member with respect to these unknown quantities and to equate the product of the coefficients of this development to o. This product will generally contain the other unknown quantities. Thus the resultant of the elimination of z alone, as we have seen, is

$$abxy + cdxy' + fgx'y + hkx'y' = 0$$

and the resultant of the elimination of y and z is

$$abcdx + fghkx' = 0.$$

These partial resultants can be obtained by means of the following practical rule: Form the constituents relating to the unknown quantities to be retained; give each of them, for a coefficient, the product of the coefficients of the constituents of the general development of which it is a factor, and equate the sum to o.

38. Theorem Concerning the Values of a Function:—*All the values which can be assumed by a function of any number of variables $f(x, y, z \dots)$ are given by the formula*

$$abc \dots k + u\,(a + b + c + \dots + k),$$

in which u is absolutely indeterminate, and a, b. $c \dots$, k are the coefficients of the development of f.

Demonstration.—It is sufficient to prove that in the equality

$$f(x, y, z \dots) = abc \dots k + u\,(a + b + c + \dots + k)$$

u can assume all possible values, that is to say, that this equality, considered as an equation in terms of u, is indeterminate.

In the first place, for the sake of greater homogeneity, we may put the second member in the form

$$u'\,abc \dots k + u(a + b + c + \dots + k),$$

for

$$abc \dots k = uabc \dots k + u'\,abc \dots k,$$

and

$$uabc \dots k < u(a + b + c + \dots + k).$$

Reducing the second member to o (assuming there are only three variables x, y, z)

$$(axyz + bxyz' + cxy'z + \ldots + kx'y'z') \times [ua'b'c' \ldots k' + u'(a' + b' + c' + \ldots + k')] + (a'xyz + b'xyz' + c'xy'z + \ldots + k'x'y'z') \times [u(a + b + c + \ldots + k) + u'abc \ldots k] = 0,$$

or more simply

$$u(a+b+c+\ldots+k)(a'xyz+b'xyz'+c'xy'z+\ldots+k'x'y'z') + u'(a'+b'+c'+\ldots+k')(axyz+bxyz'+\ldots+kx'y'z') = 0.$$

If we eliminate all the variables x, y, z, but not the indeterminate u, we get the resultant

$$u(a + b + c + \ldots + k)a'b'c' \ldots k' + u'(a' + b' + c' + \ldots + k')abc \ldots k = 0.$$

Now the two coefficients of u and u' are identically zero; it follows that u is absolutely indeterminate, which was to be proved.[1]

From this theorem follows the very important consequence that a function of any number of variables can be changed into a function of a single variable without diminishing or altering its "variability".

Corollary.—A function of any number of variables can become equal to either of its limits.

For, if this function is expressed in the equivalent form

$$abc \ldots k + u(a + b + c + \ldots + k),$$

it will be equal to its minimum $(abc \ldots k)$ when $u = 0$, and to its maximum $(a + b + c + \ldots + k)$ when $u = 1$.

Moreover we can verify this proposition on the primitive form of the function by giving suitable values to the variables.

Thus a function can assume all values comprised between its two limits, including the limits themselves. Consequently, it is absolutely indeterminate when

$$abc \ldots k = 0 \quad \text{and} \quad a + b + c + \ldots + k = 1$$

at the same time, or

$$abc \ldots k = 0 = a'b'c' \ldots k'.$$

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1 WHITEHEAD, *Universal Algebra*, Vol. I, § 33 (4).