

P. I.: 1. The simultaneous affirmation of the propositions a and not- a is false; in other words, these two propositions cannot both be true at the same time.

2. The alternative affirmation of the propositions a and not- a is true; in other words, one of these two propositions must be true.

Two propositions are said to be *contradictory* when one is the negative of the other; they cannot both be true or false at the same time. If one is true the other is false; if one is false the other is true.

This is in agreement with the fact that the terms 0 and 1 are the negatives of each other; thus we have

$$0 \times 1 = 0, \quad 0 + 1 = 1.$$

Generally speaking, we say that two terms are *contradictory* when one is the negative of the other.

17. Law of Double Negation.—Moreover this reciprocity is general: if a term b is the negative of the term a , then the term a is the negative of the term b . These two statements are expressed by the same formulas

$$ab = 0, \quad a + b = 1,$$

and, while they unequivocally determine b in terms of a , they likewise determine a in terms of b . This is due to the symmetry of these relations, that is to say, to the commutativity of multiplication and addition. This reciprocity is expressed by the *law of double negation*

$$(a')' = a,$$

which may be formally proved as follows: a' being by hypothesis the negative of a , we have

$$aa' = 0, \quad a + a' = 1.$$

On the other hand, let a'' be the negative of a' ; we have, in the same way,

$$a'a'' = 0, \quad a' + a'' = 1.$$

But, by the preceding lemma, these four equalities involve the equality

$$a = a''.$$

Q. E. D.

This law may be expressed in the following manner:

If $b = a'$, we have $a = b'$, and conversely, by symmetry.

This proposition makes it possible, in calculations, to transpose the negative from one member of an equality to the other.

The law of double negation makes it possible to conclude the equality of two terms from the equality of their negatives (if $a' = b'$ then $a = b$), and therefore to cancel the negation of both members of an equality.

From the characteristic formulas of negation together with the fundamental properties of 0 and 1, it results that every product which contains two contradictory factors is null, and that every sum which contains two contradictory summands is equal to 1.

In particular, we have the following formulas:

$$a = ab + ab', \quad a = (a + b)(a + b'),$$

which may be demonstrated as follows by means of the distributive law:

$$\begin{aligned} a &= a \times 1 = a(b + b') = ab + ab', \\ a &= a + 0 = a + bb' = (a + b)(a + b'). \end{aligned}$$

These formulas indicate the principle of the method of development which we shall explain in detail later (§§ 21 sqq.)

18. Second Formula for Transforming Inclusions into Equalities:—We can now establish two very important equivalences between inclusions and equalities:

$$(a < b) = (ab' = 0), \quad (a < b) = (a' + b = 1).$$

Demonstration.—1. If we multiply the two members of the inclusion $a < b$ by b' we have

$$(ab' < bb') = (ab' < 0) = (ab' = 0).$$

2. Again, we know that

$$a = ab + ab'.$$

Now if $ab' = 0$,

$$a = ab + 0 = ab.$$