On the other hand: 1. Add a' to each of the two members of the inclusion $a < b$; we have

$$(a' + a < a' + b) = (1 < a' + b) = (a' + b = 1).$$

2. We know that

$$b = (a + b)\ (a' + b).$$

Now, if $a' + b = 1$,

$$b = (a + b) \times 1 = a + b.$$

By the preceding formulas, an inclusion can be transformed at will into an equality whose second member is either 0 or 1. Any equality may also be transformed into an equality of this form by means of the following formulas:

$$(a = b) = (ab' + a'b = 0), \quad (a = b) = [(a + b')\ (a' + b) = 1].$$

Demonstration:

$$(a = b) = (a < b)\ (b < a) = (ab' = 0)\ (a'b = 0) = (ab' + a'b = 0),$$

$$(a = b) = (a < b)\ (b < a) = (a' + b = 1)\ (a + b' = 1) = [(a' + b')\ (a' + b) = 1].$$

Again, we have the two formulas

$$(a = b) = [(a + b)\ (a' + b') = 0], \quad (a = b) = (ab + a'b' = 1),$$

which can be deduced from the preceding formulas by performing the indicated multiplications (or the indicated additions) by means of the distributive law.

19. Law of Contraposition.—We are now able to demonstrate the *law of contraposition*,

$$(a < b) = (b' < a').$$

Demonstration.—By the preceding formulas, we have

$$(a < b) = (ab' = 0) = (b' < a').$$

Again, the law of contraposition may take the form

$$(a < b') = (b < a'),$$

which presupposes the law of double negation. It may be expressed verbally as follows: "Two members of an inclusion may be interchanged on condition that both are denied".

C. I.: "If all a is b, then all not-b is not-a, and conversely".

P. I.: "If a implies b, not-b implies not-a and conversely"; in other words, "If a is true b is true", is equivalent to saying, "If b is false, a is false".

This equivalence is the principle of the *reductio ad absurdum* (see hypothetical arguments, *modus tollens*, § 58).

20. Postulate of Existence.—One final axiom may be formulated here, which we will call the *postulate of existence*:

(Ax. IX) $$1 \not< 0,$$

whence may be also deduced $1 \neq 0$.

In the conceptual interpretation (C. I.) this axiom means that the universe of discourse is not null, that is to say, that it contains some elements, at least one. If it contains but one, there are only two classes possible, 1 and 0. But even then they would be distinct, and the preceding axiom would be verified.

In the propositional interpretation (P. I.) this axiom signifies that the true and the false are distinct; in this case, it bears the mark of evidence and of necessity. The contrary proposition, $1 = 0$, is, consequently, the type of *absurdity* (of the formally false proposition) while the propositions $0 = 0$, and $1 = 1$ are types of *identity* (of the formally true proposition). Accordingly we put

$$(1 = 0) = 0, \quad (0 = 0) = (1 = 1) = 1.$$

More generally, every equality of the form

$$x = x$$

is equivalent to one of the identity-types; for, if we reduce this equality so that its second member will be 0 or 1, we find

$$(xx' + xx' = 0) = (0 = 0), \quad (xx + x'x' = 1) = (1 = 1).$$

On the other hand, every equality of the form

$$x = x'$$

is equivalent to the absurdity-type, for we find by the same process,

$$(xx + x'x' = 0) = (1 = 0), \quad (xx' + xx' = 1) = (0 = 1).$$