$$b'x = 0, \quad a' + x = 1,$$

or

$$b'x + ax' = 0,$$

that is

$$a < x < b.$$

Now we already have, by hypothesis,

$$b < x < a;$$

so we may infer

$$b = x = a.$$

This is the case in which the value of x is completely determinate.

36. Solution of an Equation Involving One Unknown Quantity.—The solution of the equation

$$ax + bx' = 0$$

may be expressed in the form

$$x = a'u + bu',$$

u being an indeterminate, on condition that the resultant of the equation be verified; for we can prove that this equality implies the equality

$$ab'x + a'bx' = 0,$$

which is equivalent to the double inclusion

$$a'b < x < a' + b.$$

Now, by hypothesis, we have

$$(ab = 0) = (a'b = b) = (a' + b = a').$$

Therefore, in this hypothesis, the proposed solution implies the double inclusion

$$b < x < a';$$

which is equivalent to the given equation.

Remark.—In the same hypothesis in which we have

$$(ab = 0) = (b < a'),$$

we can always put this solution in the simpler but less symmetrical forms

$$x = b + a'u, \quad x = a'(b + u).$$

For

1. We have identically

$$b = bu + bu'.$$

Now

$$(b < a') < (bu < a'u).$$

Therefore

$$(x = bu' + a'u) = (x = b + a'u).$$

2. Let us now demonstrate the formula

$$x = a'b + a'u.$$

Now

$$a'b = b.$$

Therefore

$$x = b + a'u$$

which may be reduced to the preceding form.

Again, we can put the same solution in the form

$$x = a'b + u(ab + a'b'),$$

which follows from the equation put in the form

$$ab'x + a'bx' = 0,$$

if we note that

$$a' + b = ab + a'b + a'b'$$

and that

$$ua'b < a'b.$$

This last form is needlessly complicated, since, by hypothesis,

$$ab = 0.$$

Therefore there remains

$$x = a'b + ua'b'$$

which again is equivalent to

$$x = b + ua',$$

since

$$a'b = b \quad \text{and} \quad a' = a'b + a'b'.$$

Whatever form we give to the solution, the parameter u in it is absolutely indeterminate, *i. e.*, it can receive all possible values, including 0 and 1; for when $u = 0$ we have

$$x = b,$$

and when $u = 1$ we have

$$x = a',$$

and these are the two extreme values of x.

Now we understand that x is determinate in the particular case in which $a' = b$, and that, on the other hand, it is absolutely indeterminate when

$$b = 0, \quad a' = 1, \quad (\text{or } a = 0).$$

Summing up, the formula

$$x = a'u + bu'$$

replaces the "limited" variable x (lying between the limits a' and b) by the "unlimited" variable u which can receive all possible values, including 0 and 1.

Remark.[1]—The formula of solution

$$x = a'x + bx'$$

is indeed equivalent to the given equation, but not so the formula of solution

$$x = a'u + bu'$$

as a function of the indeterminate u. For if we develop the latter we find

$$ab'x + a'bx' + ab(xu + x'u') + a'b'(xu' + x'u) = 0,$$

and if we compare it with the developed equation

$$ab + ab'x + a'bx' = 0,$$

we ascertain that it contains, besides the solution, the equality

$$ab(xu' + x'u) = 0,$$

and lacks of the same solution the equality

$$a'b'(xu' + x'u) = 0.$$

Moreover these two terms disappear if we make

$$u = x$$

and this reduces the formula to

$$x = a'x + bx'.$$

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1 Poretsky. *Sept lois*, Chaps. XXXIII and XXXIV.

From this remark, PORETSKY concluded that, in general, the solution of an equation is neither a consequence nor a cause of the equation. It is a cause of it in the particular case in which

$$ab = 0,$$

and it is a consequence of it in the particular case in which

$$(a'b' = 0) = (a + b = 1).$$

But if ab is not equal to 0, the equation is unsolvable and the formula of solution absurd, which fact explains the preceding paradox. If we have at the same time

$$ab = 0 \quad \text{and} \quad a + b = 1,$$

the solution is both consequence and cause at the same time, that is to say, it is equivalent to the equation. For when $a' = b$ the equation is determinate and has only the one solution

$$x = a' = b.$$

Thus, whenever an equation is solvable, its solution is one of its causes; and, in fact, the problem consists in finding a value of x which will verify it, *i. e.*, which is a cause of it.

To sum up, we have the following equivalence:

$$(ax + bx' = 0) = (ab = 0)\sum_u (x = a'u + bu')$$

which includes the following implications:

$$(ax + bx' = 0) < (ab = 0),$$

$$(ax + bx' = 0) < \sum_u (x = a'u + bu'),$$

$$(ab = 0)\sum_u (x = a'u + bu') < (ax + bx' = 0).$$

37. Elimination of Several Unknown Quantities.— We shall now consider an equation involving several unknown quantities and suppose it reduced to the normal form, *i. e.*, its first member developed with respect to the unknown quantities, and its second member zero. Let us first concern ourselves with the problem of elimination. We can eliminate the unknown quantities either one by one or all at once.