33. Sums and Products of Functions.—It is desirable at this point to introduce a notation borrowed from mathematics, which is very useful in the algebra of logic. Let $f(x)$ be an expression containing one variable; suppose that the class of all the possible values of x is determined; then the class of all the values which the function $f(x)$ can assume in consequence will also be determined. Their sum will be represented by $\sum_x f(x)$ and their product by $\prod_x f(x)$. This is a new notation and not a new notion, for it is merely the idea of sum and product applied to the values of a function.

When the symbols $\sum$ and $\prod$ are applied to propositions, they assume an interesting significance:

$$\prod_x [f(x) = 0]$$

means that $f(x) = 0$ is true for *every* value of x; and

$$\sum_x [f(x) = 0]$$

that $f(x) = 0$ is true for *some* value of x. For, in order that a product may be equal to 1 (*i. e.*, be true), all its factors must be equal to 1 (*i. e.*, be true); but, in order that a sum may be equal to 1 (*i. e.*, be true), it is sufficient that only one of its summands be equal to 1 (*i. e.*, be true). Thus we have a means of expressing universal and particular propositions when they are applied to variables, especially those in the form: "For every value of x such and such a proposition is true", and "For some value of x, such and such a proposition is true", etc.

For instance, the equivalence

$$(a = b) = (ac = bc)\ (a + c = b + c)$$

is somewhat paradoxical because the second member contains a term (c) which does not appear in the first. This equivalence is independent of c, so that we can write it as follows, considering c as a variable x

$$\prod_x [(a = b) = (ax = bx)\ (a + x = b + x)],$$

or, the first member being independent of x,

$$(a = b) = \prod_x [(ax = bx)\ (a + x = b + x)].$$

In general, when a proposition contains a variable term, great care is necessary to distinguish the case in which it is true for *every* value of the variable, from the case in which it is true only for *some* value of the variable.[1] This is the purpose that the symbols $\prod$ and $\sum$ serve.

Thus when we say for instance that the equation

$$ax + bx' = 0$$

is possible, we are stating that it can be verified by some value of x; that is to say,

$$\sum_x (ax + bx' = 0),$$

and, since the necessary and sufficient condition for this is that the resultant $(ab = 0)$ is true, we must write

$$\sum_x (ax + bx' = 0) = (ab = 0),$$

although we have only the implication

$$(ax + bx' = 0) < (ab = 0).$$

On the other hand, the necessary and sufficient condition for the equation to be verified by every value of x is that

$$a + b = 0.$$

Demonstration.—1. The condition is sufficient, for if

$$(a + b = 0) = (a = 0)\ (b = 0),$$

we obviously have

$$ax + bx' = 0$$

whatever the value of x; that is to say,

$$\prod_x (ax + bx' = 0).$$

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[1] This is the same as the distinction made in mathematics between *identities* and *equations*, except that an equation may not be verified by any value of the variable.

2. The condition is necessary, for if

$$\prod_x (ax + bx') = 0,$$

the equation is true, in particular, for the value $x = a$; hence

$$a + b = 0.$$

Therefore the equivalence

$$\prod_x (ax + bx' = 0) = (a + b = 0)$$

is proved.[1] In this instance, the equation reduces to an identity: its first member is "identically" null.

34. The Expression of an Inclusion by Means of an Indeterminate.—The foregoing notation is indispensable in almost every case where variables or indeterminates occur in one member of an equivalence, which are not present in the other. For instance, certain authors predicate the two following equivalences

$$(a < b) = (a = bu) = (a + v = b),$$

in which u, v are two "indeterminates". Now, each of the two equalities has the inclusion $(a < b)$ as its consequence, as we may assure ourselves by eliminating u and v respectively from the following equalities:

1. $[a(b' + u') + a'bu = 0] = [(ab' + a'b)u + au' = 0].$

Resultant:

$$[(ab' + a'b)\, a = 0] = (ab' = 0) = (a < b).$$

2. $[(a + v)b' + a'bv = 0] = [b'v + (ab' + a'b)v' = 0].$

Resultant:

$$[b'(ab' + a'b) = 0] = (ab' = 0) + (a < b).$$

But we cannot say, conversely, that the inclusion implies the two equalities for *any values* of u and v; and, in fact, we restrict ourselves to the proof that this implication holds for some value of u and v, namely for the particular values

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[1] EUGEN MÜLLER, *op. cit.*