26. Disjunctive Sums.—By means of development we can transform any sum into a *disjunctive* sum, *i. e.*, one in which each product of its summands taken two by two is zero. For, let $(a + b + c)$ be a sum of which we do not know whether or not the three terms are disjunctive; let us assume that they are not. Developing, we have:

$$a + b + c = abc + abc' + ab'c + ab'c' + a'bc + a'bc' + a'b'c.$$

Now, the first four terms of this development constitute the development of a with respect to b and c; the two following are the development of $a'b$ with respect to c. The above sum, therefore, reduces to

$$a + a'b + a'b'c,$$

and the terms of this sum are disjunctive like those of the preceding, as may be verified. This process is general and, moreover, obvious. To enumerate without repetition all the a's, all the b's, and all the c's, etc., it is clearly sufficient to enumerate all the a's, then all the b's which are not a's, and then all the c's which are neither a's nor b's, and so on.

It will be noted that the expression thus obtained is not symmetrical, since it depends on the order assigned to the original summands. Thus the same sum may be written:

$$b + ab' + a'b'c, \quad c + ac' + a'bc', \ldots .$$

Conversely, in order to simplify the expression of a sum, we may suppress as factors in each of the summands (arranged in any suitable order) the negatives of each preceding summand. Thus, we may find a symmetrical expression for a sum. For instance,

$$a + a'b = b + ab' = a + b.$$

27. Properties of Developed Functions.—The practical utility of the process of development in the algebra of logic lies in the fact that developed functions possess the following property:

The sum or the product of two functions developed with respect to the same letters is obtained simply by finding the sum or the product of their coefficients. The negative of a developed function is obtained simply by replacing the coefficients of its development by their negatives.

We shall now demonstrate these propositions in the case of two variables; this demonstration will of course be of universal application.

Let the developed functions be

$$a_1xy + b_1xy' + c_1x'y + d_1x'y',$$
$$a_2xy + b_2xy' + c_2x'y + d_2x'y'.$$

1. I say that their sum is

$$(a_1 + a_2)xy + (b_1 + b_2)xy' + (c_1 + c_2)x'y + (d_1 + d_2)x'y'.$$

This result is derived directly from the distributive law.

2. I say that their product is

$$a_1a_2xy + b_1b_2xy' + c_1c_2x'y + d_1d_2x'y',$$

for if we find their product according to the general rule (by applying the distributive law), the products of two terms of different constituents will be zero; therefore there will remain only the products of the terms of the same constituent, and, as (by the law of tautology) the product of this constituent multiplied by itself is equal to itself, it is only necessary to obtain the product of the coefficients.

3. Finally, I say that the negative of

$$axy + bxy' + cx'y + dx'y'$$

is

$$a'xy + b'xy' + c'x'y + d'x'y'.$$

In order to verify this statement, it is sufficient to prove that the product of these two functions is zero and that their sum is equal to 1. Thus

$$(axy + bxy' + cx'y + dx'y')\,(a'xy + b'xy' + c'x'y + d'x'y')$$
$$= (aa'xy + bb'xy' + cc'x'y + dd'x'y')$$
$$= (0 \cdot xy + 0 \cdot xy' + 0 \cdot x'y + 0 \cdot x'y') = 0.$$

$$(axy + bxy' + cx'y + dx'y') + (a'xy + b'xy' + c'x'y + d'x'y')$$
$$= [(a + a')xy + (b + b')xy' + (c + c')x'y + (d + d')x'y']$$
$$= (1 \cdot xy + 1 \cdot xy' + 1 \cdot x'y + 1 \cdot x'y') = 1.$$

Special Case.—We have the equalities:

$$(ab + a'b')' = ab' + a'b,$$
$$(ab' + a'b)' = ab + a'b',$$

which may easily be demonstrated in many ways; for instance, by observing that the two sums $(ab + a'b')$ and $(ab' + a'b)$ combined form the development of 1; or again by *performing* the negation $(ab + a'b')'$ by means of DE MORGAN's formulas (§ 25).

From these equalities we can deduce the following equality:

$$(ab' + a'b = 0) = (ab + a'b' = 1),$$

which result might also have been obtained in another way by observing that (§ 18)

$$(a = b) = (ab' + a'b = 0) = [(a + b')(a' + b) = 1],$$

and by performing the multiplication indicated in the last equality.

THEOREM.—*We have the following equivalences:*[1]

$$(a = bc' + b'c) = (b = ac' + a'c) = (c = ab' + a'b).$$

For, reducing the first of these equalities so that its second member will be 0,

$$a(bc + b'c') + a'(bc' + b'c) = 0,$$
$$abc + ab'c' + a'bc' + a'b'c = 0.$$

Now it is clear that the first member of this equality is symmetrical with respect to the three terms *a*, *b*, *c*. We may therefore conclude that, if the two other equalities which differ from the first only in the permutation of these three letters be similarly transformed, the same result will be obtained, which proves the proposed equivalence.

Corollary.—If we have at the same time the three inclusions:

$$a < bc' + b'c, \quad b < ac' + a'c, \quad c < ab' + a'b,$$

we have also the converse inclusions, and therefore the corresponding equalities

$$a = bc' + b'c, \quad b = ac' + a'c, \quad c = ab' + a'b.$$

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1 W. STANLEY JEVONS, *Pure Logic*, 1864, p. 61.

For if we transform the given inclusions into equalities, we shall have

$$abc + ab'c' = 0, \quad abc + a'bc' = 0, \quad abc + a'b'c = 0,$$

whence, by combining them into a single equality,

$$abc + ab'c' + a'bc' + a'b'c = 0.$$

Now this equality, as we see, is equivalent to any one of the three equalities to be demonstrated.

28. The Limits of a Function.—A term x is said to be *comprised* between two given terms, a and b, when it contains one and is contained in the other; that is to say, if we have, for instance,

$$a < x, \quad x < b,$$

which we may write more briefly as

$$a < x < b.$$

Such a formula is called a *double inclusion*. When the term x is variable and always comprised between two constant terms a and b, these terms are called the *limits* of x. The first (contained in x) is called *inferior limit*; the second (which contains x) is called the *superior limit*.

THEOREM.—*A developed function is comprised between the sum and the product of its coefficients.*

We shall first demonstrate this theorem for a function of one variable,

$$ax + bx'.$$

We have, on the one hand,

$$(ab < a) < (abx < ax),$$
$$(ab < b) < (abx' < bx').$$

Therefore

$$abx + abx' < ax + bx',$$

or

$$ab < ax + bx'.$$

On the other hand,

$$(a < a + b) < [ax < (a + b)x],$$
$$(b < a + b) < [bx' < (a + b)x'].$$