and consequently the second formula of the distributive law,

$$(a + c)\ (b + c) = ab + c.$$

For

$$(a + c)\ (b + c) = ab + ac + bc + c,$$

and, by the law of absorption,

$$ac + bc + c = c.$$

This second formula implies the inclusion cited above,

$$(a + c)\ (b + c) < ab + c,$$

which thus is shown to be proved.

Corollary.—We have the equality

$$ab + ac + bc = (a + b)\ (a + c)\ (b + c),$$

for

$$(a + b)\ (a + c)\ (b + c) = (a + bc)\ (b + c) = ab + ac + bc.$$

It will be noted that the two members of this equality differ only in having the signs of multiplication and addition transposed (compare § 14).

13. Definition of 0 and 1.—We shall now define and introduce into the logical calculus two special terms which we shall designate by 0 and by 1, because of some formal analogies that they present with the zero and unity of arithmetic. These two terms are formally defined by the two following principles which affirm or postulate their existence.

(Ax. VI). There is a term 0 such that whatever value may be given to the term x, we have

$$0 < x.$$

(Ax. VII). There is a term 1 such that whatever value may be given to the term x, we have

$$x < 1.$$

It may be shown that each of the terms thus defined is unique; that is to say, if a second term possesses the same property it is equal to (identical with) the first.

The two interpretations of these terms give rise to paradoxes which we shall not stop to elucidate here, but which will be justified by the conclusions of the theory.[1]

C. I.: 0 denotes the class contained in every class; hence it is the "null" or "void" class which contains no element (Nothing or Naught). 1 denotes the class which contains all classes; hence it is the totality of the elements which are contained within it. It is called, after BOOLE, the "universe of discourse" or simply the "whole".

P. I.: 0 denotes the proposition which implies every proposition; it is the "false" or the "absurd", for it implies notably all pairs of contradictory propositions. 1 denotes the proposition which is implied in every proposition; it is the "true", for the false may imply the true whereas the true can imply only the true.

By definition we have the following inclusions

$$0 < 0, \quad 0 < 1, \quad 1 < 1,$$

the first and last of which, moreover, result from the principle of identity. It is important to bear the second in mind.

C. I.: The null class is contained in the *whole*.[2]

P. I.: The false implies the true.

By the definitions of 0 and 1 we have the equivalences

$$(a < 0) = (a = 0), \quad (1 < a) = (a = 1),$$

since we have

$$0 < a, \quad a < 1$$

whatever the value of a.

Consequently the principle of composition gives rise to the two following corollaries:

$$(a = 0)\ (b = 0) = (a + b = 0),$$
$$(a = 1)\ (b = 1) = (ab = 1).$$

Thus we can combine two equalities having 0 for a second

[1] Compare the author's *Manuel de Logistique*, Chap. I., § 8, Paris, 1905 [This work, however, did not appear].

[2] The rendering "Nothing is everything" must be avoided.

member by adding their first members, and two equalities having 1 for a second member by multiplying their first members.

Conversely, to say that a sum is "null" [zero] is to say that each of the summands is null; to say that a product is equal to 1 is to say that each of its factors is equal to 1.

Thus we have

$$(a + b = 0) < (a = 0),$$
$$(ab = 1) < (a = 1),$$

and more generally (by the principle of the syllogism)

$$(a < b)\ (b = 0) < (a = 0),$$
$$(a < b)\ (a = 1) < (b = 1).$$

It will be noted that we can not conclude from these the equalities $ab = 0$ and $a + b = 1$. And indeed in the conceptual interpretation the first equality denotes that the part common to the classes a and b is null; it by no means follows that either one or the other of these classes is null. The second denotes that these two classes combined form the whole; it by no means follows that either one or the other is equal to the whole.

The following formulas comprising the rules for the calculus of 0 and 1, can be demonstrated:

$$a \times 0 = 0, \qquad a + 1 = 1,$$
$$a + 0 = a, \qquad a \times 1 = a.$$

For

$$(0 < a) = (0 = 0 \times a) = (a + 0 = a),$$
$$(a < 1) = (a = a \times 1) = (a + 1 = 1).$$

Accordingly it does not change a term to add 0 to it or to multiply it by 1. We express this fact by saying that 0 is the *modulus* of addition and 1 the *modulus* of multiplication. On the other hand, the product of any term whatever by 0 is 0 and the sum of any term whatever with 1 is 1.

These formulas justify the following interpretation of the two terms:

C. I.: The part common to any class whatever and to the null class is the null class; the sum of any class whatever and of the whole is the whole. The sum of the null class and of any class whatever is equal to the latter; the part common to the whole and any class whatever is equal to the latter.

P. I.: The simultaneous affirmation of any proposition whatever and of a false proposition is equivalent to the latter (i. e., it is false); while their alternative affirmation is equal to the former. The simultaneous affirmation of any proposition whatever and of a true proposition is equivalent to the former; while their alternative affirmation is equivalent to the latter (i. e., it is true).

Remark.—If we accept the four preceding formulas as axioms, because of the proof afforded by the double interpretation, we may deduce from them the paradoxical formulas

$$0 < x, \quad \text{and} \quad x < 1,$$

by means of the equivalences established above,

$$(a = ab) = (a < b) = (a + b = b).$$

14. The Law of Duality.—We have proved that a perfect symmetry exists between the formulas relating to multiplication and those relating to addition. We can pass from one class to the other by interchanging the signs of addition and multiplication, on condition that we also interchange the terms 0 and 1 and reverse the meaning of the sign $<$ (or transpose the two members of an inclusion). This symmetry, or *duality* as it is called, which exists in principles and definitions, must also exist in all the formulas deduced from them as long as no principle or definition is introduced which would overthrow them. Hence a true formula may be deduced from another true formula by transforming it by the principle of duality; that is, by following the rule given above. In its application the *law of duality* makes it possible to replace two demonstrations by one. It is well to note that this law is derived from the definitions of addition and multiplication (the formulas for which are reciprocal by duality)