

A logical function may be considered as a function of all the terms of discourse, or only of some of them which may be regarded as unknown or variable and which in this case are denoted by the letters x, y, z . We shall represent a function of the variables or unknown quantities, x, y, z , by the symbol $f(x, y, z)$ or by other analogous symbols, as in ordinary algebra. Once for all, a logical function may be considered as a function of any term of the universe of discourse, whether or not the term appears in the explicit expression of the function.

24. The Law of Development.—This being established, we shall proceed to develop a function $f(x)$ with respect to x . Suppose the problem solved, and let

$$ax + bx'$$

be the development sought. By hypothesis we have the equality

$$f(x) = ax + bx'$$

for all possible values of x . Make $x = 1$ and consequently $x' = 0$. We have

$$f(1) = a.$$

Then put $x = 0$ and $x' = 1$; we have

$$f(0) = b.$$

These two equalities determine the coefficients a and b of the development which may then be written as follows:

$$f(x) = f(1)x + f(0)x',$$

in which $f(1)$, $f(0)$ represent the value of the function $f(x)$ when we let $x = 1$ and $x = 0$ respectively.

Corollary.—Multiplying both members of the preceding equalities by x and x' in turn, we have the following pairs of equalities (MACCOLL):

$$\begin{aligned} xf(x) &= ax & x'f(x) &= bx' \\ xf(x) &= xf(1), & x'f(x) &= x'f(0). \end{aligned}$$

Now let a function of two (or more) variables be developed

with respect to the two variables x and y . Developing $f(x, y)$ first with respect to x , we find

$$f(x, y) = f(1, y)x + f(0, y)x'.$$

Then, developing the second member with respect to y , we have

$$f(x, y) = f(1, 1)xy + f(1, 0)xy' + f(0, 1)x'y + f(0, 0)x'y'.$$

This result is symmetrical with respect to the two variables, and therefore independent of the order in which the developments with respect to each of them are performed.

In the same way we can obtain progressively the development of a function of 3, 4,, variables.

The general law of these developments is as follows:

To develop a function with respect to n variables, form all the constituents of these n variables and multiply each of them by the value assumed by the function when each of the simple factors of the corresponding constituent is equated to 1 (which is the same thing as equating to 0 those factors whose negatives appear in the constituent).

When a variable with respect to which the development is made, y for instance, does not appear explicitly in the function ($f(x)$ for instance), we have, according to the general law,

$$f(x) = f(x)y + f(x)y'.$$

In particular, if a is a constant term, independent of the variables with respect to which the development is made, we have for its successive developments,

$$a = ax + ax',$$

$$a = axy + axy' + ax'y + ax'y',$$

$$a = axyz + axyz' + axy'z + axy'z' + ax'yz + ax'y'z + ax'y'z' + ax'y'z'^1$$

and so on. Moreover these formulas may be directly obtained by multiplying by a both members of each development of 1.

Cor. 1. We have the equivalence

$$(a + x')(b + x) = ax + bx' + ab = ax + bx'.$$

¹ These formulas express the method of classification by dichotomy.

For, if we develop with respect to x , we have

$$ax + bx' + abx + abx' = (a + ab)x + (b + ab)x' = ax + bx'.$$

Cor. 2. We have the equivalence

$$ax + bx' + c = (a + c)x + (b + c)x'.$$

For if we develop the term c with respect to x , we find

$$ax + bx' + cx + cx' = (a + c)x + (b + c)x'.$$

Thus, when a function contains terms (whose sum is represented by c) independent of x , we can always reduce it to the developed form $ax + bx'$ by adding c to the coefficients of both x and x' . Therefore we can always consider a function to be reduced to this form.

In practice, we perform the development by multiplying each term which does not contain a certain letter (x for instance) by $(x + x')$ and by developing the product according to the distributive law. Then, when desired, like terms may be reduced to a single term.

25. The Formulas of De Morgan.—*In any development of 1, the sum of a certain number of constituents is the negative of the sum of all the others.*

For, by hypothesis, the sum of these two sums is equal to 1, and their product is equal to 0, since the product of two different constituents is zero.

From this proposition may be deduced the formulas of DE MORGAN:

$$(a + b)' = a'b', \quad (ab)' = a' + b'.$$

Demonstration.—Let us develop the sum $(a + b)$:

$$a + b = ab + ab' + ab + a'b = ab + ab' + a'b.$$

Now the development of 1 with respect to a and b contains the three terms of this development plus a fourth term $a'b'$. This fourth term, therefore, is the negative of the sum of the other three.

We can demonstrate the second formula either by a correlative argument (*i. e.*, considering the development of 0 by factors) or by observing that the development of $(a' + b')$,