39. Conditions of Impossibility and Indetermination.—The preceding theorem enables us to find the conditions under which an equation of several unknown quantities is impossible or indeterminate. Let $f(x, y, z \ldots)$ be the first member supposed to be developed, and $a, b, c \ldots, k$ its coefficients. The necessary and sufficient condition for the equation to be possible is

$$abc \ldots k = 0.$$

For, (1) if f vanishes for some value of the unknowns, its inferior limit $abc \ldots k$ must be zero; (2) if $abc \ldots k$ is zero, f may become equal to it, and therefore may vanish for certain values of the unknowns.

The necessary and sufficient condition for the equation to be indeterminate (identically verified) is

$$a + b + c \ldots + k = 0.$$

For, (1) if $a + b + c + \ldots + k$ is zero, since it is the superior limit of f, this function will always and necessarily be zero; (2) if f is zero for all values of the unknowns, $a + b + c + \ldots + k$ will be zero, for it is one of the values of f.

Summing up, therefore, we have the two equivalences

$$\sum [f(x, y, z, \ldots) = 0] = (abc \ldots k = 0).$$

$$\prod [f(x, y, z \ldots) = 0] = (a + b + c \ldots + k = 0).$$

The equality $abc \ldots k = 0$ is, as we know, the resultant of the elimination of all the unknowns; it is the consequence that can be derived from the equation (assumed to be verified) independently of all the unknowns.

40. Solution of Equations Containing Several Unknown Quantities.—On the other hand, let us see how we can solve an equation with respect to its various unknowns, and, to this end, we shall limit ourselves to the case of two unknowns

$$axy + bxy' + cx'y + dx'y' = 0.$$

First solving with respect to x,

$$x = (a'y + b'y')\,x + (cy + dy')\,x'.$$

The resultant of the elimination of x is

$$acy + bdy' = 0.$$

If the given equation is true, this resultant is true.

Now it is an equation involving y only; solving it,

$$y = (a' + c')\,y + bdy'.$$

Had we eliminated y first and then x, we would have obtained the solution

$$y = (a'x + c'x')\,y + (bx + dx')\,y'$$

and the equation in x

$$abx + cdx' = 0,$$

whence the solution

$$x = (a' + b')\,x + cdx'.$$

We see that the solution of an equation involving two unknown quantities is not symmetrical with respect to these unknowns; according to the order in which they were eliminated, we have the solution

$$x = (a'y + b'y')\,x + (cy + dy')\,x',$$
$$y = (a' + c')\,y + bdy',$$

or the solution

$$x = (a' + b')\,x + cdx,$$
$$y = (a'x + c'x')\,y + (bx + dx')\,y'.$$

If we replace the terms x, y, in the second members by indeterminates u, v, one of the unknowns will depend on only one indeterminate, while the other will depend on two. We shall have a symmetrical solution by combining the two formulas,

$$x = (a' + b')\,u + cdu',$$
$$y = (a' + c')\,v + bdv',$$

but the two indeterminates u and v will no longer be independent of each other. For if we bring these solutions into the given equation, it becomes

$$abcd + ab'c'uv + a'bd'uv' + a'cd'u'v + b'c'du'v' = 0$$

or since, by hypothesis, the resultant $abcd = 0$ is verified,

$$ab'c'uv + a'bd'uv' + a'cdu'v + b'c'du'v' = 0.$$

This is an "equation of condition" which the indeterminates u and v must verify; it can always be verified, since its resultant is identically true,

$$ab'c' . a'bd' . a'cd' . b'c'd = aa' . bb' . cc' . dd' = 0,$$

but it is not verified by any pair of values attributed to u and v.

Some general symmetrical solutions, *i. e.*, symmetrical solutions in which the unknowns are expressed in terms of several independent indeterminates, can however be found. This problem has been treated by SCHRÖDER[1], by WHITEHEAD[2] and by JOHNSON.[3]

This investigation has only a purely technical interest; for, from the practical point of view, we either wish to eliminate one or more unknown quantities (or even all), or else we seek to solve the equation with respect to one particular unknown. In the first case, we develop the first member with respect to the unknowns to be eliminated and equate the product of its coefficients to 0. In the second case we develop with respect to the unknown that is to be extricated and apply the formula for the solution of the equation of one unknown quantity. If it is desired to have the solution in terms of some unknown quantities or in terms of the known only, the other unknowns (or all the unknowns) must first be eliminated before performing the solution.

41. The Problem of Boole.—According to BOOLE the most general problem of the algebra of logic is the following[4]:

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1 *Algebra der Logik*, Vol. I, § 24.

2 *Universal Algebra*, Vol. I, §§ 35—37.

3 "Sur la théorie des égalités logiques", *Bibl. du Cong. intern. de Phil.*, Vol. III, p. 185 (Paris, 1901).

4 *Laws of Thought*, Chap. IX, § 8.