21. The Developments of 0 and of 1.—Hitherto we have met only such formulas as directly express customary modes of reasoning and consequently offer direct evidence.

We shall now expound theories and methods which depart from the usual modes of thought and which constitute more particularly the algebra of logic in so far as it is a formal and, so to speak, automatic method of an absolute universality and an infallible certainty, replacing reasoning by calculation.

The fundamental process of this method is *development*. Given the terms a, b, c . . . (to any finite number), we can develop 0 or 1 with respect to these terms (and their negatives) by the following formulas derived from the distributive law:

$$0 = aa',$$
$$0 = aa' + bb' = (a + b)(a + b')(a' + b)(a' + b'),$$
$$0 = aa' + bb' + cc' = (a + b + c)(a + b + c')(a + b' + c)$$
$$\times (a + b' + c')(a' + b + c)$$
$$\times (a' + b + c')(a' + b' + c)(a' + b' + c');$$
$$1 = a + a',$$
$$1 = (a + a')(b + b') = ab + ab' + a'b + a'b',$$
$$1 = (a + a')(b + b')(c + c') = abc + abc' + ab'c + ab'c'$$
$$+ a'bc + a'bc' + a'b'c + a'b'c';$$

and so on. In general, for any number n of simple terms, 0 will be developed in a product containing 2^n factors, and 1 in a sum containing 2^n summands. The factors of zero comprise all possible additive combinations, and the summands of 1 all possible multiplicative combinations of the n given terms and their negatives, each combination comprising n different terms and never containing a term and its negative at the same time.

The summands of the development of 1 are what BOOLE called the *constituents* (of the universe of discourse). We may equally well call them, with PORETSKY,[1] the *minima* of discourse, because they are the smallest classes into which the

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[1] See the Bibliography, page xiv.

universe of discourse is divided with reference to the n given terms. In the same way we shall call the factors of the development of o the *maxima* of discourse, because they are the largest classes that can be determined in the universe of discourse by means of the n given terms.

22. Properties of the Constituents.—The constituents or minima of discourse possess two properties characteristic of contradictory terms (of which they are a generalization); they are *mutually exclusive*, *i. e.*, the product of any two of them is o; and they are *collectively exhaustive*, *i. e.*, the sum of all "exhausts" the universe of discourse. The latter property is evident from the preceding formulas. The other results from the fact that any two constituents differ at least in the "sign" of one of the terms which serve as factors, *i. e.*, one contains this term as a factor and the other the negative of this term. This is enough, as we know, to ensure that their product be null.

The maxima of discourse possess analogous and correlative properties; their combined product is equal to o, as we have seen; and the sum of any two of them is equal to I, inasmuch as they differ in the sign of at least one of the terms which enter into them as summands.

For the sake of simplicity, we shall confine ourselves, with BOOLE and SCHRÖDER, to the study of the constituents or minima of discourse, *i. e.*, the developments of I. We shall leave to the reader the task of finding and demonstrating the corresponding theorems which concern the maxima of discourse or the developments of o.

23. Logical Functions.—We shall call a *logical function* any term whose expression is complex, that is, formed of letters which denote simple terms together with the signs of the three logical operations.[1]

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[1] In this algebra the logical function is analogous to the *integral function* of ordinary algebra, except that it has no powers beyond the first.