2. $$(ab = a + b) < (a + b < ab),$$

(Comp.) $$(a + b < ab) = (a < ab)\ (b < ab),$$
$$(a < ab)\ (ab < a) = (a = ab) = (a < b),$$
$$(b < ab)\ (ab < b) = (b = ab) = (b < a).$$

Hence
$$(ab = a + b) < (a < b)\ (b < a) = (a = b).$$

12. **The Distributive Law.**—The principles previously stated make it possible to demonstrate the *converse distributive law*, both of multiplication with respect to addition, and of addition with respect to multiplication,
$$ac + bc < (a + b)c, \qquad ab + c < (a + c)\ (b + c).$$

Demonstration:
$$(a < a + b) < [ac < (a + b)c],$$
$$(b < a + b) < [bc < (a + b)c];$$
whence, by composition,
$$[ac < (a + b)c]\ [bc < (a + b)c] < [ac + bc < (a + b)c].$$

2. $$(ab < a) < (ab + c < a + c),$$
$$(ab < b) < (ab + c < b + c),$$
whence, by composition,
$$(ab + c < a + c)\ (ab + c < b + c) < [ab + c < (a + c)\ (b + c)].$$

But these principles are not sufficient to demonstrate the *direct distributive law*
$$(a + b)\, c < ac + bc, \qquad (a + c)\ (b + c) < ab + c,$$
and we are obliged to postulate one of these formulas or some simpler one from which they can be derived. For greater convenience we shall postulate the formula

(Ax. V). $$(a + b)\, c < ac + bc.$$

This, combined with the converse formula, produces the equality
$$(a + b)\, c = ac + bc,$$
which we shall call briefly the *distributive law*.

From this may be directly deduced the formula
$$(a + b)\ (c + d) = ac + bc + ad + bd,$$

and consequently the second formula of the distributive law,

$$(a+c)\ (b+c) = ab + c.$$

For

$$(a+c)\ (b+c) = ab + ac + bc + c,$$

and, by the law of absorption,

$$ac + bc + c = c.$$

This second formula implies the inclusion cited above,

$$(a+c)\ (b+c) < ab + c,$$

which thus is shown to be proved.

Corollary.—We have the equality

$$ab + ac + bc = (a+b)\ (a+c)\ (b+c),$$

for

$$(a+b)\ (a+c)\ (b+c) = (a+bc)\ (b+c) = ab + ac + bc.$$

It will be noted that the two members of this equality differ only in having the signs of multiplication and addition transposed (compare § 14).

13. Definition of 0 and 1.—We shall now define and introduce into the logical calculus two special terms which we shall designate by 0 and by 1, because of some formal analogies that they present with the zero and unity of arithmetic. These two terms are formally defined by the two following principles which affirm or postulate their existence.

(Ax. VI). There is a term 0 such that whatever value may be given to the term x, we have

$$0 < x.$$

(Ax. VII). There is a term 1 such that whatever value may be given to the term x, we have

$$x < 1.$$

It may be shown that each of the terms thus defined is unique; that is to say, if a second term possesses the same property it is equal to (identical with) the first.