For if we transform the given inclusions into equalities, we shall have

$$abc + ab'c' = 0, \quad abc + a'bc' = 0, \quad abc + a'b'c = 0,$$

whence, by combining them into a single equality,

$$abc + ab'c' + a'bc' + a'b'c = 0.$$

Now this equality, as we see, is equivalent to any one of the three equalities to be demonstrated.

28. The Limits of a Function.—A term x is said to be *comprised* between two given terms, a and b, when it contains one and is contained in the other; that is to say, if we have, for instance,

$$a < x, \quad x < b,$$

which we may write more briefly as

$$a < x < b.$$

Such a formula is called a *double inclusion*. When the term x is variable and always comprised between two constant terms a and b, these terms are called the *limits* of x. The first (contained in x) is called *inferior limit*; the second (which contains x) is called the *superior limit*.

THEOREM.—*A developed function is comprised between the sum and the product of its coefficients.*

We shall first demonstrate this theorem for a function of one variable,

$$ax + bx'.$$

We have, on the one hand,

$$(ab < a) < (abx < ax),$$
$$(ab < b) < (abx' < bx').$$

Therefore

$$abx + abx' < ax + bx',$$

or

$$ab < ax + bx'.$$

On the other hand,

$$(a < a + b) < [ax < (a + b)x],$$
$$(b < a + b) < [bx' < (a + b)x'].$$

Therefore

$$ax + bx' < (a + b)(x + x'),$$

or

$$ax + bx' < a + b.$$

To sum up,

$$ab < ax + bx' < a + b. \qquad \text{Q. E. D.}$$

Remark 1. This double inclusion may be expressed in the following form:[1]

$$f(b) < f(x) < f(a).$$

For

$$f(a) = aa + ba' = a + b,$$
$$f(b) = ab + bb' = ab.$$

But this form, pertaining as it does to an equation of one unknown quantity, does not appear susceptible of generalization, whereas the other one does so appear, for it is readily seen that the former demonstration is of general application. Whatever the number of variables n (and consequently the number of constituents 2^n) it may be demonstrated in exactly the same manner that the function contains the product of its coefficients and is contained in their sum. Hence the theorem is of general application.

Remark 2.—This theorem assumes that all the constituents appear in the development, consequently those that are wanting must really be present with the coefficient 0. In this case, the product of all the coefficients is evidently 0. Likewise when one coefficient has the value 1, the sum of all the coefficients is equal to 1.

It will be shown later (§ 38) that a function may reach both its limits, and consequently that they are its extreme values. As yet, however, we know only that it is always comprised between them.

29. Formula of Poretsky.[2]—We have the equivalence

$$(x = ax + bx') = (b < x < a).$$

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[1] EUGEN MÜLLER, *Aus der Algebra der Logik*, Art. II.

[2] PORETSKY, "Sur les méthodes pour résoudre les égalités logiques". (*Bull. de la Soc. phys.-math. de Kazan*, Vol. II, 1884).