

## Chapter 7

# Deformation spaces of hyperbolic structures on 2-orbifolds: Teichmüller spaces of 2-orbifolds

In this section, we define the Teichmüller space of 2-orbifolds as the deformation space of hyperbolic structures. (In some sense, the space should be called a Fricke space when we are talking about hyperbolic structures but not conformal structures, following Goldman.) We discuss the geometric cutting and pasting operations and the relation to the deformation spaces. The decompositions of 2-orbifolds into elementary 2-orbifolds are introduced. Elementary 2-orbifolds are pieces that cannot be decomposed further into negative Euler characteristic 2-orbifolds. We discuss the deformation spaces for elementary 2-orbifolds. (See the beginning of Section 7.3 for definition of elementary 2-orbifolds.) Using the geometric construction, we describe the Teichmüller spaces of 2-orbifolds of negative Euler characteristic. This follows Chapter 5 of the book [Thurston (1977)]. (See also the papers [Matsumoto and Montesinos-Amilibia (1991); Ohshika (1985)].)

Recall that the boundary of an orbifold is a suborbifold. The boundary component of a 2-orbifold is either a boundary full 1-orbifold or a simple closed curve.

**Theorem 7.0.1 (Thurston).** *Let  $\Sigma$  be a closed 2-orbifold of negative Euler characteristic. The deformation space of hyperbolic structures  $\mathcal{T}(\Sigma)$  is homeomorphic to an open cell of dimension*

$$-3\chi(X_\Sigma) + 2k + l + 2n$$

where  $k$  is the number of cone-points,  $l$  the number of corner-reflectors, and  $n$  is the number of boundary full 1-orbifolds of  $\Sigma$ .

### 7.1 The definition of the Teichmüller space of 2-orbifolds

A *hyperbolic* structure on a 2-orbifold is a geometric structure modeled on  $\mathbb{H}^2$  with the isometry group  $\mathbb{P}\mathrm{SL}(2, \mathbb{R})$ . (Or it should be the disk  $B^2 \subset \mathbb{R}\mathbb{P}^2$  with  $\mathbb{P}\mathrm{O}(1, 2)$  acting on it more closely to our spirit.) The *Teichmüller* space  $\mathcal{T}(M)$  of a 2-orbifold  $M$  is the deformation space of hyperbolic structures on the 2-orbifold with geodesic boundary. As before, we reinterpret the space as

- the set of equivalence classes of diffeomorphisms  $f : M \rightarrow M'$  for a 2-orbifold  $M$  and a hyperbolic 2-orbifold  $M'$  with geodesic boundary where
- $f : M \rightarrow M'$  and  $g : M \rightarrow M''$  for hyperbolic 2-orbifolds with geodesic are equivalent if there exists a hyperbolic isometry  $h : M' \rightarrow M''$  so that  $h \circ f$  is isotopic to  $g$ .

A necessary condition for a 2-orbifold to have a hyperbolic structure with geodesic boundary is that the orbifold Euler characteristic is negative: Let the 2-orbifold have a hyperbolic structure with geodesic boundary. The 2-dimensional Gauss-Bonnet theorem states that the integral of a Gaussian curvature times the area form is  $-2\pi$  times the Euler characteristic. (See Theorem 4.4.4 in Chapter 4.)

We can prove the sufficiency by decomposition into elementary 2-orbifolds and finding explicit hyperbolic structures on these and pasting back the results. This process will be clear from the proof of Theorem 7.0.1 in this chapter.

## 7.2 The geometric cutting and pasting and the deformation spaces

Recall that the interior and boundary of a 2-orbifold in the orbifold sense may be different from the interior and boundary of the underlying surface. (See Remark 4.2.5.) Given a compact hyperbolic 2-orbifold  $\Sigma$  with geodesic boundary, we have that a geodesic segment is either transversal to the boundary components or is contained in it. A compact geodesic 1-suborbifold  $l$  without boundary points in  $\Sigma$  either is a closed geodesic in the interior or entirely in the silvered boundary component of  $|\Sigma|$  or is a segment with two silvered points as the end points which are either at silvered edges or cone-points of order two. The topological interior  $l$  is either in the interior of the topological interior of  $|\Sigma|$  or entirely in the boundary of  $|\Sigma|$ . The geometric isomorphism classes are classified by length and the topological type. Such a geodesic 1-orbifold is covered by a closed geodesic in some cover of the 2-orbifold, which is a surface. (See Section 5.1.2 also.)

Note that geodesic 1-suborbifolds are always essential. (See Section 5.2.2.2)

The *Teichmüller space*  $\mathcal{T}(I)$  for a 1-orbifold  $I$  is defined as the product of the space of lengths  $\mathbb{R}^+$ s for all components of  $I$ . We technically define  $\mathcal{T}(\emptyset)$  as a singleton.

### 7.2.1 Geometric constructions.

Recall from Chapter 5, the topological splitting and pasting constructions. In this chapter, we will do these geometrically.

Recall from Chapter 5: Let  $\Sigma$  be a 2-orbifold with boundary. The pasting map  $f$  is defined on open neighborhood  $U$  in an ambient open 2-orbifold  $S'$  of the union of the associated boundary components in  $\partial\Sigma$ . Let  $\tilde{S}'$  be the universal cover of  $S'$ . Now,  $f$  satisfies the equation  $\tilde{f} \circ \vartheta = \vartheta' \circ \tilde{f}$  where  $\tilde{f}$  is a lift of  $f$  defined on  $\tilde{U}$  the

inverse image of  $U$  in  $\tilde{S}'$  and  $\vartheta$  and  $\vartheta'$  are respective deck transformations acting on two components of the inverse images in  $\tilde{S}'$  of boundary components of  $\partial\Sigma$  to be pasted by  $f$ . In the hyperbolic structure case, it is necessary and sufficient that  $f$  is an isometry and the boundary components to be glued have the same length.

Recall that a *slide reflection* of  $\mathbb{H}^2$  is an isometry acting on a geodesic  $l$  as a nontrivial translation but exchanges the two components of  $\mathbb{H}^2 - l$ .

We will describe how to construct hyperbolic structures on a larger 2-orbifold from smaller ones. Recall the type of topological constructions with 1-orbifolds. Suppose that they are boundary components of 2-orbifolds whose components have negative Euler characteristics. We can do the following operations:

- (A)(I) Pasting or crosscapping along simple closed geodesics.
- (A)(II) Silvering or folding along a simple closed geodesic.
- (B)(I) Pasting along two geodesic full 1-orbifolds.
- (B)(II) Silvering or folding along a geodesic full 1-orbifold.

Now we suppose that the simple closed curves and 1-orbifolds are geodesic and try to obtain geometric versions of the above.

Suppose that the involved 1-orbifolds are geodesic boundary components of a hyperbolic 2-orbifold. We will look at the inverse image of the 1-orbifold in the universal cover. We consider each component of the inverse image. The above operations correspond to reglueing these components with respect to each other.

- (A)(I) For pasting two closed geodesics, it is necessary and sufficient that their lengths match. Also we have an  $\mathbb{R}$ -amount of isometries to do this. They will create hyperbolic structures inequivalent in the Teichmüller space. The cut and pasting-back constructions are so-called Fenchel-Nielsen twist. (Here the lengths of two closed geodesics have to be the same. ) By taking very good covers, the inequivalence reduces to a classical fact. (See [Johnson and Millson (1987)] for example.)
- (A)(I) For cross-capping, we have a unique isometry. The isometry has to be a unique slide reflection of distance equal to the half the length of the closed geodesic. (There is no condition on the boundary component lengths.)
- (A)(II) For folding a closed geodesics, we have an  $\mathbb{R}$ -amount of isometries to do this. They will create hyperbolic structures inequivalent in the Teichmüller space. The choice depends on the choice of two fixed points of the pasting map. The distance is half of the length of the closed geodesic. (There is no condition on the boundary component.) The inequivalence can be shown as in (A)(I) by double-covering the 2-orbifold so that the folded part lifts to a simple closed curve.
- (A)(II) For silvering, we have a unique isometry to do this; that is, the reflection about the boundary component of the universal cover will do. (There is no

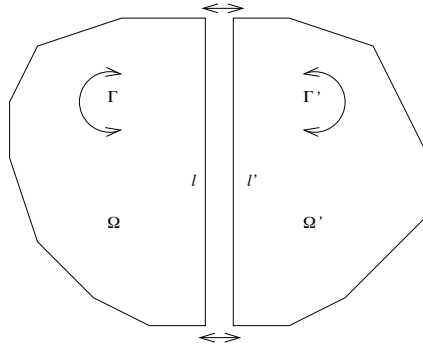


Fig. 7.1 Pasting: The actions here are isometries on the hyperbolic plane seen in the Klein model.

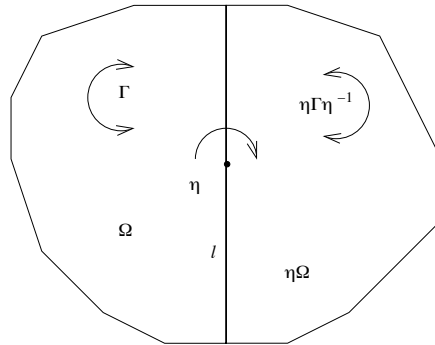


Fig. 7.2 Folding: The actions here are isometries on the hyperbolic plane seen in the Klein model.

condition on the boundary component.)

- (B)(I) For pasting along two geodesic full 1-orbifolds, it is necessary and sufficient that their lengths match. We have a unique way to do this. The lengths of the orbifolds have to be the same.
- (B)(II) For silvering and folding, we have a unique isometry to do this. (No condition)

### 7.3 The decomposition of 2-orbifolds into elementary 2-orbifolds.

Suppose that  $\Sigma$  is a compact hyperbolic 2-orbifold with  $\chi(\Sigma) < 0$  and geodesic boundary.

Simple closed geodesics and/or simple geodesic segments with endpoints in singular locus in a hyperbolic 2-orbifolds intersect minimally; i.e., they meet the minimal number of times that they can up to isotopies: a disk bounded by two geodesic segments cannot exist in  $\Sigma$ .

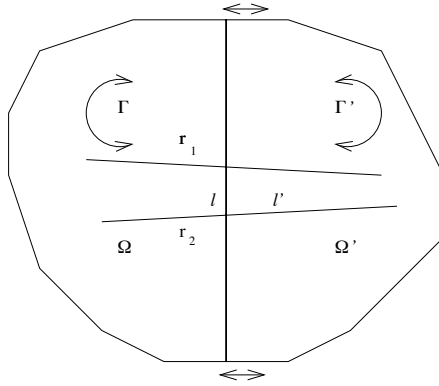


Fig. 7.3 Pasting full 1-orbifolds. The actions here are isometries on the hyperbolic plane seen in the Klein model.

Let  $c_1, \dots, c_n$  be a mutually disjoint collection of essential simple closed curves or full 1-orbifolds so that the orbifold Euler characteristic of the completion of each component of  $\Sigma - c_1 - \dots - c_n$  is negative. Then  $c_1, \dots, c_n$  are isotopic to simple closed geodesics or geodesic full 1-orbifolds  $d_1, \dots, d_n$  respectively where  $d_1, \dots, d_n$  are mutually disjoint. Here  $c_i$  is isotopic to  $d_i$  for each  $i$ , and hence  $c_i$  is a full 1-orbifold if and only if  $d_i$  is one. Also, the isotopy could be chosen simultaneously. See [Choi and Goldman (2005)] for details.

We call such a collection *decomposing* 1-orbifolds.

For example, a 2-orbifold of negative Euler characteristic based on a Möbius band admits a decomposition to an orbifold of negative Euler characteristic based on annulus by decomposing along a simple closed curve in the Möbius band.

Thus, we can decompose  $\Sigma$  into 2-orbifolds of negative Euler characteristic that cannot be applied any more geometric splitting operations; that is, there are no more 1-orbifolds decomposing it further into 2-orbifolds with negative Euler characteristic. We call such 2-orbifolds *elementary 2-orbifolds*.

A neatly embedded full 1-orbifold in a 2-orbifold is of *mirror-type* if it ends at mirror points only, is of *cone-type* if it ends at cone-points only, and is of *mixed-type* if it ends at a mirror point and a cone-point.

**Theorem 7.3.1 (Thurston).** *Let  $\Sigma$  be a compact hyperbolic 2-orbifold with  $\chi(\Sigma) < 0$  and geodesic boundary. Then there exists a mutually disjoint collection of simple closed geodesics and mirror- or cone- or mixed-type geodesic 1-orbifolds so that  $\Sigma$  decomposes along their union to a union of elementary 2-orbifolds with geodesic boundary or such elementary 2-orbifolds with some boundary 1-orbifolds silvered additionally.*

For the proof, see Chapter 5 in [Thurston (1977)] and the proof of Theorem 4.3 of [Choi and Goldman (2005)]. The basic strategy is as follows:

- For simplicity assume that  $\Sigma$  is closed and has an orientable surface as the underlying space.
- We can take a disk that contains all the cone-points of  $\Sigma$  unless  $|\Sigma|$  is homeomorphic to a 2-sphere. If there are two cone-points of order two, then we take a full 1-orbifold  $l$  ending there. Then we decompose  $\Sigma$  along  $l$  to obtain a 2-orbifold with a closed geodesic boundary. Thus, we can assume that all cone-points have order  $> 2$  with at most one exception. Unless there is just one cone-point, we can find a closed geodesic bounding all of the cone-points. Then we can decompose the surface further along the closed geodesic to obtain a pair-of-pants, an annulus with a single cone-point, or a disk with two cone-points one of which has order  $\geq 3$ .
- For each boundary component of  $\Sigma$  with corner-reflectors, we can take a closed geodesic homotopic to it bounding a 2-orbifold with negative Euler-characteristic based on an annulus unless  $\Sigma$  is a disk bounded by silvered edges and with corner-reflectors with at most one-cone point.
- The results are much easier to decompose.

### 7.3.1 Elementary 2-orbifolds.

The underlying space of an elementary 2-orbifold has to be homeomorphic to a 2-sphere, a 2-disk, an annulus, or a pair-of-pants since otherwise there is an essential simple closed curve in the interior not freely homotopic to a boundary component just by the topology.

Note that we can also alter some boundary components by silvering it and giving corner-reflector structure of order 2 at the endpoints. The results are still considered to be an elementary 2-orbifold of the same type.

We remark that a Möbius band with some singularities is not elementary as we can use a simple closed geodesic to decompose it further.

We classify elementary 2-orbifolds up to diffeomorphisms by Theorem 5.1.1 and the above decomposition methods.

- (P1) A pair-of-pants. ( $\chi = -1$ .)
  - (P2) An annulus with one cone-point of order  $n$ . ( $A(; n)$ ,  $\chi = -1 + 1/n$ .)
  - (P3) A disk with two cone-points of orders  $p, q$ , one of which is greater than 2. ( $D(; p, q)$ ,  $\chi = -1 + 1/p + 1/q$ .)
  - (P4) A sphere with three cone-points of order  $p, q, r$  where  $1/p + 1/q + 1/r < 1$ . ( $\mathbf{S}^2(; p, q, r)$ ,  $\chi = -1 + 1/p + 1/q + 1/r$ )
- 
- (A1) An annulus with one boundary component a union of a singular segment and one boundary-orbifold. (We call it *two-pronged crown* and denote it by  $A(2, 2; )$ , and we have  $\chi = -1/2$ . It has two corner-reflectors of order 2 if the boundary components are silvered.)

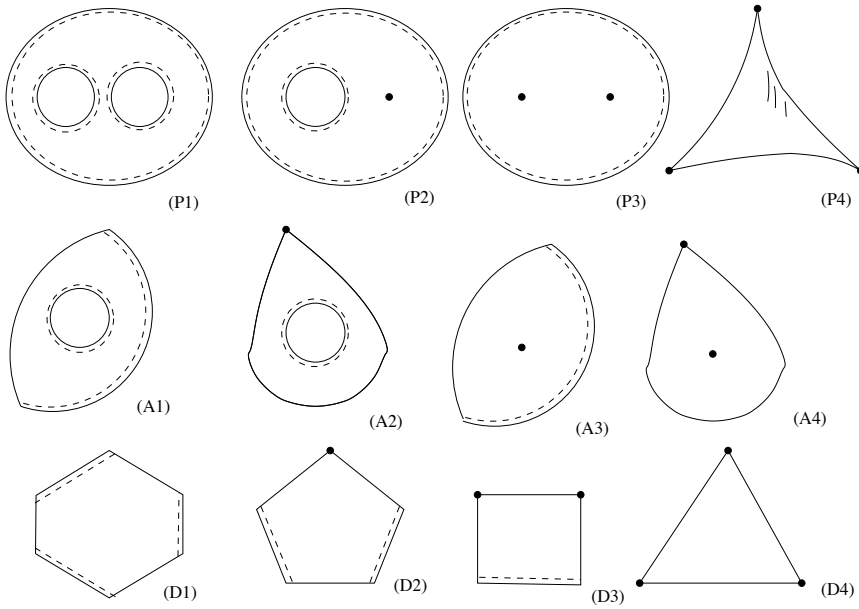


Fig. 7.4 The elementary orbifolds. Arcs with dotted arcs next to them indicate boundary components. Black points indicate singular points.

- (A2) An annulus with one boundary component of the underlying space in a singular locus with one corner-reflector of order  $n$ ,  $n \geq 2$ . (The other boundary component is a closed curve which is the boundary of the 2-orbifold. We call it a *one-pronged crown* and denote it by  $A(n; \cdot)$ , and  $\chi = -(n-1)/2n$ .)
- (A3) A disk with one singular segment and one boundary 1-orbifold and a cone-point of order  $n$  greater than or equal to three ( $D^2(2, 2; n)$ ,  $\chi = 1/n - 1/2$ .)
- (A4) A disk with one corner-reflector of order  $m$  and one cone-point of order  $n$  so that  $1/2m + 1/n < 1/2$  (with no boundary orbifold). (We have  $n \geq 3$  necessarily, and denote it by  $D^2(m; n)$ , and we have  $\chi = -1/2 + 1/n + 1/2m$ .)
- (D1) A disk with three silvered edges and three boundary 1-orbifolds. No two boundary 1-orbifolds are adjacent. (*hexagon*,  $D^2(2, 2, 2, 2, 2, 2; \cdot)$ ,  $\chi = -1/2$ )
- (D2) A disk with three silvered edges and two boundary 1-orbifolds on the boundary of the underlying space. Two boundary 1-orbifolds are not adjacent, and two silvered edges meet in a corner-reflector of order  $n$ , and the remaining silvered one a segment. (*pentagon*,  $D^2(2, 2, 2, 2, n; \cdot)$ ,  $\chi = -1/2(1 - 1/n)$ .)
- (D3) A disk with two corner-reflectors of order  $p, q$ , one of which is greater than or equal to 3, and one boundary 1-orbifold. The singular locus of the disk is a union of three silvered edges and two corner-reflectors. (*quadrilateral*,  $D^2(2, 2, p, q; \cdot)$ ,  $\chi = -1/2 + 1/2p + 1/2q$ .)
- (D4) A disk with three corner-reflectors of order  $p, q, r$  where  $1/p + 1/q + 1/r < 1$  and three silvered edges (with no boundary orbifold). (*triangle*,  $D^2(p, q, r; \cdot)$ ,

$$\chi = -1/2 + 1/2p + 1/2q + 1/2r.)$$

## 7.4 The Teichmüller spaces for 2-orbifolds

### 7.4.1 The strategy of the proof

We first prove:

**Proposition 7.4.1.** *For each elementary 2-orbifold  $S$ ,  $\mathcal{T}(S)$  is homeomorphic to  $\mathcal{T}(\partial S)$ , where  $\mathcal{T}(\partial S)$  is the product of  $\mathbb{R}^+$  for each component of  $\partial S$  corresponding to the hyperbolic-metric lengths of components of  $\partial S$ .*

Note here the rigidity of some closed elementary orbifolds, i.e., elementary orbifolds of type (P4), (A4), and (D4).

Then to obtain the deformation space of a bigger 2-orbifold, we use the above result about the Teichmüller spaces under geometric decompositions.

### 7.4.2 The generalized hyperbolic triangle theorem

A generalized triangle in the hyperbolic plane is one of following:

- (a) A hexagon: a disk bounded by six geodesic sides meeting in right angles labeled  $A, \beta, C, \alpha, B, \gamma$ .
- (b) A pentagon: a disk bounded by five geodesic sides labeled  $A, \beta, C, \alpha, B$  where  $A$  and  $B$  meet in an angle  $\gamma$ , and the rest of the angles are right angles.
- (c) A quadrilateral: a disk bounded by four geodesic sides labeled  $A, C, B, \gamma$  where  $A$  and  $C$  meet in an angle  $\beta$ ,  $C$  and  $B$  meet in an angle  $\alpha$  and the two remaining angles are right angles.
- (d) A triangle: a disk bounded by three geodesic sides labeled  $A, B, C$  where  $A$  and  $B$  meet in an angle  $\gamma$  and  $B$  and  $C$  meet in an angle  $\alpha$  and  $C$  and  $A$  meet in an angle  $\beta$ .

For generalized triangles in the hyperbolic plane, we have

$$\begin{aligned}
 \text{(a) } \cosh C &= \frac{\cosh \alpha \cosh \beta + \cosh \gamma}{\sinh \alpha \sinh \beta} \\
 \text{(b) } \cosh C &= \frac{\cosh \alpha \cosh \beta + \cos \gamma}{\sinh \alpha \sinh \beta} \\
 \text{(c) } \sinh A &= \frac{\cosh \gamma \cos \beta + \cos \alpha}{\sin \beta \sinh \gamma} \\
 \text{(d) } \cosh C &= \frac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta}
 \end{aligned} \tag{7.1}$$

In (a),  $(\alpha, \beta, \gamma)$  can be any positive numbers. In (b),  $(\alpha, \beta)$  can be any positive numbers and  $\gamma$  in  $(0, \pi/2]$ . In (c),  $(\alpha, \beta)$  can be any positive real numbers in  $(0, \pi/2]$  satisfying  $\alpha + \beta < \pi$ , and  $\gamma$  any real number. In (d),  $(\alpha, \beta, \gamma)$  can be any



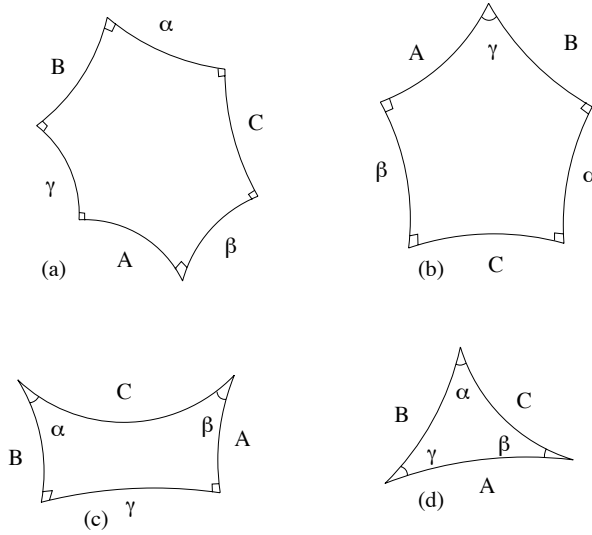


Fig. 7.5 A hexagon, a pentagon, a quadrilateral, and a triangle in the hyperbolic space with our labels.

real numbers in  $(0, \pi/2]$  satisfying  $\alpha + \beta + \gamma < \pi$ . One can use continuity arguments and some geometry to verify these. (These facts are shown in the book [Ratcliffe (2006)] for example.)

**7.4.3 The proof of Proposition 7.4.1.**

The following lemmas imply Proposition 7.4.1.

**Lemma 7.4.2.** *For elementary 2-orbifolds of type (D1), (D2), (D3), and (D4), silvered edges are labeled by the capital letters A, B, C. Assign to each vertex an angle of the form  $\pi/n$  where  $n > 1$  is an integer, for which it is a corner-reflector of that angle. Each edge labeled by Greek letters  $\alpha, \beta, \gamma$  is a boundary full 1-orbifold. Then in cases (D1), (D2), (D3), and (D4),  $\mathcal{F} : \mathcal{T}(P) \rightarrow \mathcal{T}(\partial P)$  for each of the above orbifolds  $P$  is a homeomorphism; that is,  $\mathcal{T}(P)$  is homeomorphic to an open cell of dimension 3, 2, 1, or 0 respectively.*

**Proof.** For (D1), we simply notice that we can assign the boundary lengths  $\alpha, \beta, \gamma$  freely using the equation (a). For (D2), assign  $\gamma = \pi/n$ . Then  $\alpha$  and  $\beta$  can be freely assigned. For (D3), assign  $\alpha = \pi/p$  and  $\beta = \pi/q$  for  $q > 2$ . Then  $\gamma$  can be freely assigned with  $A$  and  $B$  obtained by equation (c). Then the construction of quadrilateral is done. For (D4), we assign  $\alpha = \pi/p, \beta = \pi/q, \gamma = \pi/r$  where  $1/p + 1/q + 1/r < 1$ . Such a triangle always exists uniquely.  $\square$

For each of hyperbolic elementary orbifolds of type (P1),(P2),(P3), and (P4), there exists an isometric involution acting on each boundary component and the

quotient orbifold is of type (D1),(D2),(D3), and (D4): The involution can be constructed explicitly by considering the fundamental domains. That is, we draw shortest geodesics between the appropriate boundary components and/or cone-points to obtain an isometric pair of hexagons, one of pentagons, one of quadrilaterals and one of triangles. Then each involution is given by sending the interior of one domain to the other fixing the geodesics.

Conversely, a hyperbolic orbifold of type (D1)-(D4) is covered by one of type (P1)-(P4) by an orientable double-cover construction of Section 4.6.1.2. The hyperbolic structure is simply obtained by local-lifts of the metrics on ones on (D1)-(D4) or induced by the covering map. (See Sections 6.1 and 2.3.1.) Hence in fact, there is a homeomorphism between the deformation spaces  $\mathcal{T}(S) \rightarrow \mathcal{T}(S')$  where  $S$  double-covers  $S'$ . Furthermore  $\mathcal{T}(\partial S) \rightarrow \mathcal{T}(\partial S')$  is a homeomorphism in these cases.

Hence,  $\mathcal{F} : \mathcal{T}(S) \rightarrow \mathcal{T}(\partial S)$  is a homeomorphism for the type (P1)-(P4) orbifolds  $S$ .

**Lemma 7.4.3.** *Let  $S$  be an elementary 2-orbifold of type (A1), (A2), (A3), or (A4). Then  $\mathcal{F} : \mathcal{T}(S) \rightarrow \mathcal{T}(\partial S)$  is a homeomorphism. Thus,  $\mathcal{T}(S)$  is an open cell of dimension 2, 1, 1, or 0 when  $S$  is of type (A1), (A2), (A3) or (A4) respectively. In case (A4),  $\mathcal{T}(S)$  is a singleton.*

**Proof.** Here again elementary orbifolds of type (P1), (P2), (P3), and (P4) double-cover orbifolds of type (A1), (A2), (A3), and (A4). Here the involutions are different from the above ones. For (A1), (A3), and (A4), the involutions are about vertical axes and the perpendicular plane containing the vertical axis respectively. (See Figure 7.4.) For (A2) the involutions are about the essential simple closed curve passing the cone-point (See Figure 7.4.) The involutions are realized as isometries uniquely by considering the fundamental domains by drawing shortest geodesics of appropriate relative homotopy classes. This is again sufficient to imply the conclusions here.  $\square$

#### 7.4.4 The steps to prove Theorem 7.0.1.

We say that a 2-orbifold  $\Sigma$ , each component of which has negative Euler characteristic, is in a class  $\mathcal{P}$  if the following hold:

- (i) The deformation space of hyperbolic structures  $\mathcal{T}(\Sigma)$  is homeomorphic to an open cell of dimension

$$-3\chi(X_\Sigma) + 2k + l + 2n$$

where  $k$  is the number of cone-points,  $l$  the number of corner-reflectors, and  $n$  is the number of boundary full 1-orbifolds.

- (ii) There exists a fibration

$$\mathcal{F} : \mathcal{T}(\Sigma) \rightarrow \mathcal{T}(\partial\Sigma)$$

with fibers homeomorphic to an open cell of dimension  $\dim \mathcal{T}(\Sigma) - \dim \mathcal{T}(\partial\Sigma)$ . Here  $\mathcal{F}$  is the map induced by the restriction of the hyperbolic structures to the metric structures of  $\partial\Sigma$ .

First of all, elementary orbifolds satisfy these properties.

Let  $\Sigma$  be a compact 2-orbifold whose components are compact orbifolds of negative Euler characteristic, and it splits into an orbifold  $\Sigma'$  in  $\mathcal{P}$ . We suppose that (i) and (ii) hold for  $\Sigma'$ , and show that (i) and (ii) hold for  $\Sigma$ . Since  $\Sigma$  eventually decomposes into a union of elementary 2-orbifolds where (i) and (ii) hold, we would have completed the proof of Theorem 7.0.1 by Proposition 7.4.1.

The proofs of the above statements follow by going through each of the constructions. (For details, see [Choi and Goldman (2005)].) The dimension counting here is easy by knowing that taking diagonal drops dimensions as expected.

**(A)(I)(1)** Let the 2-orbifold  $\Sigma''$  be obtained from pasting along two closed curves  $b, b'$  in a 2-orbifold  $\Sigma'$ . The map resulting from splitting

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \rightarrow \Delta \subset \mathcal{T}(\Sigma')$$

is a principal  $\mathbb{R}$ -fibration, where  $\Delta$  is the subset of  $\mathcal{T}(\Sigma')$  where  $b$  and  $b'$  have equal lengths. Then  $\mathbb{R}$  acts by the twisting the gluing of  $b$  and  $b'$  by isometries. (The operations of cutting along a closed geodesic and re-gluing with nontrivial twists are called Fenchel-Nielsen twists in the hyperbolic surface theory.) Since

$$\mathcal{F} : \mathcal{T}(\Sigma') \rightarrow \mathcal{T}(\partial\Sigma')$$

is a fibration,  $\mathcal{F}|_{\Delta}$  is a fibration onto  $\Delta'$  the subset of  $\mathcal{T}(\partial\Sigma')$  where  $b$  and  $b'$  have the same lengths. By forgetting about  $b$  and  $b'$ , we obtain an  $\mathbb{R}$ -fibration  $\Delta' \rightarrow \mathcal{T}(\partial\Sigma')$ . Composing with  $\mathcal{SP}$ , we obtain a fibration

$$\mathcal{F} : \mathcal{T}(\Sigma'') \rightarrow \mathcal{T}(\partial\Sigma'')$$

with fibers homeomorphic to an open cell of the desired dimension.

**(A)(I)(2)** Let  $\Sigma''$  be obtained from  $\Sigma'$  by cross-capping. The resulting map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \rightarrow \mathcal{T}(\Sigma')$$

is a homeomorphism. There is an  $\mathbb{R}$ -fibration  $\mathcal{T}(\partial\Sigma') \rightarrow \mathcal{T}(\partial\Sigma'')$  by forgetting the boundary component involved in cross-capping. By composing with  $\mathcal{SP}$ , we obtain the fibration

$$\mathcal{T}(\Sigma'') \rightarrow \mathcal{T}(\partial\Sigma'').$$

**(A)(II)(1)** Let  $\Sigma''$  be obtained from  $\Sigma'$  by silvering. The clarifying map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \rightarrow \mathcal{T}(\Sigma')$$

is a homeomorphism.

**(A)(II)(2)** Let  $\Sigma''$  be obtained from  $\Sigma'$  by folding a boundary closed curve  $l'$ . The unfolding map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \rightarrow \mathcal{T}(\Sigma')$$

is a principal  $\mathbb{R}$ -fibration.

For each of these, the fibration designated by  $\mathcal{F}$  can be shown to exist as in (A)(I)(2) above.

**(B)(I)** Let  $\Sigma''$  be obtained by pasting along two full 1-orbifolds  $b$  and  $b'$  in  $\Sigma'$ . The splitting map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \rightarrow \Delta \subset \mathcal{T}(\Sigma')$$

is a homeomorphism where  $\Delta$  is a subset of  $\mathcal{T}(\Sigma')$  where the lengths of  $b$  and  $b'$  are equal.  $\mathcal{F}$  is again shown to exist as in (A)(I)(1).

**(B)(II)** Let  $\Sigma''$  be obtained by silvering or folding a full 1-orbifold. The clarifying or unfolding map

$$\mathcal{SP} : \mathcal{T}(\Sigma'') \rightarrow \mathcal{T}(\Sigma')$$

is a homeomorphism.  $\mathcal{F}$  is again shown to exist as in (A)(I)(2).

## 7.5 Notes

The Teichmüller theory for 2-orbifolds was created by Thurston in Chapter 5 of [Thurston (1977)] and were written up also in [Matsumoto and Montesinos-Amilibia (1991); Ohshika (1985)]. (See also [Kapovich (2009)].) The materials here are from the papers [Choi (2004); Choi and Goldman (2005)]. We also mention that for examples of the study of 3-dimensional orbifolds and their geometric structures, one could see the books [Cooper, Hodgson, and Kerckhoff (2000); Boileau, Maillot, Porti (2003)].