Smooth double subvarieties on singular varieties. II

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Dedicated to Professor H. Hironaka on the occasion of his 80th birthday
and to Professor S. Ishii on the occasion of her 60th birthday

Abstract.

Let $k$ be an algebraically closed field of characteristic 0. We give a brief survey on multiplicity-2 structures on varieties. Let $Z$ be a reduced irreducible nonsingular $(n - 1)$-dimensional variety such that $2Z = X \cap F$, where $X$ is a normal $n$-fold with canonical singularities, $F$ is an $(N - 1)$-fold in $\mathbb{P}^N$, such that $Z \cap \text{Sing}(X) \neq \emptyset$. Assume that $\text{Sing}(X)$ is equidimensional and $\text{codim}_X(\text{Sing}(X)) = 3$. We study the singularities of $X$ through which $Z$ passes. We also consider Fano cones. We discuss the construction of some vector bundles and the resolution property of a variety.

§1. Introduction

Multiplicity-2 structures on nonsingular varieties appear in several instances; for example, when studying nonsingular curves on a Kummer surface in $\mathbb{P}^3$, passing through some of its nodes [3]. In [1, p. 43], W. Barth gave a construction of the Horrocks–Mumford bundle assuming the existence of a nonsingular irreducible curve with certain properties. The Horrocks–Mumford bundle is a stable indecomposable rank 2 vector bundle over $\mathbb{P}^4$. A generic irreducible nonsingular curve of degree 8 and genus 5 on a Kummer surface satisfies all but one of Barth’s conditions [5, Proposition 3.5] to be the variety of jumping lines of the Horrocks–Mumford bundle in $\mathbb{P}^4$. 

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To define a multiplicity-2 structure $\tilde{Y}$ on a codimension 2 nonsingular variety $Y$ is, under some conditions, equivalent to defining a subbundle $L \subset N_{Y|\mathbb{P}^n}$.

Hulek, Okonek and Van de Ven [8] studied multiplicity-2 structures on Castelnuovo and Bordiga surfaces in $\mathbb{P}^4$ as well as on codimension-2 Castelnuovo manifolds. They also studied locally free resolutions on them as well as the stability of the normal bundle on Castelnuovo and Bordiga surfaces. Let $Y$ denote a Castelnuovo surface in $\mathbb{P}^4$ and $\tilde{Y}$ a multiplicity-2 structure on $Y$. Under suitable conditions one can construct a rank 2 vector bundle, $E$, in $\mathbb{P}^4$ with the non-reduced structure $\tilde{Y}$ as the zero-set of a section of $E$, [9].

Vogelaar [17] proved that any local complete intersection subscheme of codimension 2 of a nonsingular variety $F$ can be obtained as the dependency locus of $r - 1$ sections of a rank $r$ vector bundle over $F$ of determinant $L$ if and only if the determinant of its normal bundle twisted with $L^*$ is generated by $r - 1$ global sections, provided the vanishing of the second order cohomology of $L^*$.


We believe that our study of varieties which are complete intersections with a non reduced structure on them could be used in the construction of vector bundles in $\mathbb{P}^n$. These multiplicity-2 structures passing through the singular locus of another variety provide a better understanding of the geometry. They could also be of interest in answering Totaro’s Question: Does every algebraic variety $Y$ have the resolution property, i.e. every coherent sheaf on $Y$ is a quotient of a locally free sheaf of finite rank? [16]. If $Y$ has the resolution property, one could construct a resolution of any coherent sheaf $F$ on $Y$ by vector bundles. The question has an affirmative answer for quasiprojective varieties [10]. The answer is also affirmative for smooth and $\mathbb{Q}$-factorial varieties, since every coherent sheaf has a resolution by sums of line bundles. Payne [12] studied the question for threefolds and observed that, for a complete toric variety $X$, the resolution property implies the existence of nontrivial toric vector bundles. These are vector bundles for the dense torus $T \subset X$ whose underlying vector bundles are nontrivial. In general, there is not known way of constructing a nontrivial toric vector bundle on an arbitrary complete toric variety [12, p. 3].

All varieties are reduced and irreducible unless stated otherwise.

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§2. Curves on Kummer surfaces, Multiplicity-2 structures and
the Horrocks–Mumford bundle

Kummer surfaces appear in many different contexts: they are re­
lated to abelian surfaces and to the quadric line complex. The minimal
desingularization of a Kummer surface is a K3 surface.

**Definition 2.1.** A (16, 6) configuration is a set of 16 planes and 16
points in $\mathbb{P}^3$ such that every plane contains exactly 6 of the 16 points
and every point lies on exactly 6 of the 16 planes.

A (16, 6) configuration is non-degenerate if every two planes share
exactly two points of the configuration and every pair of points is con­
tained in exactly two planes.

**Definition 2.2.** A Kummer surface $S$ in $\mathbb{P}^3$ is a reduced irreducible
quartic surface having 16 nodes, $P_i$, $1 \leq i \leq 16$, and no other singulari­
ties.

**Definition 2.3.** The lines $P_i P_j$, $2 \leq i \leq 16$, are called special lines.
The planes forming irreducible components of the sixteen enveloping
cones of $S$ at the nodes are called special planes. The section of $S$ by
one of the special planes is a non-singular conic, counted twice; we call
this conic a special conic.

**Proposition 2.4.** The union of the 16 enveloping cones at the 16
nodes of $S$ consists of 16 planes. Each plane cuts out a conic on $S con­
taining 6 nodes. Each node lies on exactly 6 of the 16 conics. Together
the nodes of $S$ and the 16 special planes form a non-degenerate (16,6)
configuration. We call this the (16, 6) configuration associated to the
Kummer surface $S$.

**Proof.** [4, Proposition 2.16, Corollary 2.18]. Q.E.D.

Barth's construction [1] relates nonsingular curves of degree 8 and
genus 5 to the variety of jumping lines of a stable rank 2 vector bundle
in $\mathbb{P}^4$ through a fixed point $P \in \mathbb{P}^4$ (the Horrocks–Mumford bundle).
According to Barth's construction of the Horrocks–Mumford bundle, $E$,
[1, p. 43], the nonsingular curve $C$ which would be the variety of jumping
lines of $E$, has to satisfy 5 properties; we can prove that it satisfies the
following four:

- Set-theoretically, $C$ is the complete intersection of a Kummer
  surface $S_1$ and a quartic surface $S_2$ in $\mathbb{P}^3$, since $2C \simeq 4H$, [5,
  (2.91)].
- $C$ is the curve of contact of these surfaces [5, (2.74), (2.93)].
• The exact sequence

\[ 0 \to \omega_C \left( \sum_{i=1}^{16} P_i \right) \to N_C \to O_C(4) \left( - \sum_{i=1}^{16} P_i \right) \to 0 \]

splits. [5, Theorem 3.17].

• \( C \) is linearly normal [5, (3.15)],

but it does not satisfy the required fifth property as we show in the following proposition.

**Proposition 2.5.** Let \( C \) be a generic irreducible nonsingular curve of degree 8 and genus 5 on a Kummer surface \( S \), passing through its 16 nodes \( P_i, 1 \leq i \leq 16 \). If \( L = \omega_C \left( \sum_{i=1}^{16} P_i \right) \) and \( M = O_C(4) \left( - \sum_{i=1}^{16} P_i \right) \), then \( M \not\cong L(-1) \).

**Proof.** [5, (3.18)]. Q.E.D.

**Definition 2.6.** Let \( Y \) be a smooth variety in \( \mathbb{P}^n \), with ideal sheaf \( I_Y \). A non-reduced structure \( \tilde{Y} \) is a *multiplicity-2 structure* on \( Y \) if

(a) the ideal \( I_{\tilde{Y}} \) is such that \( I_{\tilde{Y}} \subset I_Y \),
(b) \( \tilde{Y} \) is locally a complete intersection,
(c) \( \tilde{Y} \) has multiplicity 2, i.e. for each point \( P \in Y \) and a general hyperplane \( H \) through \( P \) the local intersection multiplicity is

\[ i(P; \tilde{Y}, H) = \dim_k O_{P(I(\tilde{Y} \cap H))} = 2. \]

**Lemma 2.7.** To define a multiplicity-2 structure \( \tilde{Y} \) on a codimension 2 nonsingular variety \( Y \) is equivalent to defining a subbundle \( L \subset N_Y|_{\mathbb{P}^n} \), assuming that \( I_Y/I_{\tilde{Y}} \) is locally free.

**Proof.** Generalization of [8, Lemma 2]. Q.E.D.

**Example A.** Let \( X \) be the quadric cone in \( \mathbb{P}^3 \) defined by \( xy - z^2 \).

\( X \) is normal. The line \( L \), defined by \( x = z = 0 \), is a Weil divisor on \( X \) but not a Cartier divisor because it cannot be defined near the origin by one equation (the ideal \( (x, z) \) is not principal in the local ring of \( X \) at the origin). \( 2L \) is a Cartier divisor.

**Definition 2.8.** A codimension 2 variety \( Y \subset \mathbb{P}^{n+2} \) is a *Castelnuovo variety* of dimension \( n \) if \( Y \) has a resolution

\[ 0 \to O_{\mathbb{P}^{n+2}}^2 \to O_{\mathbb{P}^{n+2}}(1) \oplus O_{\mathbb{P}^{n+2}}(b) \to I_Y(b + 2) \to 0, \]

[8, p. 442].

A *Bordiga surface* is a rational surface in \( \mathbb{P}^4 \) of degree 6, [8, p. 445].
**Proposition 2.9.** Let $Y$ be a nonsingular Castelnuovo surface in $\mathbb{P}^4$ of degree $2b + 1$. If $Y$ has a multiplicity-2 structure $\tilde{Y}$ with induced canonical bundle $\omega_{\tilde{Y}}$, the this structure is given by a quotient $N_{Y|\mathbb{P}^n}^* \to \omega_Y(2 - 2b)$. In this case $\tilde{Y}$ is a complete intersection of type $(2, 2b + 1)$. The hyperquadric through $\tilde{Y}$ is unique and is singular along a line $L_0 \subset Y$.

**Proof.** [8, Prop. 12]. Q.E.D.

**Proposition 2.10.** The only Castelnuovo manifold of dimension $n \geq 3$ which admits a multiplicity-2 structure $\tilde{Y}$ such that $\tilde{Y}$ is a complete intersection is $\mathbb{P}^n$ embedded linearly.

**Proof.** [8, Prop. 15]. Q.E.D.

§3. On smooth double subvarieties on singular varieties

**Notation.** Let $X$ be a normal variety. Let $f: V \to X$ be a proper birational morphism where $V$ is a nonsingular variety. Let $D$ be a $\mathbb{Q}$-Gorenstein divisor. The pullback $f^*D$ is the divisor $f^*D = f_*^{-1}D + \sum d_iE_i$, $d_i \in \mathbb{Q}$, satisfying $E_j \cdot (f_*^{-1}D + \sum d_iE_i) = 0$, for all $E_j \in \text{Exc}f$, where $f_*^{-1}D$ is the strict transform of $D$, [11, 4-6-3].

**Definition 3.1.** A normal variety $X$ of dimension $n$ has only canonical singularities (resp. terminal singularities, resp. log terminal singularities, resp. log canonical singularities) if

(a) the canonical divisor $K_X$ is $\mathbb{Q}$-Cartier, that is, there exists $e \in \mathbb{N}$ such that $eK_X$ is a Cartier divisor. The index of the singularity is

$$\text{index}(K_X) = \min\{e \in \mathbb{N} : eK_X \text{is a Cartier divisor}\}.$$

(b) Consider a projective divisorial resolution $f: V \to X$, where $V$ is a nonsingular variety. In the ramification formula

$$K_V = f^*K_X + \sum a_iE_i$$

all the coefficients for the exceptional divisors are nonnegative, that is $a_i \geq 0$, (resp. $a_i > 0$, resp. $a_i > -1$, resp. $a_i \geq -1$) for all $i$.

**Definition 3.2.** (a) Let $(O_{X,P}, M_P)$ be the local ring of a point $P \in X$ of a $k$-scheme. Let $V \subset M_P$ be a finite dimensional $k$-vector space which generates $M_P$ as an ideal of
By a general hyperplane through $P$ we mean the subscheme $H \subseteq U$ defined in a suitable open neighbourhood $U$ of $P$ by the ideal $(v)O_X$, where $v \in V$ is a $k$-point of a certain dense Zariski open set in $V$, [13, (2.5)]. By a general linear variety of codimension $r$ through $P$ we mean the subscheme $L \subseteq U$ defined in a suitable open neighbourhood $U$ of $P$ by the ideal $(v_1, \ldots, v_r)O_X$, where $v_1, \ldots, v_r \in V$ are $k$-points of a certain dense Zariski open set in $V$.

(b) Let $X$ be a singular $n$-fold. We say that a point $Q \in \text{Sing}(X)$ is a general point of $\text{Sing}(X)$ if, for a general hyperplane $H$ such that $Q \in H$ and for some a divisorial resolution $f : V \to X$, the preimage $f^{-1}(Q)$ of $Q$ and the strict transform $f^{-1}_*(X \cap H)$ satisfy that $f^{-1}(Q) \subseteq f^{-1}_*(X \cap H)$.

**Remark B.** Saying that $P \in X$ Cohen–Macaulay and canonical of index 1 is equivalent to saying that $P \in X$ rational Gorenstein, [13, p. 286].

**Definition 3.3.** (a) Let $X$ be a threefold. A point $P \in X$ is called a compound Du Val singularity or a cDV point if, for some hyperplane section $H$ through $P$, $P \in H$ is a Du Val singularity. Equivalently, $P \in X$ is cDV if it is locally analytically isomorphic to the hypersurface singularity given by $f + tg$, where $g \in k[x, y, z, t]$ is arbitrary and $f \in k[x, y, z]$ represents a Du Val singularity, [13, (2.1)].

(b) Let $X$ be an $n$-dimensional normal variety and $P$ a point of $X$. Let $P$ be an $n$-fold isolated singularity (that is, the spectrum of an equicharacteristic local noetherian complete ring of Krull dimension $n$, without zero divisors, whose closed point $P$ is singular). Let $\pi : \tilde{X} \to X$ be the minimal desingularization of $X$ at $P$. The genus of a normal singularity $P$ is defined to be $\dim_k(R^{n-1}(\pi_*O_{\tilde{X}})_P)$. If the genus is 0, the singularity is said to be rational. If the genus is 1, it is elliptic.

**Proposition 3.4.** Let $X$ be an $n$-dimensional variety, $n \geq 2$.

(a) If $P \in X$ is a rational Gorenstein point then, for a general hyperplane section $H$ through $P$, $P \in H$ is elliptic or rational Gorenstein.

(b) If there exists a hyperplane section $H$ through $P$ such that $P \in H$ is a rational Gorenstein then $P \in X$ is a rational Gorenstein. In particular, cDV points are canonical.

**Proof.** [13, (2.6)]. Q.E.D.
Note C (Generalized Reid’s Method). Let $X$ be a normal variety of dimension $n$ in $\mathbb{P}^N$. To study canonical and terminal singularities of the $n$-fold $X$, we reduce by one its dimension by taking a general hyperplane section meeting Sing$(X)$. We use the information on the hyperplane section to analyze the original singularity of $X$, [11, p. 198], [14]. We keep repeating this procedure as follows:

Let $H_0$ be a general hyperplane through Sing$(X)$.

Let $H_{r+1}$, $0 \leq r \leq n - 3$, be a general hyperplane through the singular locus of $X_r = X \cap H_0 \cap \cdots \cap H_r$.

$\dim(X_r) = n - r - 1$.

Let $L_{k+1}$ be a general linear variety of codimension $k + 1$ in $\mathbb{P}^N$, $0 \leq k \leq n - 3$ such that Sing$(X) \cap L_{k+1} \neq \emptyset$. Let $W_k = X \cap L_{k+1}$.

Note that, if $L_{k+1} = H_0 \cap \cdots \cap H_k$, $X_k = W_k$, [7, Note 3.3].

This method of studying singularities by taking hyperplane sections encounter serious problems when studying isolated singularities. Note that, by Proposition 3.4, if $P \in X$ is a rational Gorenstein point then, for a general hyperplane section $H$ through $P$, $P \in H$ is elliptic or rational Gorenstein.

Remark D. Note that to study canonical terminal singularities, log-terminal and log-canonical of the $n$-fold $X$, we could reduce the problem to study $X \cap Y$, where $Y$ is a general nonsingular variety [7, (3.8)].

Proposition 3.5. Let $X$ be a normal singular $n$-fold with only canonical singularities. Let $W_r$ be as in Note C. Assume that

$$\text{codim}_{W_r}(\text{Sing}(W_r)) = 2,$$

for all $r$, $0 \leq r \leq n - 3$. Every point of $X$ has an analytic neighbourhood which is (nonsingular or) isomorphic to $P \times \mathbb{A}^{n-2}$, where $P$ is a Du Val surface singularity.

Proof. [7, (5.2)]. Q.E.D.

Note E. Let $C$ be an irreducible nonsingular curve $2C = V \cap W$, where $V$ and $W$ are two surfaces and $W$ has at most rational double points. Let us suppose that $C$ passes through a rational double point $P$ of $W$. Let $W$ be the minimal desingularization of $W$ at $P$, $\pi : \tilde{W} \to W$. Let $E_k$, $1 \leq k \leq n$, be the irreducible components of the exceptional divisor. The total transform $\pi^*(2C) = \sum_{j=1}^{n} \beta_j E_j + 2E$, where $E$ is the strict transform of $C$, $\beta_j \in \mathbb{N}$.

Proposition 3.6. Let $C$ be an irreducible nonsingular curve $2C = V \cap W$, where $V$ and $W$ are two surfaces and $W$ has only rational double
points as singularities. Assume that $C$ passes through a rational double point $P$ of $W$. $P$ cannot be either of type $A_{2r}$, $r \in \mathbb{N}$, or type $E_6$, or $E_8$. For $C$ to pass only through one singularity of type $A_{2r+1}$, $r \in \mathbb{N}$, we must have $(\sum_{j=1}^{2r+1} \beta_j E_j)^2 = -(2r + 2)$. For $C$ to pass only through one singularity of type $E_7$, we must have $(\sum_{j=1}^{7} \beta_j E_j)^2 = -6$. For $C$ to pass only through one singularity of type $D_n$, $n \geq 4$, we must have that either $(\sum_{j=1}^{n} \beta_j E_j)^2 = -4$, or, for $n = 2k$, $k \in \mathbb{N}$, $k \geq 3$, $(\sum_{j=1}^{n} \beta_j E_j)^2 = -n$.

Proof. [6, Theorem 0.9]. Q.E.D.

**Proposition 3.7.** Let $Z$ be a reduced irreducible nonsingular $(n-1)$-dimensional variety such that $2Z = X \cap Y$, where $X$ is an $n$-fold and $Y$ is an $(N - 1)$-fold in $\mathbb{P}^N$, $X$ normal with canonical singularities and such that $Z \cap \text{Sing}(X) \neq \emptyset$. Let $W_r$ be as in Note C. Assume that $\text{codim}_X(\text{Sing}(W_r)) = 2$, for all $r$, $0 \leq r \leq n - 4$. Then $Z$ has empty intersection with canonical singularities of $X$ which have analytical neighbourhoods isomorphic to $P \times A^{n-2}$, where $P$ is a rational surface singularity of types $A_{2k}$, $k \in \mathbb{N}$, $E_6$ and $E_8$. For $Z$ to have non-empty intersection with canonical singularities of $X$ which have analytical neighbourhoods isomorphic to $P \times A^{n-2}$, where $P$ is a rational surface singularity of type $A_{2k+1}$, $k \in \mathbb{N}$ we must have $(\sum_{j=1}^{2k+1} \beta_j E_j)^2 = -(2k + 2)$, where $E_j$, $1 \leq j \leq 2k + 1$, are the irreducible components of the exceptional divisor supported on $\pi^{-1}(P)$ for $\pi : W_{n-3} \to W_{n-3}$ the minimal resolution of $P \in W_{n-3} \cap Y$. For $P$ to be of type $E_7$, we must have $(\sum_{j=1}^{7} \beta_j E_j)^2 = -6$, where $E_k$, $1 \leq k \leq 7$, are the irreducible components of the exceptional divisor as above. For $P$ to be of type $D_n$, $n \geq 4$, we must have that either $(\sum_{j=1}^{n} \beta_j E_j)^2 = -4$, or, for $n = 2k$, $k \in \mathbb{N}$, $k \geq 3$, $(\sum_{j=1}^{n} \beta_j E_j)^2 = -n$, where $E_k$, $1 \leq k \leq n$, are the irreducible components of the exceptional divisor as above.

Proof. [7, Corollary 7.2]. Q.E.D.

**Proposition 3.8.** Let $Z$ be a reduced irreducible nonsingular $(n-1)$-dimensional variety such that $2Z = X \cap Y$, where $X$ is an $n$-fold and $Y$ is an $(N - 1)$-fold in $\mathbb{P}^N$, $X$ normal with canonical singularities and such that $Z \cap \text{Sing}(X) \neq \emptyset$. Assume that $\text{codim}_X(\text{Sing}(X)) = 3$. Let $W_r$ be as in Note C, for all $r$, $0 \leq r \leq n - 4$. Then, $\text{Sing}(W_{n-4})$ is a union of canonical isolated singularities $P$’s. Let us assume that there exists a hyperplane section $H'$ through $P$ such that $W_{n-4} \cap H'$ is a normal surface with rational double points. Then $Z$ has empty intersection with canonical singularities of $X$ which have analytical neighbourhoods isomorphic to $P \times A^{n-3}$, where $P$ is a rational surface singularity in
Smooth double subvarieties

Sing\((W_{n-4} \cap H')\) of types \(A_{2k}\), \(k \in \mathbb{N}\), \(E_6\) and \(E_8\). For \(Z\) to have non-empty intersection with canonical singularities of \(X\) which have analytical neighbourhoods isomorphic to \(P \times A^{n-3}\), where \(P\) is a rational surface singularity in \(\text{Sing}(W_{n-4} \cap H')\) of type \(A_{2k+1}\), \(k \in \mathbb{N}\), \(E_6\) and \(E_7\).

For \(Z\) to have non-empty intersection with canonical singularities of \(X\) which have analytical neighbourhoods isomorphic to \(P_x A^{n-3}\), where \(P\) is a rational surface singularity in \(\text{Sing}(W_{n-4} \cap H')\) of type \(A_{2k+1}\), \(k \in \mathbb{N}\), we must have \((\sum_{j=1}^{2k+1} \beta_j E_j)^2 = -(2k + 2)\), where \(E_j\), \(1 \leq j \leq 2k + 1\), are the irreducible components of the exceptional divisor supported on \((\pi_{W_{n-4} \cap H'})^{-1}(P)\) for \(\pi_{W_{n-4} \cap H'} : (W_{n-4} \cap H') \to W_{n-4} \cap H'\) the minimal resolution of \(P\), \(P \in W_{n-4} \cap H' \cap Y\), or \(P\) to be of type \(E_7\), we must have \((\sum_{j=1}^{7} \beta_j E_j)^2 = -6\), where \(E_k\), \(1 \leq k \leq 7\), are the irreducible components of the exceptional divisor as above. For \(P\) to be of type \(D_n\), \(n \geq 4\), we must have that either \((\sum_{j=1}^{n} \beta_j E_j)^2 = -4\), or, for \(n = 2k\), \(k \in \mathbb{N}\), \(k \geq 3\), \((\sum_{j=1}^{2k} \beta_j E_j)^2 = -n\), where \(E_k\), \(1 \leq k \leq n\), are the irreducible components of the exceptional divisor as above.

**Proof.** Since \(\dim(W_{n-4}) = 3\), \(\dim(\text{Sing}(W_{n-4})) = 0\). Thus, \(\text{Sing}(W_{n-4})\) is a union of isolated canonical singularities \(P\)'s. We assume that there exists a hyperplane section \(H'\) through \(P\) such that \(W_{n-4} \cap H'\) is a normal surface with rational double points. Given \(2Z = X \cap Y\) we intersect it with \(H_0, H, 0 \leq r \leq n-4\), as follows:

\[2Z \cap H_0 \cap \cdots \cap H_{n-4} \cap H' = Y \cap X \cap H_0 \cap \cdots \cap H_{n-4} \cap H'.\]

We obtain a nonsingular curve \(C\) such that \(2C = Y \cap X \cap H_0 \cap \cdots \cap H_{n-4} \cap H'\) and that \(C \cap \text{Sing}(W_{n-4} \cap H') \neq \emptyset\). We apply Proposition 3.6 to obtain the result.

Q.E.D.

**Definition 3.9.** A *Fano variety* \(X\) is a normal projective variety with log terminal singularities such that the anticanonical divisor \(-K_X\) is an ample \(\mathbb{Q}\)-Cartier divisor. Let \(H \in \text{Pic}(X)\) be a primitive ample divisor class. The *Fano index* \(s = i(X)\) is defined by \(K_X = -sH; s \leq \dim X + 1\).

**Lemma 3.10.** Let \(Y\) be a smooth projectively normal subvariety of \(\mathbb{P}^N\), with hyperplane divisor \(H\) such that \(K_Y\) linearly equivalent to \(qH\), for \(q \in \mathbb{Q}\). Let \(X\) be the cone in \(\mathbb{P}^{N+1}\) over \(Y\). Let \(\tilde{X}\) be the \(\mathbb{P}^1\)-bundle \(\pi : \mathbb{P}(O_Y \oplus O_Y(H)) \to Y\). Let \(Y_0\) be the section corresponding to the quotient \(O_Y(H)\) of \(O_Y \oplus O_Y(H)\), such that \(Y_0|Y \simeq -H\). Let \(f : \tilde{X} \to X\) the contraction of \(Y_0\). We have that

\[K_{\tilde{X}} = f^*K_X + (-1 - q)H.\]

Thus, the singularities of \(X\) are log terminal if and only if \(q < 0\). \(X\) is a *Fano variety* if and only if \(Y\) is a Fano variety.

**Proof.** [2, p. 95].

Q.E.D.
Corollary 3.11. Let $Y$ be a smooth projectively normal subvariety of $\mathbb{P}^N$, with hyperplane divisor $H$ such that $K_Y$ linearly equivalent to $qH$, for $q \in \mathbb{Q}$. Let $X$ be the cone in $\mathbb{P}^{N+1}$ over $Y$. Thus, the singularities of $X$ are terminal (resp. canonical, resp. log canonical) if and only if $q < -1$ (resp. $q \leq -1$, resp. $q \leq 0$).

Proof. Immediate from Lemma 3.10 and Definition 3.1. Q.E.D.

Example F. Let us consider the canonical Fano 4-fold $X$ obtained as follows. Let us embed $\mathbb{P}^1 \times \mathbb{P}^3$ into $\mathbb{P}^{19}$ by the line bundle $H = O(1,2)$. Let $Y$ be a hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^3$. Let $X$ be the projective cone over $Y$. $K_{\mathbb{P}^1 \times \mathbb{P}^3} = -2H$, $K_Y = -H$, $K_X = -2H$. $X$ is a canonical Fano 4-fold, with a canonical singularity at the vertex of the cone. Let $Z$ be a reduced irreducible nonsingular threefold such that $2Z = X \cap Y$, where $X$ is the 4-fold and $Y$ is a hypersurface in $\mathbb{P}^{19}$, $X$ normal with canonical singularities and such that $Z \cap \text{Sing}(X) \neq \emptyset$. We consider a linear variety of dimension 2, $W$, through $P \in Z \cap \text{Sing}(X)$, $W$ sufficiently general. $P' \in W \cap Z \cap \text{Sing}(X)$ is an elliptic surface singularity. Note that the multiplicity of the vertex of the cone is greater than 2.

References


Smooth double subvarieties


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