Chapter 6

Geometry of orbifolds: geometric structures on orbifolds

In this section, we introduce the geometric structures on orbifolds. The definition is given by the method of atlases of charts, making use of (G, X)-pseudo group structures in Section 2.3. We show that geometric orbifolds are always good by using the foliation theory, an important result due to Thurston (See Chapter 5 of the book [Thurston (1977)].) Then we discuss developing maps, global charts, and associated holonomy homomorphisms. These can also be used as definitions of geometric structures. We also introduce the approach using flat bundles and transverse sections to define the geometric structures. (See Section 2.4.) These observations were first due to Goldman (1987) for manifolds. The article [Goldman (2010)] contains a general introduction to geometric structures on manifolds.

Next, we introduce the deformation spaces of geometric structures on orbifolds using the above three approaches as were done by Goldman for manifolds. We finally mention the local homeomorphism theorem from the deformation space to the representation space.

6.1 The definition of geometric structures on orbifolds

Let (G, X) be a pair defining a geometry. That is, G is a Lie group acting on a manifold effectively and transitively. Let M be a connected n-orbifold with boundary, possibly empty. We have three ways to define a (G, X)-geometric structure on M:

- Atlases of charts.
- A developing map from the universal covering space.
- A cross-section of the flat orbifold X-bundle.

6.1.1 *An atlas of charts approach*

Given an imbedding $f : U \to V$ between two domains U and V in $\mathbb{R}^n$ with groups G_1 and G_2 acting on them respectively, we denote by $f^* : G_1 \to G_2$ the homomorphism determined by sending $\vartheta \in G_1$ to the element of G_2 agreeing with $f \circ \vartheta \circ f^{-1}$ in an open subset provided this is always uniquely determined.

An X-*chart* of a model open set form the triple (U, K, ϕ) of M is simply a h-equivariant homeomorphism from U to an open subset of X where h is an injective homomorphism $K \to G$. Given an atlas of charts for M, for each chart (U, K, ϕ) in the atlas, we suppose that we find an X-chart $\rho : U \to X$ and an injective homomorphism $h : K \to G$ so that ρ is an equivariant map. Let (U, K, ϕ) and (V, H, ψ) be two charts with the inclusion map $\iota : \psi(V) \to \phi(U)$. For an embedding $\tilde{\iota} : (V, H, \psi) \to (U, K, \phi)$ of charts lifting ι, if we have

$$\rho \circ \tilde{\iota} = g \circ \rho', h'(\cdot) = gh(\tilde{\iota}^*(\cdot))g^{-1} \text{ for some } g \in G,$$

then ι or $\tilde{\iota}$ are said to be a (G, X)-*map*. Two X-charts (V, H, ψ) and (U, K, ϕ) in an atlas of X-charts are (G, X)-*compatible* if given any point $x \in \psi(V) \cap \phi(U)$ in M, we have an X-chart (W, K, η) so that $\eta(W)$ is a neighborhood of x in $\psi(V) \cap \phi(U)$ and the embedding of $\eta(W)$ in each of $\phi(U)$ and $\psi(V)$ is a (G, X)-map.

If we simply identify with open subsets of X, the above simplifies greatly and $\tilde{\iota}$ is a restriction of an element of g and $\tilde{\iota}^*$ is a conjugation by g also.

This gives us a way to build an orbifold from open subset pieces of X. A maximal such atlas of compatible X-charts is called *a* (G, X)-*structure* on M.

(Note that this gives no condition on $\partial\mathcal{O}$. Sometimes, it will be necessary to put restrictions to work with deformation spaces. A priori, one does not know what the boundary condition should be.)

An (G, X)-*map* $f : M \to N$ is a smooth map so that for each x and $y = f(x)$, there are charts (U, K, ϕ) and (V, H, ψ) so that f sends $\phi(U)$ into $\psi(V)$ and lifts to an immersion $\tilde{f} : U \to V$ so that

$$\rho' \circ \tilde{f} = g \circ \rho \text{ and } h'(\tilde{f}^*(\cdot)) = gh(\cdot)g^{-1} \text{ for } g \in G.$$

In other words, f is a restriction of an element g of G up to charts with a homomorphism $K \to H$ induced by a conjugation by an element g of G.

Let M be an orbifold. Note that an orbifold-immersion $f : M \to N$ to an orbifold N with a (G, X)-structure μ induces a (G, X)-structure on M so that f becomes a (G, X)-map. M is said to have a (G, X)-*structure induced by* f to be denoted by $f^*(\mu)$. (See Section 2.3.1 also.)

Theorem 6.1.1 (Thurston). *Let M be an n-orbifold with boundary, possibly empty. An (G, X)-orbifold M is always good.*

Proof. Basically we build the space of germs of local (G, X)-maps from M to X which is a principal bundle and is a manifold: M is covered by open sets that can be identified with open sets in X. For a local finite subgroup K of G acting on $U \subset M$ identified with an open subset of X, let K act on $G \times U$ by $k(g, u) = (kg, ku)$ for $u \in U, g \in G, k \in K$. For each (U, K, ϕ), we build $G(U) = (G \times U)/K$ and a projection $G(U) \to U/K$. For any inclusion $V \to U$ for open sets $U, V \subset M$, we obtain $G(V) \to G(U)$ induced by inclusion maps. We paste these together to obtain $G(M)$. Then $G(M)$ is a manifold since K acts on $G \times U$ freely. The foliation given by pasting $g_0 \times U$ in $G(U)$ is a foliation by open manifolds with the same dimension as M. Each leaf of the foliation covers M forming a manifold cover of M.

□

If G is a subgroup of a linear group, then M is very good by Selberg's lemma provided M has finitely generated fundamental group. Thus M is a quotient orbifold $\tilde{M}/\Gamma$ where Γ is finite and contains copies of all of the local group.

6.1.2 *The developing maps and holonomy homomorphisms*

Let a connected orbifold M admit a (G, X)-structure. Let $\tilde{M}$ denote the universal cover of M with a deck transformation group $\pi_1(M)$. Then $\tilde{M}$ is a manifold and we obtain a *developing map* $D : \tilde{M} \to X$ by first finding an initial chart $\rho : U \to X$ and continuing by extending maps by patching. We use a nice cover of $\tilde{M}$ and extend. The map is well-defined independently of which path of charts one took to arrive at a given chart: To show this, we consider a homotopy of paths and consider mutually intersecting three X-charts simultaneously and the map can be consistently defined on their union.

Since we can change the initial chart to $k \circ \rho$ for any $k \in G$, it follows that $k \circ D$ is an another developing map and conversely any developing map is of such a form.

Given a deck transformation $\gamma : \tilde{M} \to \tilde{M}$, we see that $D \circ \gamma$ is a developing map also and hence equals $h(\gamma) \circ D$ for some $h(\gamma) \in G$. Let $\pi_1(M)$ denote the group of deck transformations of $\tilde{M}$. The map $h : \pi_1(M) \to G$ is a homomorphism, so-called the holonomy homomorphism.

The pair (D, h) is said to be the *developing pair.* The development pair is determined up to an action of G given by $(D, h(\cdot)) \to (g \circ D, gh(\cdot)g^{-1})$.

Conversely, a developing map (D, h) gives us X-charts: For each open chart (U, K, ψ), we lift to a component of $p^{-1}(U)$ in $\tilde{M}$ and obtain a restriction of D to the component. This gives us X-charts. A different choice of components gives us the compatible charts. Local group actions and embeddings satisfy the desired properties. Thus, a development pair completely determines the (G, X)-structure on M.

6.1.3 *The definition as flat bundles with transversal sections*

Given a (G, X)-orbifold M with X-charts, we form a G-bundle $G(M)$ as in the proof of Theorem 6.1.1. This is a principal G-bundle. We form an associated X-bundle $X(M)$ using the G-action on X: $X(M) = G(M) \times X/G$ where G acts on the right on $G(M)$ and left on X and G acts on $G(M) \times X$ on the right by

$$g : (u, x) \to (ug, g^{-1}(x)), g \in G, u \in G(M), x \in X.$$

A flat G-bundle is an object obtained by patching open sets $G \times U$ by the left action of G as in the proof of Theorem 6.1.1, and so is a flat X-bundle defined as above.

6.1.3.1 *Flat X-bundles*

One can also define a notion of foliation on n-dimensional orbifolds. Given an n-orbifold M and each model triple (U, K, ϕ), we give a smooth submersion $U \to \mathbb{R}^i$ for some $1 \leq i \leq n-1$ equivariant with respect to a homomorphism $K \to L$ for a finite group acting on $\mathbb{R}^i$ smoothly. The fibers of the maps is said to be leaves. For any embedding $(V, J, \psi) \to (U, K, \phi)$, the leaves of the charts are compatible. A leaf of a foliation is also defined as in the manifold cases as maximal $n-i$-dimensional subset that is a union of images of leaves of model triples.

A foliation in the manifold $G(M)$ with leaves transversal to fibers induces a foliation in $G(M) \times X$ with leaves transversal to fibers and hence a foliation in the orbifold $X(M)$ with leaves transversal to fibers. This corresponds to a flat G-connection. A flat G-connection on $X(M)$ is a way to identify each fiber of $X(M)$ with X locally-consistently. A flat G-connection on $X(M)$ gives us a flat G-connection on $X(\tilde{M})$. Since $\tilde{M}$ is a simply-connected manifold, $X(\tilde{M})$ can be identified with $X \times \tilde{M}$ as an X-bundle where we can regard sets of form $x \times \tilde{M}$ as leaves for the flat connections. $X(\tilde{M})$ covers $X(M)$ and hence

$$X(M) = (X \times \tilde{M})/\pi_1(M)$$

where the connection corresponds to foliations with leaves of type $x \times \tilde{M}$. Hence this gives us a representation $h : \pi_1(M) \to G$ so that for any $\gamma \in \pi_1(M)$, the corresponding action in $X \times \tilde{M}$ is given by $(x, m) \to (h(\gamma)x, \gamma(x))$.

Conversely, given a representation h, we can build $X \times \tilde{M}$ and act by $\gamma(x, m) = (h(\gamma)x, \gamma(m))$ to obtain a flat X-bundle $X(M)$. (This theory is completely analogous to Section 2.4.2.2. See also the books [Kobayashi and Nomizu (1997); Bishop and Crittendon (2002)] for details.)

6.1.3.2 *Flat X-bundles with transversal sections*

A development pair (D, h) of M gives us a flat X-bundle $X(M)$ with a section $s : M \to X(M)$. We obtain a section $D' : \tilde{M} \to X \times \tilde{M}$ transversal to the foliation by taking $D'(x) = (D(x), x)$ for $x \in \tilde{M}$. The transversality of D' to the constant foliation is actually equivalent to the immersive property of D. The left-action of $\pi_1(M)$ gives us a section $s : M \to X(M)$ transversal to the foliation.

On the other hand, given a transversal section $s : M \to X(M)$, we obtain a transversal section $s' : \tilde{M} \to X \times \tilde{M}$. By a projection to X, we obtain an immersion $D : \tilde{M} \to X$ so that $D \circ \gamma = h(\gamma) \circ D$ for some $h(\gamma)$ in G. The map $h : \pi_1(M) \to G$ is a homomorphism. Hence we obtain a development pair.

6.1.4 *The equivalence of three notions.*

Given an atlas of X-charts, i.e., a (G, X)-structure, we determine a development pair (D, h). Given a development pair (D, h), we determine an atlas of X-charts, i.e., a (G, X)-structure. Given a development pair (D, h), we determine a flat X-bundle $X(M)$ with a transversal section $M \to X(M)$. Given a section $s : M \to X(M)$ to a flat X-bundle, we determine a development pair (D, h). Thus, these three classes of definitions are equivalent.

6.2 The definition of the deformation spaces of (G, X)-structures on orbifolds

Consider the set $\mathcal{M}_{(G,X)}(M)$ of all (G, X)-structures on a connected orbifold M. We introduce an equivalence relation: two (G, X)-structures μ_1 and μ_2 are *equivalent* if there is an isotopy $\phi : M \to M$ from the identity map I_M so that $\phi^*(\mu_1) = \mu_2$. The deformation space of (G, X)-structures on M is defined as $\mathcal{M}_{(G,X)}/\sim$. (Currently, we just have a set.)

We reinterpret the space as

- the set of equivalence classes of diffeomorphisms $f : M \to M'$ for M an orbifold and M' a (G, X)-orbifold
- where $f : M \to M' \sim g : M \to M''$ if there exists a (G, X)-diffeomorphism $h : M' \to M''$ so that $h \circ f$ is isotopic to g.

6.2.1 *The isotopy-equivalence space.*

First, we identify $\pi_1(M)$ with $\pi_1(M \times I)$. Consider the set of diffeomorphisms $f : \tilde{M} \to \tilde{M}'$ equivariant with respect to an isomorphism $f_* : \pi_1(M) \to \pi_1(M')$ for a (G, X)-orbifold M'. We introduce an equivalence relation on this set: Given $f : \tilde{M} \to \tilde{M}'$ and $g : \tilde{M} \to \tilde{M}''$, we say that they are *equivalent* if there exists a (G, X)-map $\phi : \tilde{M}' \to \tilde{M}''$ so that $\phi \circ f$ is isotopic to g by an isotopy $\tilde{M} \times I \to \tilde{M}''$ equivariant with respect to both $\phi_* \circ f_*$ and g_* which are equal. Denote this set by $\mathcal{D}_I(M)$.

We claim that $\mathcal{D}_I(M)$ is in one-to-one correspondence with $\mathcal{M}_{(G,X)}/\sim$: Given an element f of the first space, we obtain an induced diffeomorphism $\hat{f} : M \to M'$ for a (G, X)-manifold M'. The equivariant isotopy goes to an isotopy. So this is a well-defined map. The inverse is given by lifting a diffeomorphism $g : M \to M'$ for a (G, X)-manifold M' to the universal covers.

The space $\mathcal{S}(M)$ is defined as follows: Consider the set of triples of form $(D, \tilde{f} : \tilde{M} \to \tilde{M}')$ where $f : M \to M'$ is a diffeomorphism for orbifolds M and M', $\tilde{f} : \tilde{M} \to \tilde{M}'$ is a lift of f, and $D : \tilde{M}' \to X$ is an immersion equivariant with respect to a homomorphism $h : \pi_1(M') \to G$. We define $(D, \tilde{f}) \sim (D', \tilde{f}' : \tilde{M} \to \tilde{M}'')$ if there is a diffeomorphism $\tilde{\phi} : \tilde{M}' \to \tilde{M}''$ so that $D' \circ \tilde{\phi} = D$ and an isotopy

$H : \tilde{M} \times I \to \tilde{M}''$ equivariant with respect to $\tilde{f}'_* : \pi_1(M) \to \pi_1(M'')$ so that $\tilde{\phi} \circ \tilde{f} = H_0$ and $\tilde{f}' = H_1$. We finally give a topology on this space by the C^1-topology on the space of maps $\tilde{M} \to X$ restricting to the space of maps of form $D \circ \tilde{f} : \tilde{M} \to X$. (Here the C^1-topology means the compact C^1-topology.)

There is a G-action on $\mathcal{S}(M)$ given by sending D to $g \circ D$ for $g \in G$.

6.2.2 *The topology of the deformation space*

Theorem 6.2.1. *Let M be a connected orbifold. There is a natural action of G on $\mathcal{S}(M)$ given by $g(D, \tilde{f}) = (g \circ D, \tilde{f}), g \in G$. The quotient space $\mathcal{S}(M)/G$ is in one-to-one correspondence with the deformation space $\mathcal{M}_{(G,X)}/\sim$. This space has the quotient topology from the C^1-topology of $\mathcal{S}(M)$.*

Proof. We show that $\mathcal{D}_I(M)$ is in one-to-one correspondence to $\mathcal{S}(M)/G$.

We first obtain a map $\mathcal{D}_I(M) \to \mathcal{S}(M)/G$: Given an element $\tilde{f} : \tilde{M} \to \tilde{M}'$, we have a developing map $D : \tilde{M}' \to X$ equivariant with respect to $h : \pi_1(M') \to G$. Also, given $\tilde{f}' : \tilde{M} \to \tilde{M}''$, we have a developing map $D' : \tilde{M}'' \to X$ equivariant with respect to $h' : \pi_1(M'') \to G$. If $\tilde{f} : \tilde{M} \to \tilde{M}'$ and $\tilde{f}' : \tilde{M} \to \tilde{M}''$ are equivalent, then there is a (G, X)-diffeomorphism $M' \to M''$ and hence two global charts $D' \circ \tilde{f}$ and $D'' \circ \tilde{f}'$ differ only by an element of G.

Conversely, we obtain a map $\mathcal{S}(M)/G \to \mathcal{D}_I(M)$: given $(D, \tilde{f})$, we obviously obtain a (G, X)-structure on M' If $(D, \tilde{f})$ and $(D', \tilde{f}')$ are equivalent, then there is a diffeomorphism $\phi : M' \to M''$ so that $D' \circ \tilde{\phi} = g \circ D$ for a lift $\tilde{\phi}$ of ϕ. This means $\phi' : M' \to M''$ is a (G, X)-diffeomorphism. The above two maps are clearly inverses of each other. □

We will denote by $\mathcal{D}_{(G,X)}(M)$ the space $\mathcal{S}(M)/G$ with the topology given in the theorem.

6.2.3 *The local homeomorphism theorem*

6.2.3.1 *The representation space*

Suppose that $\pi_1(M)$ is finitely-generated. In particular if M is a compact n-orbifold, this is true. Denote by $g_1, \ldots, g_n$ the set of generators and $R_1, \ldots, R_m, \ldots$ be the set of relations.

The set of homomorphisms $\pi_1(M) \to G$ is to be identified with a subset of G^n by sending a homomorphism h to $(h(g_1), \ldots, h(g_n))$. This is clearly an injective map. This image is described as an algebraic subset defined by polynomial relations given by $R_1, \ldots, R_m, \ldots$; that is, each R_i yields $R_i(h(g_1), \ldots, h(g_n)) = \mathrm{I}$, which gives us a system of polynomial equations. (The polynomial relations will always be finitely many.) This follows since if the relations are satisfied, then we can obtain the representation conversely. Denote the space by $\mathbf{Hom}(\pi, G)$, which is an algebraic set.

There is an action of G on $\mathbf{Hom}(\pi_1(M), G)$ given by the conjugation action $(g \star h)(\cdot) = gh(\cdot)g^{-1}$. We denote by $\mathbf{Rep}(\pi_1(M), G)$ the quotient space $\mathbf{Hom}(\pi_1(M), G)/G$.

6.2.3.2 *The map* hol

We define $\mathrm{hol}' : \mathcal{S}(M) \to \mathbf{Hom}(\pi_1(M), G)$ by sending the equivalence class of $(D, \tilde{f})$ to a homomorphism $h \circ \tilde{f}_* : \pi_1(M) \to G$

This induces $\mathrm{hol} : \mathcal{D}_{(G,X)}(M) \to \mathbf{Rep}(\pi_1(M), G)$. We denote by $\mathbf{Hom}(\pi, G)^s$ the subset of $\mathbf{Hom}(\pi_1(M), G)$ where the conjugation action of G given by $h(\cdot) \to gh(\cdot)g^{-1}, g \in G$ is stable; i.e., the orbits are closed and the stabilizers are finite. (See Section 1 of [Johnson and Millson (1987)].) We denote by $\mathcal{S}^s_{(G,X)}(M)$ the inverse image of this set under hol' and a G-invariant set. Denote by $\mathcal{D}^s_{(G,X)}(M)$ the image of this set under the quotient map

$$\mathcal{S}(M) \to \mathcal{S}(M)/G$$

and denote by $\mathbf{Rep}(\pi, G)^s$ the quotient image of $\mathbf{Hom}(\pi, G)^s$.

When M is disconnected as in Chapter 7, the deformation space $\mathcal{D}_{(G,X)}(M)$ is defined as the product space $\prod_{i=1}^n \mathcal{D}_{(G,X)}(M_i)$ for components $M_1, \dots, M_n$ and $\mathbf{Rep}(\pi_1(M), G)$ is also defined as the product space $\prod_{i=1}^n \mathbf{Rep}(\pi_1(M_i), G)$. Also, similarly, we define

$$\mathcal{D}^s_{(G,X)}(M) := \prod_{i=1}^n \mathcal{D}^s_{(G,X)}(M_i), \ \mathbf{Rep}(\pi_1(M), G)^s := \prod_{i=1}^n \mathbf{Rep}(\pi_1(M_i), G)^s.$$

The main purpose of this section is to state:

Theorem 6.2.2. *Suppose that M is a closed n-orbifold. Then* hol *restricts to a local homeomorphism*

$$\mathcal{D}^s_{(G,X)}(M) \to \mathbf{Rep}(\pi_1(M), G)^s.$$

It is sufficient to prove for the case when M is connected. We just give an informal discussion here since the proof is very complicated (see [Choi (2004)] for details): We send the equivalence class of $(D, \tilde{f})$ to the associated homomorphism $h : \pi_1(M) \to G$. First, it is easy to show that hol' is continuous: If $D' \circ \tilde{f}'$ is sufficiently close to $D \circ f$ in a sufficiently large compact subset of $\tilde{M}$, then the holonomy $h'(g_i)$ of generators g_i are as close to the original $h(g_i)$ as needed.

Conversely, given a geometric structure corresponding to h, if one deforms h by a small amount, then we can change the geometric structure correspondingly by considering local models and changing them slightly and patching up the differences in a consistent way. Finally, we have to show that such a change of a geometric structure is unique up to isotopies.

6.3 Notes

The local homeomorphism result, introduced by Weil (1960, 1962), was a very important and subtle result for the study of deformations of (G, X)-structures on manifolds. For manifolds, Thurston (1977) (and Ehresmann) gave a proof. Later J. Morgan gave a series of lectures on it, which is written up by Walter Lok in Section 1.1 of [Lok (1984)]. Also, Canary, Epsten, and Green gave a short proof of it also (Canary, Epstein, Green, 1987). See also Chapter 7 of [Kapovich (2009)].

Actually, we can find a short transversal section proof given by Goldman (1987) in the manifold cases. It should be possible to modify this proof for the orbifold cases as well. But the proof is conceptually not different.

The main part of this chapter is from the papers [Choi (2004); Choi and Goldman (2005)]. Chapter 6 of the book [Kapovich (2009)] also devotes some pages to geometric orbifolds. The principal bundles, transversal sections, and flat connections are very interconnected and we think that this gives a very pleasant picture of geometric structures and shows that the notion of geometric structures is intrinsic to nature.