Approximation of Expectation of Diffusion Process and Mathematical Finance

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§1. Introduction

Let \((\Omega, \mathcal{F}, P)\) be a probability space and let \(\{(B^1(t), \ldots, B^d(t) ; t \in [0, \infty))\}\) be a \(d\)-dimensional Brownian motion. Let \(B^0(t) = t, t \in [0, \infty)\). Let \(V_0, V_1, \ldots, V_d \in C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)\). Here \(C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)\) denotes the space of \(\mathbb{R}^n\)-valued smooth functions defined in \(\mathbb{R}^N\) whose derivatives of any order are bounded. We regard elements in \(C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)\) as vector fields on \(\mathbb{R}^N\).

Now let \(X(t, x), t \in [0, \infty), x \in \mathbb{R}^N\), be the solution to the Stratonovich stochastic integral equation

\[
X(t, x) = x + \sum_{i=0}^{d} \int_0^t V_i(X(s, x)) \circ dB^i(s).
\]

Then there is a unique solution to this equation. Moreover we may assume that with probability one \(X(t, x)\) is continuous in \(t\) and smooth in \(x\).

In many fields, it is important to compute \(E[f(X(T, x))]\) numerically, where \(f\) is a function defined in \(\mathbb{R}^N\). Let \(u(t, x) = E[f(X(t, x))]\), \(t > 0, x \in \mathbb{R}^N\). Then \(u\) satisfies the following PDE:

\[
\begin{align*}
\frac{\partial u}{\partial t}(t, x) &= Lu(t, x), \\
u(0, x) &= f(x).
\end{align*}
\]

Here \(L = \frac{1}{2} \sum_{i=1}^{N} V_i^2 + V_0\). So to compute \(E[f(X(T, x))]\) is the same to compute the solution \(u(T, x)\) to PDE. However, in mathematical finance, if we think of the problem of pricing of Europian options, there are sometimes following difficulties.

(1) \(L\) can be degenerate. Moreover, \(L\) may not satisfy even the Hörmander condition.

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(2) \( f \) may not be continuously differentiable.

Bally and Talay \cite{1} showed that under the Hörmander condition, Euler-Maruyama approximation gives a good approximation, even if the function \( f \) is only bounded measurable. In this paper, we introduce a new method to compute \( E[f(X(T,x))] \) numerically. Our method works when the function \( f \) is Lipschitz continuous. Our main tools are Malliavin calculus and stochastic Taylor approximation based on Lie algebra. Such stochastic Taylor expansion was initiated by Ben Arous \cite{2}, and has been studied by many authors (\cite{3}, \cite{8}, \cite{10}, also see \cite{9}).

§2. Notation and Results

Let \( A = \{\emptyset\} \cup \bigcup_{k=1}^{\infty}\{0,1,\ldots,d\}^k \) and for \( \alpha \in A \), let \( |\alpha| = 0 \) if \( \alpha = \emptyset \), let \( |\alpha| = k \) if \( \alpha = (\alpha^1,\ldots,\alpha^k) \in \{0,1,\ldots,d\}^k \), and let \( ||\alpha|| = |\alpha| + \text{card}\{1 \leq i \leq |\alpha| ; \alpha^i = 0\} \). For \( \alpha, \beta \in A \), we define \( \alpha \ast \beta \in A \) by \( \alpha \ast \beta = (\alpha^1,\ldots,\alpha^k,\beta^1,\ldots,\beta^\ell) \) if \( \alpha = (\alpha^1,\ldots,\alpha^k) \in \{0,1,\ldots,d\}^k \) and \( \beta = (\beta^1,\ldots,\beta^\ell) \in \{0,1,\ldots,d\}^\ell \). Then \( A \) becomes a semigroup with respect the product \( \ast \) with the identity \( \emptyset \).

Let \( A_0 \) and \( A_1 \) denote \( A \setminus \{\emptyset\} \) and \( A \setminus \{\emptyset, (0)\} \), respectively. Also, for each \( m \geq 1 \), \( A(m) \), let \( A_0(m) \) and \( A_1(m) \) denote \( \{\alpha \in A ; ||\alpha|| \leq m\} \), \( \{\alpha \in A_0 ; ||\alpha|| \leq m\} \) and \( \{\alpha \in A_1 ; ||\alpha|| \leq m\} \) respectively.

Let \( B^{\alpha\emptyset}(t) \), \( t \in [0,\infty) \), \( \alpha \in A \), be inductively defined by

\[
B^{\alpha\emptyset} = 1, \quad B^{\alpha(i)} = B^i(t), \quad i = 0,1,\ldots,d
\]

and

\[
B^{\alpha\ast(i)}(t) = \int_0^t B^{\alpha}(s) \circ dB^i(s), \quad i = 0,1,\ldots,d.
\]

We define a vector field \( V_{[\alpha]} \), \( \alpha \in A \), inductively by

\[
V_{[\emptyset]} = 0, \quad V_{[i]} = V_i, \quad i = 0,1,\ldots,d
\]

\[
V_{[\alpha\ast(i)]} = [V_{\alpha}, V_i], \quad i = 0,1,\ldots,d.
\]

Now we assume the following throughout the paper.

(UFG) There is an integer \( \ell \) such that for any \( \alpha \in A_1 \), there are \( \varphi_{\alpha,\beta} \in C_b^{\infty}(\mathbb{R}^N) \), \( \alpha \in A_1 \), \( \beta \in A_1(\ell) \), satisfying the following.

\[
V_{[\alpha]} = \sum_{\beta \in A_1(\ell)} \varphi_{\alpha,\beta} V_{[\beta]}.
\]

Remark.
Approximation of Expectation of Diffusion Process

(1) Let us think of $C^\infty_b(R^N)$-module $M = \sum_{\alpha \in \mathcal{A}_0} C^\infty_b(R^N)V[\alpha]$. Then the assumption (UFG) is equivalent to the assumption that $M$ is finitely generated as a $C^\infty_b(R^N)$-module.

(2) The following condition (UH) (Uniform Hörmander condition) implies the assumption (UFG).

(UH) There are an integer $\ell$ and a constant $c > 0$ such that

$$\sum_{\alpha \in A_1(\ell)} (V[\alpha], \xi)^2 \geq c|\xi|^2, \quad \text{for all } x, \xi \in R^N$$

Let $V_\alpha, \alpha \in A,$ be differential operators given by

$$V_\alpha = \text{Identity, if } \alpha = \emptyset,$$

and

$$V_\alpha = V_{\alpha_1} \cdots V_{\alpha_k}, \quad \text{if } \alpha = (\alpha_1, \ldots, \alpha_k).$$

Let us define a semi-norm $\| \cdot \|_{V,n}, n \geq 1,$ on $C^0_b(R^N; R)$ by

$$\|f\|_{V,n} = \sum_{k=1}^{n} \sum_{\alpha_1, \ldots, \alpha_k \in A_1 \atop \|\alpha_1, \ldots, \alpha_k\| = n} \|V[\alpha_1] \cdots V[\alpha_k]f\|_{\infty}.$$

Now let us define a semigroup of linear operators $\{P_t\}_{t \in [0, \infty)}$ by

$$(P_t f)(x) = E[f(X(t, x))], \quad t \in [0, \infty), \quad f \in C^\infty_b(R^N).$$

Then we can prove the following by using a similar argument in Kusuoka-Stroock [7] (also see [5] for the details).

**Theorem 1.** For any $n, m \geq 0$ and $\alpha_1, \ldots, \alpha_{n+m} \in A_1,$ there is a constant $C > 0$ such that

$$\|V[\alpha_1] \cdots V[\alpha_n] P_t V[\alpha_{n+1}] \cdots V[\alpha_{n+m}] f\|_{\infty} \leq \frac{C}{\ell^{\|\alpha_1, \ldots, \alpha_{n+m}\|/2}} \|f\|_{\infty}, \quad f \in C^\infty_b(R^N).$$

**Corollary 2.** For any $n \geq 0$ and $\alpha_1, \ldots, \alpha_n \in A_1,$ there is a constant $C > 0$ such that

$$\|V[\alpha_1] \cdots V[\alpha_n] P_t f\|_{\infty} \leq \frac{C t^{1/2}}{\ell^{\|\alpha_1, \ldots, \alpha_n\|/2}} \|\nabla f\|_{\infty}, \quad f \in C^\infty_b(R^N).$$
Definition 3. We say that a family of random variables \( \{ Z_\alpha ; \alpha \in \mathcal{A}_0 \} \) is \( m \)-moment similar, \( m \geq 1 \), if \( Z_{(0)} = 1 \),

\[ E[|Z_\alpha|^n] < \infty \quad \text{for any } n \geq 1, \alpha \in \mathcal{A}_0, \]

and if

\[ E[Z_{\alpha_1} \cdots Z_{\alpha_k}] = E[B^{\alpha_1}(1) \cdots B^{\alpha_k}(1)] \]

for any \( k = 1, 2, \ldots, m \) and \( \alpha_1, \ldots, \alpha_k \in \mathcal{A}_0 \) with \( \|\alpha_1\| + \cdots + \|\alpha_k\| \leq m \).

Let \( H : \mathbb{R}^N \to \mathbb{R}^N \) be given by \( H(x) = (x_1, x_2, \ldots, x_N), x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \).

Our main result is the following.

Theorem 4. Let \( m \) be an integer and suppose that a family of random variables \( \{ Z_\alpha ; \alpha \in \mathcal{A}_0 \} \) is \( m \)-moment similar. Let \( Q(s) \) be a Markov operator in \( C_b(\mathbb{R}^N) \)

\[ Q(s)f(x) = E \left[ f \left( \sum_{k=0}^{m} \frac{1}{k!} \sum_{\|\alpha_1\| + \cdots + \|\alpha_k\| \leq m} s(\|\alpha_1\| + \cdots + \|\alpha_k\|)/2 \right. \right. \]

\[ \times \left. \left. (P_{\alpha_1}^0 \cdots P_{\alpha_k}^0)(V_{[\alpha_1]} \cdots V_{[\alpha_k]} H)(x) \right) \right] \]

for \( f \in C_b(\mathbb{R}^N) \) and \( x \in \mathbb{R}^N \). Here

\[ P_{\alpha}^0 = |\alpha|^{-1} \sum_{k=1}^{\|\alpha\|} \frac{(-1)^{k-1}}{k} \sum_{\beta_1 \cdots \beta_k = \alpha} Z_{\beta_1} \cdots Z_{\beta_k}. \]

Then for any \( n \geq 1 \) there is a constant \( C > 0 \) such that

\[ \|P_s f - Q(s)f(x)\|_\infty \leq C \left( \sum_{k=m+1}^{n(m+1)} s^{k/2} \|f\|_{V_{k}} + s^{(m+1)/2} \|\nabla f\|_\infty \right), \]

\( s \in (0, 1], f \in C_b^\infty(\mathbb{R}^N, \mathbb{R}). \)

Let \( T > 0 \) and \( \gamma > 0 \). Let \( t_k = t_k^{(n)} = k^{\gamma} T/n^\gamma \), \( n \geq 1, k = 0, 1, \ldots, n \), and let \( s_k = s_k^{(n)} = t_k - t_{k-1}, k = 1, \ldots, n \). Then we have the following.
**Theorem 5.** Let $m \geq 1$ and $Q(s)$, $s > 0$ be as in Theorem 4. Then we have the following.

For $\gamma \in (0, m - 1)$, there is a constant $C > 0$ such that

$$
\|P_T f - Q(s_n)Q(s_{n-1}) \cdots Q(s_1)f\|_\infty \leq Cn^{-\gamma/2}\|\nabla f\|_\infty,
$$

$$
f \in C_b^\infty(\mathbb{R}^N), \ n \geq 1.
$$

For $\gamma = m - 1$, there is a constant $C > 0$ such that

$$
\|P_T f - Q(s_n)Q(s_{n-1}) \cdots Q(s_1)f\|_\infty \leq Cn^{-(m-1)/2}\log(n + 1)\|\nabla f\|_\infty,
$$

$$
f \in C_b^\infty(\mathbb{R}^N), \ n \geq 1.
$$

For $\gamma > m - 1$, there is a constant $C > 0$ such that

$$
\|P_T f - Q(s_n)Q(s_{n-1}) \cdots Q(s_1)f\|_\infty \leq Cn^{-(m-1)/2}\|\nabla f\|_\infty,
$$

$$
f \in C_b^\infty(\mathbb{R}^N), \ n \geq 1.
$$

§3. Example of 5-moment similar family

Let $\eta_i$, $i = 1, \ldots, d$ and $\eta_{ij}$, $1 \leq i < j \leq d$, are independent random variables such that

$$
P(\eta_i = 0) = \frac{1}{2}, \quad P\left(\eta_i = \pm \sqrt{2 \pm \sqrt{2}}\right) = \frac{1}{8},
$$

and

$$
P(\eta_{ij} = \pm 1) = \frac{1}{2}.
$$

Then we see that

$$
E[\eta_i] = E[\eta_i^3] = 0, \quad E[\eta_i^2] = 1, \quad E[\eta_i^4] = 3,
$$

and

$$
E[\eta_{ij}] = 0, \quad E[\eta_{ij}^2] = 1.
$$

Now let us define random variables $\{Z_\alpha ; \alpha \in A_0\}$ as follows.

(1) The case where $\|\alpha\| = 1$.

$$
Z_i = \eta_i, \quad i = 1, \ldots, d.
$$
(2) The case where $\|\alpha\| = 2$.

$$Z_0 = 1,$$

$$Z_{ij} = \begin{cases} \frac{1}{2}(\eta_i \eta_j + \eta_{ij}), & 1 \leq i < j \leq d, \\ \frac{1}{2}(\eta_i \eta_j - \eta_{ji}), & 1 \leq j < i \leq d, \\ \frac{1}{2} \eta_i \eta_j, & 1 \leq i = j \leq d. \end{cases}$$

(3) The case where $\|\alpha\| = 3$.

$$Z_{i0} = Z_{0i} = \frac{1}{2} \eta_i, \quad Z_{iii} = \frac{1}{6} \eta_i^3, \quad 1 \leq i \leq d,$$

$$Z_{iij} = Z_{jii} = \frac{1}{4} \eta_i, \quad Z_{iji} = 0, \quad 1 \leq i \neq j \leq d,$$

and $Z_\alpha = 0$ in other cases.

(4) The case where $\|\alpha\| = 4$.

$$Z_\alpha = E[B^\alpha],$$

that is

$$Z_{iijj} = \frac{1}{8}, \quad 1 \leq i, j \leq d,$$

$$Z_{0ii} = Z_{i0i} = \frac{1}{4}, \quad 1 \leq i \leq d,$$

$$Z_{00} = \frac{1}{2},$$

and $Z_\alpha = 0$ in the other case.

(5) The case where $\|\alpha\| \geq 5$.

$$Z_\alpha = 0.$$

Then the family of random variables $\{Z_\alpha ; \alpha \in A_0\}$ is 5-moment similar.

§4. Preparation from Algebra

We say that a polynomial $p$ of $x_\alpha, \alpha \in A_0$, is $m$-homogeneous, $m \geq 0$, if

$$p(\varepsilon^{\|\alpha\|} x_\alpha, \alpha \in A_0) = \varepsilon^m p(x_\alpha, \alpha \in A_0), \quad \varepsilon > 0.$$

Let $\mathcal{U}$ be the free algebra generated by $\{v_0, v_1, \ldots, v_d\}$ over $\mathbb{R}$. Then the algebra $\mathcal{U}$ can be extended to the algebra $\overline{\mathcal{U}}$ of formal power series
in \( \{v_0, v_1, \ldots, v_d\} \). We define \( v^\alpha \in \mathcal{U}, \alpha \in \mathcal{A} \), by \( v^0 = 1 \), and by \( v^\alpha = v^{\alpha_1} \cdots v^{\alpha_k} \), if \( \alpha = (\alpha^1, \ldots, \alpha^k) \). Then \( \mathcal{U} \) is the complete direct sum of the space \( Rv^\alpha, \alpha \in \mathcal{A} \). We define convergence in \( \mathcal{U} \) by \( \sum_{\alpha \in \mathcal{A}} a_{\alpha,n}v^\alpha \to \sum_{\alpha \in \mathcal{A}} a_{\alpha}v^\alpha, n \to \infty \), if \( a_{\alpha,n} \to a_{\alpha} \) for any \( \alpha \in \mathcal{A} \).

For \( x, y \in \mathcal{U} \), let \([xy] = xy - yx\). For \( \alpha \in \mathcal{A} \), let \( v^{[\alpha]} \in \mathcal{U} \) denote 0, if \( \alpha = 0 \), \( v_i \), if \( \alpha = i \in \{0, 1, \ldots, d\} \), and \([\cdots([v_{\alpha_1}v_{\alpha_2}]v_{\alpha_3}]\cdots, v_{\alpha_k}] \), if \( \alpha = (\alpha^1, \ldots, \alpha^k) \) and \( k \geq 2 \). Let \( \mathcal{U}^C \) be the closure of \( \sum_{\alpha \in \mathcal{A}} Rv^{[\alpha]} \) in \( \mathcal{U} \). Then \( \mathcal{U}^C \) is closed under Lie product \([ \ ]\) (see Jacobson [4, p.168]).

We use the following two theorems (see Jacobson [J, pp.167–174]).

**Theorem 6** (Friedrichs). Let \( \delta \) be a continuous homomorphism from \( \tilde{\mathcal{U}} \) into \( \tilde{\mathcal{U}} \otimes \tilde{\mathcal{U}} \) determined by \( \delta(1) = 1 \otimes 1 \) and \( \delta(v_i) = v_i \otimes 1 + 1 \otimes v_i \), \( i = 0, 1, \ldots, d \). Then for \( x \in \tilde{\mathcal{U}} \), \( x \in \mathcal{U}^C \) if and only if \( \delta(x) = x \otimes 1 + 1 \otimes x \).

**Theorem 7.** Let \( \sigma \) be a linear continuous operator from \( \tilde{\mathcal{U}} \) into \( \mathcal{U}^C \) given by \( \sigma(v^\alpha) = |\alpha|^{-1}v^{[\alpha]} \), \( \alpha \in \mathcal{A} \). Then the restriction of \( \sigma \) on \( \mathcal{U}^C \) is identity.

Let \( \mathcal{B}_{\tilde{\mathcal{U}}} \) be a Borel algebra over \( \tilde{\mathcal{U}} \). Let \( (\Omega, \mathcal{F}, P) \) be a complete probability space. One can define \( \tilde{\mathcal{U}} \)-valued random variables and their expectaions etc. naturally. Let \( \{\mathcal{F}_t\}_{t \in [0, \infty)} \) be a filtration satisfying a usual hypothesis, \( (B^1(t), \ldots, B^d(t)), t \in [0, \infty) \), be a \( d \)-dimensional \( \{\mathcal{F}_t\}_{t \in [0, \infty)} \)-Brownian motion, and \( B^0(t) = t, t \in [0, \infty) \). We say that \( X(t) \) is a \( \tilde{\mathcal{U}} \)-valued continuous semimartingale, if there are continuous semimartingales \( X_\alpha, \alpha \in \mathcal{A} \), such that \( X(t) = \sum_{\alpha \in \mathcal{A}} X_\alpha(t)v^\alpha \). For \( \tilde{\mathcal{U}} \)-valued continuous semimartingale \( X(t), Y(t) \), we can define \( \tilde{\mathcal{U}} \)-valued continuous semimartingales \( \int_0^t X(s) \circ dY(s) \) and \( \int_0^t \circ dX(s)Y(s) \) by

\[
\begin{align*}
\int_0^t X(s) \circ dY(s) & = \sum_{\alpha, \beta \in \mathcal{A}} \left( \int_0^t X_\alpha(s) \circ dY_\beta(s) \right)v^\alpha v^\beta, \\
\int_0^t \circ dX(s)Y(s) & = \sum_{\alpha, \beta \in \mathcal{A}} \left( \int_0^t Y_\beta(s) \circ dX_\alpha(s) \right)v^\alpha v^\beta,
\end{align*}
\]

where

\[
X(t) = \sum_{\alpha \in \mathcal{A}} X_\alpha(t)v^\alpha, \quad Y(t) = \sum_{\beta \in \mathcal{A}} Y_\beta(t)v^\beta.
\]

Then we have

\[
X(t)Y(t) = X(0)Y(0) + \int_0^t X(s) \circ dY(s) + \int_0^t \circ dX(s)Y(s).
\]
Since \( \mathbf{R} \) is regarded a vector subspace in \( \overline{\mathbf{U}} \), we can define \( \int_0^t X(s) \circ dB^i(s) \), \( i = 0, 1, \ldots, d \), naturally. We can similarly think of \( \overline{\mathbf{U}} \otimes \overline{\mathbf{U}} \)-valued semimartingales and stochastic calculus for them.

Now let us consider SDE on \( \overline{\mathbf{U}} \)

\[
X(t) = 1 + \sum_{i=0}^d \int_0^t X(s) v_i \circ dB^i(s), \quad t \geq 0.
\]

One can easily solve this SDE and obtain

\[
X(t) = 1 + \sum_{\alpha \in A_0} B^{\alpha}(t) v^\alpha.
\]

We also have the following.

**Proposition 8.** Let \( p_{\alpha}^0, \alpha \in A_0 \), be \( \|\alpha\| \)-homogeneous polynomials in \( x_\beta, \beta \in A_0 \), given by

\[
p_{\alpha}^0(x_\beta, \beta \in A_0) = |\alpha|^{-1} \sum_{k=1}^{\|\alpha\|} \frac{(-1)^{k-1}}{k} \sum_{\beta_1, \ldots, \beta_k \in A_0, \beta_1 \cdots \beta_k = \alpha} x_{\beta_1} \cdots x_{\beta_k}.
\]

Then

\[
\log X(t) = \sum_{\alpha \in A_0} p_{\alpha}^0(B^{\alpha}(t), \beta \in A_0) v^\alpha.
\]

In other words,

\[
X(t) = 1 + \sum_{\alpha \in A_0} B^{\alpha}(t) v^\alpha = \exp \left( \sum_{\alpha \in A_0} p_{\alpha}^0(B^{\alpha}(t), \beta \in A_0) v^\alpha \right).
\]

**Proof.** Note that

\[
\delta(X(t)) = 1 \otimes 1 + \sum_{i=0}^d \int_0^t \delta(X(s))(v_i \otimes 1 + 1 \otimes v_i) \circ dB^i(s),
\]

and

\[
X(t) \otimes X(t) = 1 \otimes 1 + \int_0^t \circ d(X(s) \otimes 1)(1 \otimes X(s))
\]

\[
+ \int_0^t (X(s) \otimes 1) \circ d(1 \otimes X(s))
\]

\[
= 1 \otimes 1 + \sum_{i=0}^d \int_0^t X(s) \otimes X(s)(v_i \otimes 1 + 1 \otimes v_i) \circ dB^i(s).
\]
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Since one can easily see the uniqueness of such SDE on \( \bar{U} \otimes \bar{U} \), we have
\[
\delta(X(t)) = X(t) \otimes X(t).
\]
For any \( u \in \bar{U} \) with of the form \( u = \sum_{\alpha \in A_0} a_{\alpha} v^\alpha \), we have
\[
\exp(u) \otimes \exp(u) = \exp(u \otimes 1 + 1 \otimes u),
\]
which implies
\[
\log((1 + u) \otimes (1 + u)) = \log(1 + u) \otimes 1 + 1 \otimes \log(1 + u).
\]
So we have
\[
\delta(\log X(t)) = \log(\delta X(t)) = \log X(t) \otimes 1 + 1 \otimes \log X(t).
\]
So by Theorem 6 we see that \( \log X(t) \in \bar{U}^C \) \( P \)-a.s. On the other hand,
\[
\log X(t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left( \sum_{\alpha_1, \ldots, \alpha_k \in A_0} B^\alpha(t) \cdots B^{\alpha_k}(t) v^{\alpha_1 \cdots \alpha_k} \right).
\]
So acting the linear operator \( \sigma \) in Theorem 7, we have our assertion.

Q.E.D.

**Proposition 9.** There are polynomials \( q_0^\alpha, \alpha \in A_0 \), in \( x_\beta, \beta \in A_0 \), such that
\[
\log \left( \exp(-x_0 v_0) \exp \left( \sum_{\alpha \in A_0} x_\alpha v^{[\alpha]} \right) \right) = \sum_{\alpha \in A_1} q_0^\alpha (x_\beta, \beta \in A_0) v^{[\alpha]}
\]
for any \( x_\beta \in \mathbb{R}, \beta \in A_0 \). Moreover, \( q_0^0 = 0 \) and \( q_0^\alpha \) is \( \| \alpha \| \)-homogeneous for each \( \alpha \in A_1 \).

**Proof.** Similarly to the proof of Proposition 8, we see that
\[
\log \left( \exp(-x_0 v_0) \exp \left( \sum_{\alpha \in A_0} x_\alpha v^{[\alpha]} \right) \right) \in \bar{U}^C. \]
Since we have
\[
\exp(-x_0 v_0) \exp \left( \sum_{\alpha \in A_0} x_\alpha v^{[\alpha]} \right) = 1 + \sum_{\alpha \in A_1} x_\alpha v^{[\alpha]}
\]
\[
+ \sum_{\ell+k \geq 2} \sum_{\alpha_1, \ldots, \alpha_k \in A_0} \frac{1}{\ell! k!} (-x_0)^\ell x_{\alpha_1} \cdots x_{\alpha_k} v_0^\ell v^{[\alpha_1]} \cdots v^{[\alpha_k]}.
\]
Note that \( v_0^\ell v^{[\alpha_1]} \cdots v^{[\alpha_k]} \in U_{2\ell + \| \alpha_1 \| \cdots \| \alpha_k \|}. \) So acting the linear operator \( \sigma \) in Theorem 7 again, we have our assertion.

Q.E.D.
§5. Basic Estimates

For \( n \geq 0 \) let \( \varphi_n \) denote a map from \( \overline{U} \) into the space of differential operators in \( \mathbb{R}^N \) of order \( n \) given by

\[
\varphi_n \left( \sum_{\alpha \in \mathcal{A}} a_\alpha v^\alpha \right) = \sum_{\alpha \in \mathcal{A}(n)} a_\alpha V_\alpha, \quad a_\alpha \in \mathbb{R}, \alpha \in \mathcal{A}.
\]

Note that if \( u \in \overline{U}^L \), then \( \varphi_n(u) \) is a vector field.

First we observe the following.

**Proposition 10.** For any \( U \in C^\infty_b(\mathbb{R}^N; \mathbb{R}^N) \),

\[
\left\| f(\exp(U)(\cdot)) - \sum_{k=0}^{n} \frac{1}{k!} U^k f \right\|_\infty \leq \frac{1}{(n+1)!} \| U^{n+1} f \|_\infty
\]

for any \( f \in C^\infty_b(\mathbb{R}^N) \) and \( n \geq 1 \).

**Proof.** One can prove the following inductively.

\[
f(\exp(tU)(x)) = \sum_{k=0}^{n} \frac{t^k}{k!} (U^k f)(x) + \int_0^t \frac{(t-s)^n}{n!} (U^{n+1} f)(\exp(sU)(x)) \, ds.
\]

Then we have our assertion. Q.E.D.

As corollaries of the above Proposition, we have the following.

**Proposition 11.** For any \( u = \sum_{\alpha \in \mathcal{A}_1} a_\alpha v^{[\alpha]} \in \overline{U}^L \), and \( n \geq 1 \) we have

\[
\left\| f(\exp(\varphi_n(u))(\cdot)) - (\varphi_n(\exp(u)) f)(\cdot) \right\|_\infty \leq \sum_{k=n+1}^{n(n+1)} \max\{ |a_\alpha|^{1/\|\alpha\|} ; \alpha \in \mathcal{A}_1(n) \}^k \| f \|_{V,k}
\]

for any \( f \in C^\infty_b(\mathbb{R}^N) \).

**Proposition 12.** For any \( u = \sum_{\alpha \in \mathcal{A}_0} a_\alpha v^{[\alpha]} \in \overline{U}^L \), and \( n \geq 1 \) we have a constant \( C \) depending only on \( d \) and \( n \) such that

\[
\left\| f(\exp(\varphi_n(u))(\cdot)) - (\varphi_n(\exp(u)) f)(\cdot) \right\|_\infty \leq C \sum_{k=n+1}^{n(n+1)} \max\{ |a_\alpha|^{1/\|\alpha\|} ; \alpha \in \mathcal{A}_0(n) \}^k \sum_{\alpha \in \mathcal{A} ; \|\alpha\| = k} \| V_\alpha f \|_\infty
\]
for any \( f \in C_b^\infty(\mathbb{R}^N) \).

Also, we have the following.

**Proposition 13.** For any \( u^{(i)} = \sum_{\alpha \in \mathcal{A}_0} a^{(i)}_\alpha u^{[\alpha]} \in \mathcal{U}^c \), \( i = 1, 2 \), and \( n \geq 1 \), we have a constant \( C \) depending only on \( d \) and \( n \) such that

\[
\|f(\exp(\varphi_n(u^{(1)})) (\exp(\varphi_n(u^{(2)}))(\cdot))) - (\varphi_n(\exp(u^{(2)}) \exp(u^{(1)})) f)(\cdot)\|_\infty \\
\leq C \sum_{k=n+1}^{2n(n+1)} \max \{ ||\alpha||^{1/||\alpha||} ; \alpha \in \mathcal{A}_0(2n), i = 1, 2 \}^k \sum_{\alpha \in \mathcal{A}, ||\alpha||=k} ||V_\alpha f||_\infty
\]

for any \( f \in C_b^\infty(\mathbb{R}^N) \).

**Proof.** Note that

\[
f(\exp(\varphi_n(u^{(1)}))(\exp(\varphi_n(u^{(2)}))(x))) - (\varphi_n(\exp(u^{(2)}) \exp(u^{(1)})) f)(x) \\
= f(\exp(\varphi_n(u^{(1)}))(\exp(\varphi_n(u^{(2)}))(x))) \\
- (\varphi_n(\exp(u^{(1)})) f)(\exp(\varphi_n(u^{(2)}))(x))) \\
+ (\varphi_n(\exp(u^{(1)})) f)(\exp(\varphi_n(u^{(2)}))(x))) \\
- \varphi_n(\exp(u^{(2)}))(\varphi_n(\exp(u^{(1)})) f)(x)) \\
+ \varphi_n(\exp(u^{(2)}))(\varphi_n(\exp(u^{(1)})) f)(x)) \\
- (\varphi_n(\exp(u^{(2)}) \exp(u^{(1)})) f)(x).\]

Then we have our assertion from previous two propositions. Q.E.D.

§6. Moment Equivalent Families

Let \((\Omega, \mathcal{F}, P)\) be a probability space.

**Definition 14.** We say that families of random variables \( \{Z_\alpha ; \alpha \in \mathcal{A}_0\} \) and \( \{Z'_\alpha ; \alpha \in \mathcal{A}_0\} \) are \( m \)-moment equivalent, \( m \geq 1 \), if

\[ E[|Z_\alpha|^n] < \infty, \quad E[|Z'_\alpha|^n] < \infty, \quad \text{for any } n \geq 1 \text{ and } \alpha \in \mathcal{A}_0, \]

and

\[ E[Z_{\alpha_1} \cdots Z_{\alpha_k}] = E[Z'_{\alpha_1} \cdots Z'_{\alpha_k}] \]

for any \( k = 1, 2, \ldots, m \) and \( \alpha_1, \ldots, \alpha_k \in \mathcal{A}_0 \) with \( ||\alpha_1|| + \cdots + ||\alpha_k|| \leq m \).

The main result in this section is the following.
Theorem 15. Let \( m \geq 1 \). Let \( \{ Z_{\alpha}^{(1)} ; \alpha \in A_0 \} \) and \( \{ Z_{\alpha}^{(2)} ; \alpha \in A_0 \} \) are \( m \)-moment equivalent families of random variables such that \( Z_{(0)}^{(1)} = Z_{(0)}^{(2)} = 1 \). Let \( Z^{(i)}(\varepsilon), \varepsilon > 0, \) be a \( U_{\varepsilon}^{C} \)-valued random variable given by \( Z^{(i)}(\varepsilon) = \sum_{\alpha \in A_0} \varepsilon^{\|\alpha\|} Z_{\alpha}^{(i)} v[\alpha] \).

Then for any \( n \geq 1 \), there is a constant \( C > 0 \) depending only on \( n \) and moments of \( Z_{\alpha}^{(i)}, i = 1, 2, \alpha \in A_0(n) \), such that

\[
\sup_{x \in \mathbb{R}^N} \left| E[f(\exp(\varphi_n(Z^{(1)}(\varepsilon)))(x))] - E[f(\exp(\varphi_n(Z^{(2)}(\varepsilon)))(x))] \right|
\leq C \left( \sum_{k=m+1}^{n(m+1)} \varepsilon^k \|f\|_{V,k} + \varepsilon^{n+1} \|\nabla f\|_\infty \right), \quad \varepsilon \in (0, 1], f \in C^\infty_b(\mathbb{R}^N; \mathbb{R}).
\]

To prove this theorem we need some preparations.

First we have the following combining Propositions 12 and 13.

Proposition 16. Let \( \{ Z_\alpha ; \alpha \in A_0 \} \) is a family of random variables such that \( Z_0 = 1 \). Let \( Z(\varepsilon) = \sum_{\alpha \in A_0} \varepsilon^{\|\alpha\|} Z_{\alpha}^{(i)} v[\alpha] \). Then for any \( n \geq 1 \) and \( p \in [1, \infty) \), there is a constant \( C > 0 \) depending only on \( n, p, \) and moments of \( Z_\alpha, \alpha \in A_0(n) \), such that

\[
\sup_{x \in \mathbb{R}^N} E \left[ \left| f(\exp(\varphi_n(Z(\varepsilon)))(\exp(-\varepsilon^2 V_0)(x))) - f\left( \exp\left( \varphi_n\left( \sum_{\alpha \in A_0} \varepsilon^{\|\alpha\|} q_\alpha^0(Z_\beta, \beta \in A_0) v[\alpha] \right) \right)(x) \right) \right|^p \right]^{1/p}
\leq C \sum_{\alpha \in A_0} \varepsilon^{\|\alpha\|} \|V_\alpha f\|_\infty, \quad \varepsilon \in (0, 1], f \in C^\infty_b(\mathbb{R}^N; \mathbb{R}).
\]

Here polynomials \( q_\alpha^0, \alpha \in A_1 \), are as in Proposition 9.

As a corollary we have the following.

Corollary 17. Let us assume the same as the previous proposition. Then for any \( n \geq 1 \) and \( p \in [1, \infty) \), there is a constant \( C > 0 \)
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depending only on n, p and moments of $Z_\alpha$, $\alpha \in A_0(n)$, such that

\[
\sup_{x \in \mathbb{R}^N} E \left[ \left| \exp(\phi_n(Z(\varepsilon)))(\exp(-\varepsilon^2 V_0)(x)) 
- \exp \left( \phi_n \left( \sum_{\alpha \in A_0} \varepsilon \| \alpha \| q_0^0(Z_\beta, \beta \in A_0) v[\alpha] \right) \right) \right|^p \right]^{1/p} 
\leq C \varepsilon^{n+1} \sum_{\alpha \in A_0, n+1 \leq \| \alpha \| \leq 2n(n+1)} \| V_\alpha H \|_{\infty}, \quad \varepsilon \in (0, 1].
\]

Proof. Let $\psi \in C_b^\infty(\mathbb{R}; \mathbb{R})$ such that $\psi(t) = t$, $|t| < 1$, and $0 \leq \psi'(t) \leq 1$, $t \in \mathbb{R}$. Let $f_{\ell,j} \in C_b^\infty(\mathbb{R}^N; \mathbb{R})$, $\ell \geq 1$, $j = 1, \ldots, N$, be given by $f_{\ell,j}(x) = \ell \psi(\ell^{-1} x_j)$. Then we see that

\[
\sup_{t \geq 1, j = 1, \ldots, N} \| \nabla f_{\ell,j} \|_{\infty} < \infty,
\]

and

\[
\max_{j = 1, \ldots, N} \| \nabla^k f_{\ell,j} \|_{\infty} \to 0, \quad \ell \to \infty, \quad k \geq 2.
\]

So we see that

\[
\sup_{t \geq 1, j = 1, \ldots, N} \| V_\alpha f_{\ell,j} \|_{\infty} < \infty, \quad \alpha \in A_0.
\]

Therefore applying the previous proposition for $f_{\ell,j}$ and letting $\ell \uparrow \infty$, we have our assertion. Q.E.D.

Similarly by using Proposition 12, we have the following.

**Proposition 18.** Let us assume the same as the previous proposition. Then for any $n \geq 1$ and $p \in [1, \infty)$, there is a constant $C > 0$ depending only on $n$, $p$ and moments of $Z_\alpha$, $\alpha \in A_0(n)$, such that

\[
\sup_{x \in \mathbb{R}^N} E \left[ |\exp(\phi_n(Z(\varepsilon)))(\exp(Z(\varepsilon)))H(x)|^p \right]^{1/p} 
\leq C \varepsilon^{n+1} \sum_{\alpha \in A_0, n+1 \leq \| \alpha \| \leq 2n(n+1)} \| V_\alpha H \|_{\infty}, \quad \varepsilon \in (0, 1].
\]
Now let us prove Theorem 15. Note that

$$
\sup_{x \in \mathbb{R}^N} |E[f(\exp(\varphi_n(Z^{(1)}(\varepsilon)))(x))] - E[f(\exp(\varphi_n(Z^{(2)}(\varepsilon)))(x))]| = \sup_{x \in \mathbb{R}^N} |E[f(\exp(\varphi_n(Z^{(1)}(\varepsilon)))(\exp(-\varepsilon^2 V_0)(x)))] - E[f(\exp(\varphi_n(Z^{(2)}(\varepsilon)))(\exp(-\varepsilon^2 V_0)(x)))]|
$$

On the other hand, by Corollary 17, we have

$$
\sup_{x \in \mathbb{R}^N} E \left[ f(\exp(\varphi_n(Z^{(i)}(\varepsilon)))(\exp(-\varepsilon^2 V_0)(x))) - f\left(\exp\left(\varphi_n\left(\sum_{\alpha \in A_0} \varepsilon^{\|\alpha\|} q_0^0(Z_{\beta}^{(i)}, \beta \in A_0)v^{[\alpha]}(x)\right)\right)\right)\right] 
\leq C\varepsilon^{n+1} \left( \sum_{\alpha \in A_0} \|V_{\alpha} H\|_{\infty} \right) \|\nabla f\|_{\infty},
$$

$$
\varepsilon \in (0, 1), \ f \in C_b^\infty(\mathbb{R}^N; \mathbb{R}).
$$

Also, since $q_0^{(0)} = 0$, by Proposition 11 we have

$$
\sup_{x \in \mathbb{R}^N} E \left[ f\left(\exp\left(\varphi_n\left(\sum_{\alpha \in A_0} \varepsilon^{\|\alpha\|} q_0^0(Z_{\beta}^{(i)}, \beta \in A_0)v^{[\alpha]}(x)\right)\right)\right] 
- E\left[\left(\varphi_n\left(\sum_{\alpha \in A_0} \varepsilon^{\|\alpha\|} q_0^0(Z_{\beta}^{(i)}, \beta \in A_0)v^{[\alpha]}(x)\right)\right)f(x)\right] 
\leq C\varepsilon^{n+1} \left( \sum_{k=n+1}^{n(n+1)} \|f\|_{V,k} \right), \ \varepsilon \in (0, 1), \ f \in C_b^\infty(\mathbb{R}^N; \mathbb{R}).
$$

Note that $\{q_0^0(Z_{\beta}^{(i)}, \beta \in A_0); \alpha \in A_0\}, \ i = 1, 2$, are $m$-moment equivalent to each other, we see that

$$
E\left[\left(\varphi_m\left(\sum_{\alpha \in A_0} \varepsilon^{\|\alpha\|} q_0^0(Z_{\beta}^{(1)}, \beta \in A_0)v^{[\alpha]}(x)\right)\right)f(x)\right] 
= E\left[\left(\varphi_m\left(\sum_{\alpha \in A_0} \varepsilon^{\|\alpha\|} q_0^0(Z_{\beta}^{(2)}, \beta \in A_0)v^{[\alpha]}(x)\right)\right)f(x)\right].
$$

Therefore we have our theorem.
§7. SDE

Let \(X(t, x)\) be the solution of SDE (1). Also, let \(\tilde{X}(t)\) be the solution to SDE (2) in \(\tilde{U}\). Then we have the following.

**Proposition 19.** For any \(n \geq 1\), there is a constant \(C\) depending only on \(d\) and \(n\) such that

\[
\sup_{x \in \mathbb{R}^N} E \left[ |f(X(t, x)) - (\varphi_n(\tilde{X}(t))f)(x)|^2 \right]^{1/2} 
\leq C\left(\frac{n+1}{2}\right) \sum_{\alpha \in \mathcal{A}, \|\alpha\| = n+1, n+2} \|\mathcal{V}_\alpha f\|_\infty, \quad t \in (0, 1], f \in C^\infty_b(\mathbb{R}^N; \mathbb{R}).
\]

**Proof.** Note that

\[
f(X(t, x)) = f(x) + \sum_{i=0}^d \int_0^t (V_i f)(X(t, x)) \circ dB^i(t).
\]

So we have

\[
f(X(t, x)) = \sum_{\alpha \in \mathcal{A}(n)} (\mathcal{V}_\alpha f)(x)B^\alpha(t) + R(t, x).
\]

Here

\[
R(t, x) = \sum' \int_0^t \int_0^{s_k} \int_0^{s_{k-1}} \cdots \int_0^{s_1} \int_0^{s_0} (V_i V_\alpha f)(X(s, x)) dB^i(s)
\]

and \(\sum'\) is the summation with respect to \(\alpha = (\alpha^1, \ldots, \alpha^d) \in \mathcal{A}(n)\) and \(i = 0, 1, \ldots, d\), with \(\|i\| \cdot \|\alpha\| \geq n + 1\). Since

\[
\int_0^t (V_i V_\alpha f)(X(s, x)) \circ dB^i(s)
\]

\[
= \int_0^t (V_i V_\alpha f)(X(s, x)) dB^i(s) + (1 - \delta_{0, i}) \frac{1}{2} \int_0^t (V_i^2 V_\alpha f)(X(s, x)) ds.
\]

we see that there is a constant \(C(d, n)\) depending only on \(d\) and \(n\) such that

\[
\sup_{x \in \mathbb{R}^N} E[|R(t, x)|^2]^{1/2} 
\leq C(d, n)t^{(n+1)/2} \max\{\|\mathcal{V}_\alpha f\|_\infty ; \alpha \in \mathcal{A}, \|\alpha\| = n+1, n+2\}.
\]
Since $X(t, \cdot) : \mathbb{R}^N \to \mathbb{R}^N$ is a diffeomorphism, we can think of the push-forward $X(t)^*$. Then we have

$$X(t)^* = \text{Identity} + \sum_{i=0}^{d} \int_{0}^{t} X(s)^* V_i \circ dB^i(s)$$

as linear operators in $C^\infty(\mathbb{R}^N)$. So we have

$$\sum_{\alpha \in \mathcal{A}(n)} B^{\alpha}(t)V_\alpha = \varphi_n(\bar{X}(t)).$$

This proves our assertion. Q.E.D.

Combining the previous proposition with Propositions 8 and 12, and applying the argument in Corollary 17, we have the following.

**Proposition 20.** For any $n \geq 1$, there is a constant $C > 0$ depending only on $n$ and $d$ such that

$$\sup_{x \in \mathbb{R}^N} E\left[ \left| f(X(s,x)) \right|^2 \right]^{1/2} \leq C \left( \sum_{\alpha \in \mathcal{A}, n+1 \leq \|\alpha\| \leq n(n+2)} s^{\|\alpha\|/2} \|V_\alpha f\|_{\infty} \right),$$

$s \in (0, 1]$, $f \in C_b^\infty(\mathbb{R}^N; \mathbb{R})$.

In particular for any $n \geq 1$, there is a constant $C' > 0$ depending only on $n$ and $d$ such that

$$\sup_{x \in \mathbb{R}^N} E\left[ \left| X(s,x) \right|^2 \right]^{1/2} \leq C'^{s(n+1)/2} \sum_{\alpha \in \mathcal{A}_0, n+1 \leq \|\alpha\| \leq 2n(n+1)} \|V_\alpha H\|_{\infty}, \ s \in (0, 1].$$

Here $p^{\beta}_\alpha$ are polynomials in Proposition 8.
§8. Proof of Theorems

By Theorem 15 and Proposition 20, we have the following.

**Theorem 21.** Let \( m \geq 1 \). Let \( \{Z_\alpha ; \alpha \in A_0\} \) be \( m \)-moment similar family of random variables. Then for any \( n \geq 1 \), there is a constant \( C > 0 \) depending only on \( n \) and moments of \( Z_\alpha, \alpha \in A_0 \) such that

\[
\sup_{x \in \mathbb{R}^N} \left| E[f(X(s, x))] - E\left[ f\left( \exp\left( \varphi_n\left( \sum_{\alpha \in A_0} s^{\|\alpha\|/2} p_0^0(Z_\beta, \beta \in A_0) v^{[\alpha]}(x) \right) \right) \right) \right] \leq C \left( \sum_{k=m+1}^{n(m+1)} s^{k/2} \|f\|_{V_k} \right) + s^{(n+1)/2} \left( \sum_{\alpha \in A_0} \|V_\alpha H\|_{\infty} \right) \|f\|_{\infty},
\]

for \( \varepsilon \in (0, 1] \), \( f \in C^\infty_b(\mathbb{R}^N; \mathbb{R}) \).

Now Theorem 4 is an easy consequence of Theorems 15, 21 and Proposition 18.

Now let us prove Theorem 5. By Theorem 4, Corollary 2 and the argument in [6], we have the following.

**Proposition 22.** For any \( a \geq 1 \), there is a constant \( C > 0 \) such that

\[
\|P_{t+s} f - Q(s) P_t f\|_{\infty} \leq \frac{C s^{(m+1)/2}}{t^{m/2}} \|f\|_{\infty}
\]

for any \( s, t \in (0, a] \) and \( f \in C^\infty_b(\mathbb{R}^N; \mathbb{R}) \) with \( s \leq at \).

By this proposition, under the assumption in Theorem 5, we have

\[
\|P_T f - Q(s_n) Q(s_{n-1}) \cdots Q(s_1) f\|_{\infty} \leq \sum_{k=1}^n \|Q(s_n) \cdots Q(s_{k+1}) (P_{sk} - Q(s_k)) P_{t_{k-1}} f\|_{\infty} \leq \sum_{k=2}^n \|P_{t_{k-1} + s_k} f - Q(s_k) P_{t_{k-1}} f\|_{\infty} + \|P_{s_1} f - Q(s_1) f\|_{\infty}.
\]
It is easy to see that there is a constant $C > 0$ such that
\[ \| P_s f - f \|_\infty \leq C s^{1/2} \| \nabla f \|_\infty \]
and
\[ \| Q(s) f - f \|_\infty \leq C s^{1/2} \| \nabla f \|_\infty \]
for any $s \in (0, 1]$ and $f \in C^\infty_b(\mathbb{R}^N, \mathbb{R})$. So we see that there is a constant $C > 0$ such that
\[
\| P_T f - Q(s_n) Q(s_{n-1}) \cdots Q(s_1) f \|_\infty \\
\leq C n^{-\gamma/2} \left( 1 + \sum_{k=1}^{n-1} \frac{k^{(m+1)(\gamma-1)/2}}{k^{m\gamma/2}} \right) \| \nabla f \|_\infty \\
= C n^{-\gamma/2} \left( 1 + \sum_{k=1}^{n-1} k^{(\gamma-m-1)/2} \right) \| \nabla f \|_\infty .
\]
This implies our theorem.

References

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