Therefore

$$ax + bx' < (a + b)(x + x'),$$

or

$$ax + bx' < a + b.$$

To sum up,

$$ab < ax + bx' < a + b. \qquad \text{Q. E. D.}$$

*Remark* 1. This double inclusion may be expressed in the following form:[1]

$$f(b) < f(x) < f(a).$$

For

$$f(a) = aa + ba' = a + b,$$
$$f(b) = ab + bb' = ab.$$

But this form, pertaining as it does to an equation of one unknown quantity, does not appear susceptible of generalization, whereas the other one does so appear, for it is readily seen that the former demonstration is of general application. Whatever the number of variables $n$ (and consequently the number of constituents $2^n$) it may be demonstrated in exactly the same manner that the function contains the product of its coefficients and is contained in their sum. Hence the theorem is of general application.

*Remark* 2.—This theorem assumes that all the constituents appear in the development, consequently those that are wanting must really be present with the coefficient 0. In this case, the product of all the coefficients is evidently 0. Likewise when one coefficient has the value 1, the sum of all the coefficients is equal to 1.

It will be shown later (§ 38) that a function may reach both its limits, and consequently that they are its extreme values. As yet, however, we know only that it is always comprised between them.

**29. Formula of Poretsky.**[2]—We have the equivalence

$$(x = ax + bx') = (b < x < a).$$

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[1] EUGEN MÜLLER, *Aus der Algebra der Logik*, Art. II.

[2] PORETSKY, "Sur les méthodes pour résoudre les égalités logiques". (*Bull. de la Soc. phys.-math. de Kazan*, Vol. II, 1884).

*Demonstration.*—First multiplying by $x$ both members of the given equality [which is the first member of the entire secondary equality], we have

$$x = ax,$$

which, as we know, is equivalent to the inclusion

$$x < a.$$

Now multiplying both members by $x'$, we have

$$0 = bx',$$

which, as we know, is equivalent to the inclusion

$$b < x.$$

Summing up, we have

$$(x = ax + bx') < (b < x < a).$$

Conversely,

$$(b < x < a) < (x = ax + bx').$$

For

$$(x < a) = (x = ax),$$
$$(b < x) = (bx' = 0).$$

Adding these two equalities member to member [the second members of the two larger equalities],

$$(x = ax)\ (0 = bx) < (x = ax + bx').$$

Therefore

$$(b < x < a) < (x = ax + bx'),$$

and thus the equivalence is proved.

**30. Schröder's Theorem.**[1]—The equality

$$ax + bx' = 0$$

signifies that $x$ lies between $a'$ and $b$.

*Demonstration:*

$$(ax + bx' = 0) = (ax = 0)\ (bx' = 0),$$
$$(ax = 0) = (x < a'),$$
$$(bx' = 0) = (b < x).$$

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[1] SCHRÖDER, *Operationskreis des Logikkalkuls* (1877), Theorem 20.