1, p_2, \ldots, p_n}$. Hence
\[ \sum c_{p_1, p_2, \ldots, p_n} z_{p_1, p_2, \ldots, p_n - 1, \ldots, p_n} = 0 \quad (i = 1, 2, \ldots, n), \]
or
\[ \sum c_{q_1, q_2, \ldots, q_i + 1, \ldots, q_n} z_{q_1, q_2, \ldots, q_n} = 0 \quad (i = 1, 2, \ldots, n). \]
These $n$ equations in $z_1, z_2, \ldots, z_n$ are therefore equivalent to one only; and the continued ratio $c_{q_1+1}, c_{q_2}, \ldots, c_{q_n}$ of $c_{q_1}, c_{q_2+1}, \ldots, c_{q_n}$ is the same for all sets of values of $q_1, q_2, \ldots, q_n$ whose sum is $l$. Equating to $a_1: a_2: \ldots: a_n$, it follows that $c_{p_1+p_2+\ldots+p_n} = l + 1$. Hence it follows that $(a_1, a_2, \ldots, a_n)$ is a solution of the equations $F_1 = F_2 = \ldots = F_n = 0$.

11. The Product Theorem. If $F_u$ is the product of two polynomials $F_u', F_u''$, the resultant $R$ of $F_1, F_2, \ldots, F_u$ is the product of the resultants $R', R''$ of $F_1, F_2, \ldots, F_u'$ and $F_1, F_2, \ldots, F_u''$.

For in the general case $R'$ and $R''$ are irreducible, and if either vanishes $R$ vanishes. Hence $R$ is divisible by $R'R''$. Also it can be easily verified that the leading terms of $R$ and $R'R''$ are identical. Hence $R = R'R''$.

This result can easily be extended to the case in which any or all of $F_1, F_2, \ldots, F_u$ resolve into two or more factors.

If $F_1, F_2, \ldots, F_u$ are all members of the module $(F_1', F_2', \ldots, F_u')$ the resultant $R$ of $F_1, F_2, \ldots, F_u$ is divisible by the resultant $R'$ of $F_1', F_2', \ldots, F_u'$. For if $R' = 0$ then $R = 0$.

12. Solution of Equations by means of the Resultant. The method of the resultant for solving equations can only be applied in what is called the principal case, that is, in the case in which the number $r$ of the equations is not greater than the number $n$ of the unknowns, and the resultant $F'_0$ of the equations with respect to $x_1, x_2, \ldots, x_r$ (after a linear substitution of the unknowns) does not vanish identically. When $F'_0$ vanishes identically the method of the resultant fails, but the equations can be solved by the method of the resolvent, due to Kronecker, as explained later. The method of the resolvent is also applicable to any number of equations whether greater or less than the number of unknowns.

Homogeneous Equations. Let the equations be $F_1 = F_2 = \ldots = F_r = 0$ of degrees $l_1, l_2, \ldots, l_r$, where $r \leq n$. We assume that their resultant $F'_0$ with respect to $x_1, x_2, \ldots, x_{r-1}$ does not vanish. We regard $x_1, x_2, \ldots, x_r$ as the unknowns, the solutions being functions of $x_{r+1}, \ldots, x_n$. But instead of solving for one of the unknowns $x_1, x_2, \ldots, x_r$ we solve for a linear combination of them, viz. for $x = u_1x_1 + u_2x_2 + \ldots + u_rx_r$, where $u_1, u_2, \ldots, u_r$ are undetermined quantities. Let $F_u$ stand for $x = u_1x_1 - u_2x_2 - \ldots - u_rx_r$. Then we regard $F_1 = F_2 = \ldots = F_r = F_u = 0$.

* Called the Liouville substitution.
as the given system of equations with \(x_1, \ldots, x_r, x\) as unknowns, and their resultant \(F'_0(u)\) with respect to \(x_1, x_2, \ldots, x_r\) gives the equation \(F'_0(u) = 0\) for \(x\).

**Definition.** \(F'_0(u)\) is called the \(u\)-resultant of \((F_1, F_2, \ldots, F_r)\).

Applying the reasoning of § 9 it is seen that \(F'_0(x)\) is the resultant (with respect to \(x_1, \ldots, x_{r-1}, x_0\)) of \(F_1, F_2, \ldots, F_r\) when \(x_r, x_{r+1}, \ldots, x_n\) are changed to \(x_r x_0, x_{r+1} x_0, \ldots, x_n x_0\), and is a homogeneous polynomial in \(x_r, x_{r+1}, \ldots, x_n\) of degree \(L = l_1 l_2 \ldots l_r\), viz.

\[
F'_0 = R_{r+1} x_L + \ldots,
\]

where \(R_{r+1}\) is the resultant of \((F_1, F_2, \ldots, F_r)_{x_{r+1}=\ldots=x_n=0}\), and does not vanish; for a homogeneous substitution beforehand between \(x_r, x_{r+1}, \ldots, x_n\) only would be carried through to \(F'_0\).

Similarly \(F'_0(u)\) is the resultant (with respect to \(x_1, \ldots, x_r, x_0\)) of \(F_1, F_2, \ldots, F_r, F_u\) when \(x_r, x_{r+1}, \ldots, x_n\) are changed to \(x_r x_0, x_{r+1} x_0, \ldots, x_n x_0\), and is a homogeneous polynomial \(R'_{r+1} x^L + \ldots\) in \(x, x_{r+1}, \ldots, x_n\) where \(R'_{r+1}\) is the resultant of \((F_1, F_2, \ldots, F_r, F_u)_{x_{r+1}=\ldots=x_n=0}\). It is easily seen* that \(R'_{r+1} = R_{r+1}\). Hence

\[
F'_0 = R_{r+1} x_L + \ldots, \quad \text{where } R_{r+1} \neq 0.
\]

To each solution \(x_r = x_{ri}\) of \(F'_0 = 0\) corresponds a solution \((x_{1i}, x_{2i}, \ldots, x_{ri})\) of the equations \(F'_1 = F'_2 = \ldots = F'_r = 0\) for \(x_1, x_2, \ldots, x_r\) (§ 10). There are therefore \(L\) solutions altogether, and they are all finite, since \(R_{r+1} \neq 0\).

Similarly to each of the \(L\) solutions \(x = x_i\) of \(F'_0 = 0\) there corresponds a solution \((x_{1i}, x_{2i}, \ldots, x_{ri}, x_i)\) of \(F'_1 = \ldots = F'_r = F'_u = 0\); and as regards \((x_{1i}, x_{2i}, \ldots, x_{ri})\) the \(L\) solutions must be the same as those obtained by solving \(F'_0 = 0\). Hence it follows that

\[
x_i = u_1 x_{1i} + u_2 x_{2i} + \ldots + u_r x_{ri},
\]

where \(x_{1i}, x_{2i}, \ldots, x_{ri}\) are independent of \(u_1, u_2, \ldots, u_r\). Hence

\[
F'_0 = R_{r+1} \Pi (x - u_1 x_{1i} - \ldots - u_r x_{ri}) \quad (i = 1, 2, \ldots, L).
\]

Thus \(F'_0\) is a product of \(L\) factors which are linear in \(x, u_1, u_2, \ldots, u_r\), and the coefficients of \(u_1, u_2, \ldots, u_r\) in each factor supply a solution of the equations \(F'_1 = F'_2 = \ldots = F'_r = 0\).

Also the number of solutions is either \(L = l_1 l_2 \ldots l_r\) or infinite, the latter being the case when \(F'_0\) vanishes identically.

* By introducing \(a\) as coefficient of \(x\) in \(F_u\) it is seen that \(R'_{r+1}\) is divisible by \(a^L\) by considering weight with respect to \(x\). Also the whole coefficient of \(a^L\) in \(R'_{r+1}\) is \(R_{r+1}\) (§ 8). Hence \(R'_{r+1} = a^L R_{r+1} = R_{r+1}\).
If $D_u$ is the determinant for $(F_1, F_2, \ldots, F_r, F_n)$, regarding $x_1, x_2, \ldots, x_r, x_0$ as the variables, like the $D$ of § 6, we have $D_u = AF_0(u)$. The extraneous factor $A$ depends only on the coefficients of $(F_1, F_2, \ldots, F_r)_{x_r=0}$, that is, of $(F_1, F_2, \ldots, F_r)_{x_{r+1}=\ldots=x_n=0}$. Hence $A$ is a pure constant, independent of $x_{r+1}, \ldots, x_n$ and of $u_1, u_2, \ldots, u_r$, and we may take $D_u = 0$ as the equation for $x$.

**Definition.** The number of times a linear factor $x - u_1x_1 - \ldots - u_rx_r$ is repeated in $F_0(u)$ or $D_u$ is called the *multiplicity* of the solution $(x_1, x_2, \ldots, x_r)$. This term has a definite geometrical interpretation; it is the number of solutions or points, in the general case distinct, which come into coincidence with a particular solution or point in the particular example considered.

In the case of $n$ homogeneous equations in $n$ unknowns such that $R \neq 0$, the complete solution consists of the non-proper solution $(0, 0, \ldots, 0)$ with multiplicity $L = l_1l_2\ldots l_n$.

**Non-homogeneous Equations.** In the case of non-homogeneous equations a linear substitution beforehand affects only $x_1, x_2, \ldots, x_n$ and not the variable $x_0$ of homogeneity. Hence it is possible for $R_{r+1}$ to vanish identically, while $F_0$ and $F_0(u)$ do not, no matter what the original substitution may be. In this case there is a diminution in the number of finite solutions for $x$, but not in the number of linear factors of $F_0(u)$. To a factor $u_1x_1 + u_2x_2 + \ldots + u_rx_r$ of $F_0(u)$ not involving $x$ corresponds what is called an infinite solution of $F_1 = F_2 = \ldots = F_r = 0$ in the ratio $x_1 : x_2 : \ldots : x_r$. Infinite solutions are however nonexistent in the theory of modular systems (§ 42). An extreme case is that in which $F_0(u)$ does not vanish identically, but is independent of $x$, when all the $L$ solutions are at infinity.

It may happen that a system of non-homogeneous equations has only a finite number of finite solutions while the resultant $F_0$ vanishes identically. In such a case the method of the resultant fails to give the solutions.

**Example.** The equations $x_1^2 = x_2 + x_1x_3 = x_3 + x_1x_2 = 0$ have the finite solution $x_1 = x_2 = x_3 = 0$; but the resultant vanishes identically because the corresponding homogeneous equations

$$x_1^2 = x_0x_2 + x_1x_3 = x_0x_3 + x_1x_2 = 0$$

are satisfied by $x_0 = x_1 = 0$, a system of two independent equations only.