Chapter 5

Topology of 2-orbifolds: 2-orbifold topological constructions

We now wish to concentrate on 2-orbifolds to illustrate more concretely. In many cases, the theory is much easier to understand. Also, we study the topological constructions of 2-orbifolds. We will follow the papers [Choi and Goldman (2005); Scott (1983)].

We first classify smooth 2-orbifolds with possibly empty boundary up to diffeomorphisms. Next 1-dimensional suborbifolds are classified. We discuss the Euler characteristic and the Riemann-Hurwitz formula. We classify the bad orbifolds by discussing about the good, very good, and bad 2-orbifolds. (At present, we can do this for 2-orbifolds only. For higher dimensions, these may not be appropriate terminologies even.)

In the rest of the chapter, we discuss topological cut-and-paste methods applicable to 2-orbifolds.

5.1 The properties of 2-orbifolds

Recall that the singular points of a two-dimensional orbifold fall into three types (See Figure 4.7):

(i) The mirror point: $\mathbb{R}^2/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts by reflections on the y-axis.
(ii) The cone-points of order n: $\mathbb{R}^2/\mathbb{Z}_n$ where $\mathbb{Z}_n$ acting by rotations by angles $2\pi m/n$ for integers m.
(iii) The corner-reflector of order n: $\mathbb{R}^2/D_n$ where D_n is the dihedral group generated by reflections about two lines meeting at an angle π/n.

From this, we obtain that the underlying space of a 2-orbifold is a surface with corner.

The singular strata associated with conjugate local groups are as follows: a silvered point belongs to a 1-dimensional strata, called a *silvered arc*. The other types have isolated points as strata. Recall that boundary of a 2-orbifold is a suborbifold. The silvered arc may have an end point in the boundary of the 2-orbifold and it may end in a corner-reflector of order ≥ 2 also but not at a cone-point by the local group considerations.

- On the boundary of a surface with a corner, one can choose a collection of mutually disjoint maximal smooth open arcs ending at corners. If two half-arcs in the distinguished arcs end at a corner-point, then the corner-point is a *distinguished one*. If only one of the chosen arc ends at the corner, the corner-point is *ordinary*. The diffeomorphism type of the surface with the collection of chosen arcs will be called the *boundary pattern*.
- Recall Example 4.2.4: given a surface with corner and a collection of discrete points in its interior and the boundary pattern, we can put an orbifold structure on it so that the selected interior points become cone-points and the distinguished corner-points the corner-reflectors of given order and the ordinary end points and points of chosen arcs the silvered points.

Theorem 5.1.1. *Any* 2*-orbifold is obtained from a smooth surface with corner by silvering some arcs and putting cone-points and corner-reflectors. The smooth orbifold topology of* 2*-orbifold is classified by the underlying smooth topology of the surface with corner and the cardinality and orders of cone-points, corner-reflectors, and the boundary pattern of silvered arcs.*

Proof. Given a 2-orbifold, we can forget the orbifold structure and we obtain a smooth surface with corner. Thus, we can obtain the orbifold back by doing the construction as above.

Given two 2-orbifolds Σ_1 and Σ_2 with the same boundary pattern of silvered arcs and corner reflectors and cone-points, we remove the neighborhoods of small corner-reflectors and cone-points diffeomorphic to disks or disks intersected with $\mathbb{R}^2_+$ to obtain Σ_1' and Σ_2'. Then there is a diffeomorphism $f : \Sigma_1' \to \Sigma_2'$ considering Σ_1' and Σ_2' as surfaces with corner. Only silvered arcs remain as the singular sets. f can be isotoped to an orbifold-diffeomorphism.

We can extend f radially at each neighborhood of corner-reflectors and cone-points where we have to smooth the maps radially. We obtain a smooth orbifold diffeomorphism. □

A *full* 1*-orbifold* is an orbifold with the underlying space homeomorphic to an interval and where both endpoints are singular. A boundary full 1-orbifold is a full 1-orbifold in the boundary of 2-orbifold. (Recall Remark 4.2.5.)

Let Σ be a surface with corners. Thurston's notation of a closed 2-orbifold with base space Σ is given by $\Sigma(j_1, j_2, \ldots; k_1, k_2, \ldots)$ where $j_1, j_2, \ldots$ indicate the orders of the cone-points and $k_1, k_2, \ldots$ indicate the orders of the corner-reflectors. This will determine the orbifold diffeomorphism class by additionally specifying on which boundary components of the underlying space Σ the corner-reflectors and the boundary full 1-orbifolds are and in what way by combinatorially ordering these on the boundary components.

From now on, a *special singular point* will mean either a cone-point or a corner-reflector.

5.1.1 *The triangulation of 2-orbifolds*

For 2-orbifolds, the Riemannian metric and triangulation can be approached in much simpler manner. (See Chapter 4 for the full generality.)

Proposition 5.1.2. *One can put a Riemannian metric on a 2-orbifold so that the boundary of the underlying cornered surface is a union of geodesic arcs and each corner-reflector has angles π/n for its order n and each cone-point has angles $2\pi/m$ for its order m. One can give a triangulation by smooth triangles so that slivered arcs and boundary curves are in the union of 1-skeletons and corner-reflectors and cone-points are in 0-skeletons.*

Proof. First construct such a Riemannian metric on the boundary by putting such metrics on the boundary by using a broken geodesic in the Euclidean plane and a singular Euclidean metric around the cone points. (See Example 4.2.4.) We require that the corner-reflector of order n has angle π/n and the cone points of order m have a Euclidean angle $2\pi/m$. Put a Riemannian metric on the surface with neighborhoods of singular points removed so that the boundary is geodesic. We extend the metric to the interior using partition of unity.

By removing open balls around cone-points and corner-reflectors, we obtain a smooth surface with corners and without singular points.

Now we can find a smooth triangulation so that the interior of each edge of a triangle is completely inside the boundary with the corner points removed. Finally we extend the triangulation by cone-construction to the interiors of the removed balls. □

5.1.2 *The classification of 1-dimensional suborbifolds of 2-orbifolds*

A compact 1-orbifold is either a closed arc homeomorphic to a circle, a segment, a segment with one silvered endpoint, or a segment with two silvered endpoint.

Recall that a *neatly* embedded suborbifold is an embedded suborbifold so that its boundary is in the boundary of the ambient orbifold and each point of the boundary has a neighborhood in the orbifold modeled on a half space H^n with the suborbifold neighborhood modeled on another half space H^m embedded in it for $0 \leq m \leq n$. (See Section 4.4.2.) A properly and neatly embedded 1-orbifold in a 2-orbifold with boundary either avoids the singular sets in its topological interior or is entirely contained in a singular set. In the former case we have:

- No silvered-point case: An embedded closed arc avoiding boundary or singular points or an arc segment with two endpoints in the boundary avoiding singularities.

- One silvered-point case: An arc segment with silvered endpoint at a cone-point of order two or in the interior of a silvered arc and the other endpoint in the boundary.
- Two silvered-points case: An arc segment with silvered endpoints at cone-points of order two or in the interiors of silvered arcs. (This is a full 1-orbifold, which may or may not be the boundary one.)

By a *silvered edge*, we mean a maximal arc whose interior is silvered. (The endpoints must be also singular.)

If the neatly embedded 1-orbifold is in the singular set, we classify them as below:

- a silvered embedded closed arc,
- a maximal segment in a silvered edge with two end points in the boundary of the orbifold,
- a segment in a silvered edge with one endpoint in a corner reflector of order two and the other in the boundary, and
- a segment in a silvered edge with two endpoints in two corner-reflectors of order two. (This is a full 1-orbifold but not the boundary one.)

(It might be useful to recall Remark 4.2.5.)

These are all possible compact neatly embedded 1-orbifolds and we will assume that our 1-orbifolds are of these types only.

5.1.3 *The orbifold Euler-characteristic for orbifolds due to Satake*

Let $\mathcal{O}$ be a compact n-dimensional orbifold with boundary. Then $\mathcal{O}$ admits a finite triangulation by Theorem 4.5.4. Thus, the underlying space of $\mathcal{O}$ has a cell-decomposition. We defined the Euler characteristic to be

$$\chi(\mathcal{O}) = \sum_{c_i} (-1)^{\dim(c_i)} \frac{1}{|\Gamma(c_i)|},$$

where c_i ranges over the open cells and $|\Gamma(c_i)|$ is the order of the group $\Gamma(c_i)$ associated with a point of c_i. (The order is independent of the point in a given open cell.)

Proposition 5.1.3. *Let X be an n-dimensional orbifold for $n \geq 1$. If X is finitely covered by another orbifold X', then $\chi(X') = r\chi(X)$ where r is the number of sheets for regular points.*

Proof. Each point of a strata with a local group G' in X has the inverse image equal to the set of points $p_i, i = 1, \ldots, n$ in a respective stratum S_i in X' with local group G'_i so that

$$r = \sum_{i=1}^{n} \frac{|G'|}{|G'_i|}.$$

This is verified by taking a nearby regular point and counting the inverse images near p_is.

Thus, the inverse image of an open cell c with the local group $\Gamma(c)$ consists of a union of open cells $c_1, \ldots, c_n$ with local groups $\Gamma(c_i)$ of same dimension where we have

$$\frac{r}{|\Gamma(c)|} = \sum_{i=1}^{n} \frac{1}{|\Gamma(c_i)|}.$$

□

The Euler-characteristic of a compact 1-orbifold is as follows: a circle 0, a segment 1, a segment with one silvered-point 1/2, a full 1-orbifold 0.

A *separating* 1-orbifold Y in a 2-orbifold Σ is a 1-dimensional suborbifold in Σ with the topological interior $|Y|^o$ of the underlying space $|Y|$ of Y is in the topological interior $|\Sigma|^o$ of the underlying space $|\Sigma|$ of Σ. so that $|\Sigma| - |Y|$ has two components and moreover, for each point x of Y, the local group H_x of x is included isomorphic to the local group G_x of x in Σ for the model triple embedding $(I, H_x, \phi) \to (U, G_x, \psi)$ where (I, H_x, ϕ) is a model-triple for x in Y and (U, G_x, ψ) is a model-triple for x in Σ.

In fact, this is true for the following two cases only:

- Y is a simple closed curve and has no singularity and lies in the interior of the underlying surface $|\Sigma|$, or
- Y has two silvered points in the interior of silvered arcs in the boundary of $|\Sigma|$ and the interior of $|Y|$ is in the interior of $|\Sigma|$.

In this case, $\Sigma - Y$ then completes with respect to the path-metric into a union of two suborbifolds Σ_1 and Σ_2.

Assuming this, we have the following additivity formula:

$$\chi(\Sigma) = \chi(\Sigma_1) + \chi(\Sigma_2) - \chi(Y). \tag{5.1}$$

The formula is to be verified by counting open cells with weights since the orders of singular points in the boundary orbifold equal the ambient orders.

5.1.4 *The generalized Riemann-Hurwitz formula*

Suppose that a 2-orbifold Σ with or without boundary has the compact underlying space X_Σ and m cone-points of order q_i and n corner-reflectors of order r_j and n_Σ boundary full 1-orbifolds.

Then the following generalized *Riemann-Hurwitz formula* is very useful:

$$\chi(\Sigma) = \chi(X_\Sigma) - \sum_{i=1}^{m} \left(1 - \frac{1}{q_i}\right) - \frac{1}{2} \sum_{j=1}^{n} \left(1 - \frac{1}{r_j}\right) - \frac{1}{2} n_\Sigma, \tag{5.2}$$

where q_i, $i = 1, \ldots, m$, are the orders of cone points and r_j, $j = 1, \ldots, n$, are the orders of corner-reflectors and n_Σ is the number of boundary full 1-orbifolds, i.e., the full 1-orbifolds in the boundary of the orbifold Σ.

We prove this formula by a doubling argument and cutting and pasting using equation 5.1. (See [Scott (1983)] for details):

We double the 2-orbifold Σ to Σ'. (See Section 4.6.1.2.) Now we have only closed curve boundary components and cone-points. Then $\chi(\Sigma')$ equals

$$\chi(X_{\Sigma'}) - 2\sum_{i=1}^{m}\left(1 - \frac{1}{q_i}\right) - \sum_{j=1}^{n}\left(1 - \frac{1}{r_j}\right)$$

as can be verified by decomposing Σ' by cutting out disks around the cone-points. We have $\chi(\Sigma) = \chi(\Sigma')/2$ by Proposition 5.1.3, and $\chi(X_{\Sigma'})$ equals $2\chi(X_\Sigma) - n_\Sigma$.

To explain more: while we cannot yet do the full cutting and pasting constructions for 2-orbifolds which we do from Section 5.2.1 to the end of this chapter, we can do this when the 1-orbifolds are separating;

5.1.5 *A geometrization of* 2*-orbifolds* : *a partial result*

Proposition 5.1.4. *Let S be a* 2*-orbifold whose underlying space is a disk with at least one special singularity and has nonempty (orbifold) boundary or a disk or* 2*-sphere with at least three special singular points. Then S is very good and so is regularly covered by a compact surface.*

Proof. First, cover S by a double-cover $\hat{S}$ if S contains silvered points. (See Proposition 4.4.3.) Otherwise let $\hat{S}$ be S. Then $\hat{S}$ has only cone-points. Let n be the number of the cone-points, and let $p_1, p_2, \ldots, p_n$ denote their orders. The underlying space is a sphere or a planar surface. Now, the boundary of $\hat{S}$ is a disjoint union of simple closed curves if the underlying space is a planar surface. Let k be the number of boundary components. Then we have either $n \geq 3$ or have $n \geq 1$ and $k \geq 1$.

If $\hat{S}$ has just one singular point that has to be a cone-point, with one boundary component, then $\hat{S}$ is regularly covered by a smooth disk without singularity and we are done. Assume we have either $n \geq 3$, have $n = 2$ and $k \geq 1$, or have $n = 1$ and $k \geq 2$.

We can construct an orbifold structure on a planar subsurface $\mathcal{P}$, diffeomorphic to $|\hat{S}|$, on a 2-sphere $\mathbf{S}^2$ in $\mathbb{R}^3$ so that the cone-points are on the xy-plane, each boundary circle is symmetric with respect to the reflection on the xy-plane, and the reflection on the xy-plane restricts to an orbifold-involution. Denote the resulting orbifold by $\hat{\mathcal{P}}$. By constructing $\hat{\mathcal{P}}$ to have the same number of cone-points as $\hat{S}$, we see that $\hat{S}$ is diffeomorphic to $\hat{\mathcal{P}}$ by Theorem 5.1.1. Now, it follows that the orbifold $\hat{S}$ covers an orbifold Σ with n corner-reflectors on a disk D^2 with orders $p_1, p_2, \ldots, p_n$ and with k number of boundary full 1-orbifold.

One can construct a geodesic polygon P with angles $\pi/p_1, \pi/p_2, \ldots, \pi/p_n$ and $2k$ angles of $\pi/2$ on the 2-sphere, a Euclidean plane, or a hyperbolic plane depending on whether $\sum_{i=1}^{n} \pi(1-1/p_i) + k\pi$ is smaller than 2π, equal to 2π or greater than 2π. (See Proposition 3.2.2.) (Note that P has at least three vertices.) Then Σ is diffeomorphic to the quotient orbifold of a domain in a 2-sphere, a Euclidean plane, or a hyperbolic plane by the generated reflection group. Thus, $\hat{S}$ admits a spherical, Euclidean, or hyperbolic structure with geodesic boundary. By Selberg's lemma (Corollary 4 in Chapter 7 of the book [Ratcliffe (2006)]), the group $\pi_1(\hat{S})$ has a finite-index normal subgroup that is torsion-free consisting of orientation-preserving isometries. The corresponding covering is an orientable surface $\hat{S}'$ since the group is torsion-free and acts on a sphere, a Euclidean space or a hyperbolic space. Moreover, we can choose the surface $\hat{S}'$ covering regularly $\hat{S}$ by taking the intersection of conjugates of the finite index subgroup $\pi_1(\hat{S}')$ in the fundamental group $\pi_1(\hat{S})$. Finally, $\hat{S}$ covers S regularly. $\square$

5.1.6 *Good, very good, and bad 2-orbifolds*

The purpose of this section is to prove Theorem 5.1.5.

It is fairly easy to distinguish between the good and bad 2-orbifolds as Thurston (1977) shows. We will prove this here.

Since we know the existence of the universal cover of orbifolds from Chapter 4, we can cover any 2-orbifold S with a simply-connected 2-orbifold $\tilde{S}$.

Let S be a compact 2-orbifold with possibly empty boundary. We divide into two cases: $\chi(|S|) \leq 0$ and $\chi(|S|) > 0$ for the underlying space $|S|$, a cornered surface, of S.

If $\chi(|S|) \leq 0$, then the (topological) fundamental group of $|S|$ is infinite, and we obtain a noncompact (topological) cover of $|S|$ which is also an orbifold-cover of S as well. Hence the universal covering $\tilde{S}$ of S is noncompact. Suppose that $\tilde{S}$ has some singular points. Since we can do a doubling-operation otherwise, we see that $\tilde{S}$ has only cone-points. However, $\tilde{S}$ cannot have a cone-point: Otherwise, we can remove a disk-neighborhood D of any cone-point of order say k for an integer $k > 1$. Since $|\tilde{S}|$ is homeomorphic to a disk, and $|\tilde{S}| - D$ has an infinite cyclic fundamental group, we can cover $\tilde{S} - D$ by a k-fold cyclic cover. Hence, by pasting in a disk, we again obtain a nontrivial covering orbifold of $\tilde{S}$, which is absurd. Thus, $\tilde{S}$ is a surface and S is a good orbifold.

Now suppose that $\chi(|S|) > 0$. Thus, $|S|$ is homeomorphic to a sphere, a projective plane, or a disk.

Suppose that S is a disk with at least one cone-point or a corner-reflector with nonempty (orbifold) boundary or a disk or a 2-sphere with at least three cone-points and/or corner-reflectors. By Proposition 5.1.4, S is good.

Suppose that S is a projective plane with at least two cone-points. Then the double-cover of S is good by Proposition 5.1.4 again. Now, suppose that S is a projective plane with one cone-point. Then the double-cover of S is a sphere with two cone-points of identical orders, and is covered by a 2-sphere.

Suppose that S is a disk with empty boundary and has at most two special singular points. Then the double-cover is a sphere with two to four cone-points. If the number of cone points is greater than or equal to three, then S is good. Suppose that we have at most two cone-points. Then S is a disk with one-cone point or a disk with one or two corner-reflectors. In the first case, S is covered by a sphere with two cone-points of identical orders, and S is covered by a 2-sphere. Suppose that S is a disk with two corner-reflectors of identical orders. Then S is covered by a sphere with two cone-points of identical orders. We are left with a disk with one corner-reflector or two corner-reflectors of different orders.

Suppose that S is a 2-sphere with one or two cone-points. If the orders are identical in the second case, then S is regularly covered by a 2-sphere and is good. We are left with the case when S is a 2-sphere with one or two cone-points of distinct orders.

A sphere Σ with cone points of orders p and q with p and q relatively prime is not covered by a manifold since the fundamental group is trivial by the Van Kampen theorem. (See Section 4.7.1.3.) A sphere with one cone point is also not covered by a manifold by the same reason.

The universal cover of a sphere with one cone-point, a sphere with two cone-points of distinct orders, a disk with one cone-point, or a disk with two corner-reflectors of distinct orders is covered by a sphere with one-cone point or two cone-points of relatively prime orders p and q. Hence, we conclude that a sphere with one cone-point, a sphere with two cone-points of distinct orders, a disk with one corner-reflector and a disk with two corner-reflectors of distinct orders are bad, and they are the only bad 2-orbifolds.

We will now continue to show that compact 2-orbifolds are very good except for bad ones.

Theorem 5.1.5. *A sphere with one or two cone-points with orders m and n where $m \neq n$ is a bad orbifold. So is a disk with silvered edges and one or two corner-reflectors of order m and n where $m \neq n$ are bad. Except for these, every other compact 2-orbifold is good. Furthermore, compact good orbifolds are very good. In fact, we can assume that the finite covering is always regular.*

Proof. The first parts were proved in above paragraphs.

We need to show the final statement only by the above discussions. By double-covering, the 2-orbifolds can be assumed to be orientable and have cone-points only as singular points. Let $\mathcal{O}$ denote a 2-orbifold.

If the underlying space is of Euler characteristic ≥ 1, then there is a covering by an orbifold whose underlying space is a sphere or a disk. This was studied above and was shown to be bad or is good. The good ones are very good according to Proposition 5.1.4.

Now suppose that the Euler characteristic of the underlying space is ≤ 0. There exists a disk D containing all the cone-points. Let $\mathcal{D}$ denote the corresponding orbifold. By Proposition 5.1.4, there is a finite regular covering surface S.

The boundary component of D is covered by m boundary components of S and each boundary component of S covers the boundary component of D by n-fold coverings for identical n. The closure of the complement of D is a surface S' of negative Euler characteristic and has infinite homology $H_1(S')$. Suppose that $\partial S'$ has other component than ∂D, then we can find a homomorphism from the group $\pi_1(S')$ of deck transformations of $\tilde{S}'$ to $\mathbb{Z}_n$ so that a simple closed curve in ∂D maps to the generator. Then the kernel of the homomorphism gives us a finite regular covering S'' of the complement S'. We see that S'' has a boundary component mapping to S' in an n-fold way. Hence by attaching copies of S'' for each boundary component of S, we obtain a very good cover of the original orbifold.

Suppose that $\partial S'$ has no other component than ∂D. We can explicitly find a homomorphism $\pi_1(S') \to S_{2n}$ where S_{2n} is a permutation group of $2n$ elements where a simple closed curve in $\partial S'$ is mapped to an order n element: Let the homotopy class of $\partial S'$ is written as $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$ for generators $a_i, b_i, i = 1, \ldots, g$, $g \geq 1$, of $\pi_1(S')$. Then we find a homomorphism $\phi : \pi_1(S') \to S_{2n}$ by

- $\phi(a_1) = (1, 2, \cdots, n)$,
- $\phi(b_1) = (1, n+1)(2, n+2) \cdots (n, 2n)$, and
- $\phi(a_i) = \phi(b_i) = \mathrm{I}$ for $i \geq 2$.

Again, we can find S'' as above. (We thank the referee for supplying this part.)

By construction, this is a regular covering. (The proof follows the article [Scott (1983)].) □

Remark 5.1.6. The universal cover $\tilde{S}$ of a noncompact 2-orbifold S cannot have singularity: We can assume that $\tilde{S}$ is orientable and hence has only cone-points as singular points. If there is a cone-point, then we can find a cylic branched cover of $\tilde{S}$ by taking a properly embedded arc from the cone-point leaving every compact subset. This is absurd. Thus, noncompact 2-orbifolds are always good.

A noncompact 2-orbifold may not be very good since there is an example with a sequence of cone points of strictly increasing orders. However, noncompact but precompact 2-orbifolds are always very good by Theorem 5.1.5.

5.2 Topological operations on 2-orbifolds: constructions and decompositions

We will now study the question of how to construct and decompose 2-orbifolds:

- Definition of splitting and sewing of 2-orbifolds
- Regular neighborhoods of 1-orbifolds

- Reinterpretation of splitting and sewing.
- Identification interpretations of splitting and sewing

5.2.1 *The definition of splitting and sewing 2-orbifolds*

We will assume that the 2-orbifolds are very good from now on. (Actually, it is sufficient for our purpose that neighborhoods of the involved 1-orbifolds are very good.)

Let S be a very good 2-orbifold so that its underlying space X_S is a pre-compact open surface with a path-metric admitting a compactification to a surface with boundary. Such a metric always exists and the topology of the compactification is unique up to homeomorphism type. Let $\hat{S}$ be a very good cover, that is, a finite regular cover, of S, so that S is orbifold-diffeomorphic to $\hat{S}/F$ where F is a finite group acting on $\hat{S}$.

Since $X_{\hat{S}} = \hat{S}$ is also pre-compact and has a path-metric, we complete it to obtain a compact surface $X'_{\hat{S}}$ with respect to the path metric. The action of the group F extends to $\hat{S}$ by the path-metric. Then $X'_{\hat{S}}/F$ with the quotient orbifold structure is said to be the *orbifold-completion* of S.

- Let S be a 2-orbifold with an embedded circle or a full 1-orbifold l in the interior of S. Obtaining the orbifold-completion $\hat{S}$ of $S - l$ with respect to the path-metric is called *splitting* S along l. Since $S - l$ has an embedded copy in $\hat{S}$, we see that there exists a map $\hat{S} \to S$ sending the copy to $S - l$. Let $\hat{l}$ denote the union of boundary components of $\hat{S}$ corresponding to l under the map.
- Conversely, S is said to be obtained from *sewing* $\hat{S}$ *along* $\hat{l}$.
- If the interior of the underlying space of l lies in the interior of the underlying space of S, then the components of $\hat{S}$ are said to be the *decomposed components of* S *along* l, and we also say that S *decomposes* into $\hat{S}$ along l.
- Of course, if l is a union of disjoint embedded circles or full 1-orbifolds, the same definitions hold.

There are two distinguished classes of splitting and sewing operations:

A simple closed curve boundary component can be made into a set of mirror points and conversely in a unique manner by Proposition 4.4.3.

A boundary full 1-orbifold can be made into a 1-orbifold of mirror points with two corner-reflectors of order two and conversely in a unique manner: (an end point has a neighborhood which is a quotient space of a dihedral group of order four acting on the open ball generated by two reflections.) The forward process is called *silvering* and the reverse process *clarifying.*

5.2.2 *Regular neighborhoods of 1-orbifolds*

5.2.2.1 *The classification of Euler-characteristic zero orbifolds*

Let A be a compact annulus with nonempty boundary. The quotient orbifold of an annulus has a zero Euler characteristic.

Proposition 5.2.1. *The compact 2-orbifolds with nonempty boundary and of zero Euler characteristic is as follows:*

(1) *an annulus,*
(2) *a Möbius band,*
(3) *an annulus with one boundary component silvered* (*a* silvered annulus),
(4) *a disk with two cone-points of order two with no mirror points* (*a* $(;2,2)$-disk),
(5) *a disk with two boundary* 1*-orbifolds, two silvered edges* (a silvered strip),
(6) *a disk with one cone-point of order* 2 *and one boundary full* 1*-orbifold* (*a* bigon with a cone-point of order two), *which has only one silvered edge,*
(7) *a disk with two corner-reflectors of order two and one boundary full* 1*-orbifold* (*a* half-square), *which has three silvered edges.*

Proof. The underlying space should have a nonnegative Euler characteristic by the Riemann-Hurwitz formula. If the Euler characteristic of the space is zero, there are none of cone-points, corner-reflectors, and boundary full 1-orbifolds and we obtain cases in (1), (2), or (3).

Suppose now that the underlying space is homeomorphic to a disk. If there is no singular point in the boundary, then (4) holds as there has to be exactly two cone-points of order two by the Riemann-Hurwitz formula. If there are two boundary full 1-orbifolds, then no singular points in the interior and no corner-reflector can exist; thus, (5) holds.

Suppose that exactly one boundary full 1-orbifold exists. If a cone-point exists, then it has to be a unique one and of order two, and (6) holds. If there are no cone-points, but corner-reflectors, then exactly two corner-reflectors of order two and no more. (7) holds. □

5.2.2.2 *Regular neighborhoods of* 1*-orbifolds*

Let l be a circle or a 1-orbifold in the interior of a 2-orbifold S so that $\pi_1(l)$ injects into $\pi_1(S)$. The image is clearly infinite and is not homotopic to a single point. In this case, l is said to be *essential.*

Let l be an essential 1-orbifold. Then l has a deck-transformation-group invariant neighborhood of zero Euler characteristic considering its very good cover and the deck-transformation-group invariant tubular neighborhoods. Since the inverse image of l consists of closed curves which represent generators, we deduce that l is contained in the neighborhood as follows.

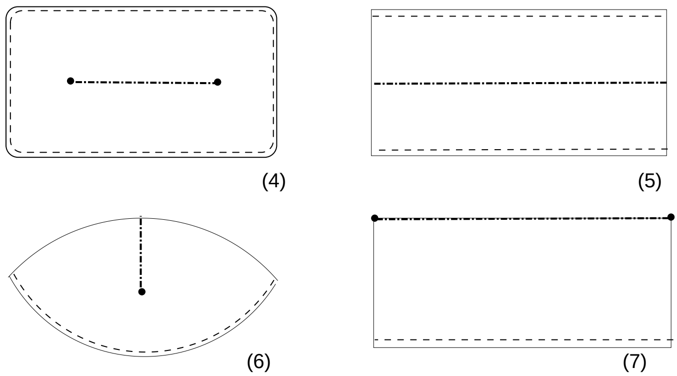

Fig. 5.1 For each orbifold, the arcs with dashed arc nearby are the boundary components and the thicker dotted arc is the 1-orbifold that the 2-orbifold is a regular neighborhood of. The black dot indicates the cone-point of order two or corner-reflectors of order two

- For (1) and (2), l is the closed curve representing the generator of the fundamental group;
- For (3), l is the mirror set that is a boundary component of the underlying space;
- For (4), l is the arc connecting the two cone-points unique up to isotopy.
- For (5), l is an arc connecting two interior points of two silvered edges respectively;
- For (6), l is an arc connecting an interior point of an silvered edge and an cone-point of order two;
- For (7), the silvered edge in the topological boundary connecting the two corner-reflectors of order two.

Given a 1-orbifold l and a neighborhood N of it in some ambient 2-orbifold as in Proposition 5.2.1, we say that N a *regular neighborhood* if the pair (N, l) is diffeomorphic to one of the above.

Proposition 5.2.2. *A* 1*-orbifold embeded as a suborbifold in a good* 2*-orbifold has a regular neighborhood which is unique up to isotopy.*

Proof. The existence is proved above. The uniqueness up to isotopy is proved as follows: In fact, regular neighborhoods are tubular neighborhoods if we use the Riemannian metric on the orbifolds. (See the end of Section 4.4.2 for details.) Each tubular neighborhood fibers over a 1-orbifold with fibers connected 1-orbifolds in the orbifold sense. We can isotopy any tubular neighborhood into any other tubular neighborhood by contracting in the fiber directions. To prove the uniqueness up to isotopy, we can modify the proof of Theorem 5.3 in Chapter 4 of the book [Hirsch (1976)] to be adopted to an annulus with a finite group acting on it and an embedded circle. $\square$

5.2.3 *Splitting and sewing on 2-orbifolds reinterpreted*

Let l be a 1-orbifold embedded in the interior of a 2-orbifold S. If one removes l from the interior of a regular neighborhood, we obtain either a union of one or two open annuli, or a union of one or two open silvered strips. In (2)-(4), an open annulus results. For (1), a union of two open annuli results. For (6)-(7), an open silvered strip results. For (5), we obtain a union of two open silvered strips. The results can be easily path-completed to be unions of one or two compact annuli or unions of one or two silvered strips respectively.

We complete $S - l$ in this manner: We take a closed regular neighborhood N of l in S. We remove $N - l$ to obtain the above types and complete it by the path metric and re-identify with $S - l$ to obtain a compactified 2-orbifold. This process is the splitting of S along l.

Conversely, we describe sewing: Take an open annular 2-orbifold N which is a regular neighborhood of a 1-orbifold l:

- Suppose that l is a circle. We obtain $U = N - l$ which is a union of one or two annuli.
- Take a 2-orbifold S' with a union l' of one or two boundary components which are circles.
- Take an open regular neighborhood of l' and remove l' to obtain V.
- Suppose that U and V are the identical suborbifolds. We identify $S' - l'$ and N along U and V.
- This gives us a 2-orbifold S, and S is obtained from S' by sewing along l'.

- Suppose that l is a closed curve and corresponds to a 1-orbifold l' in S'. We obtain (1),(2),(3)-type neighborhoods of l'' in this way. The operation in case (1) is said to be *pasting*, in case (2) *cross-capping*, and in case (3) *silvering* along simple closed curves.
- Suppose that l is a full 1-orbifold. $U = N - l$ is either an open annulus or a union of one (resp. two) silvered strips.
- The former happens if N is of type (4) and the latter if N is of type (5)-(7).
- In case (4), take a 2-orbifold S' with a boundary component l' a circle. Then we identify U with a regular neighborhood of l' with l' removed to obtain an orbifold S. Then l corresponds a full 1-orbifold l'' in S in a one-to-one manner. l'' has a type-(4) regular neighborhood. The operation is said to be *folding* along a simple closed curve. (See Section 4.2.2.)
- In the remaining cases, take a 2-orbifold S' with a union l' of one (resp. two) boundary full 1-orbifolds. Take a regular neighborhood N of l' and remove l' to obtain V. Identify U with V for $S' - l'$ and $N - l$ to obtain S. Then S is obtained from S' by sewing along l'. Again l corresponds to a full 1-orbifold l'' in S in a one-to-one manner.
- We obtain (5),(6), and (7)-type neighborhoods of l'' in this way, where the operations are said to be *pasting*, *folding*, and *silvering* along full 1-orbifolds

respectively.

- In other words, silvering is the operation of removing a regular neighborhood and replacing by a silvered annulus or a half square. Clarifying is an operation of removing the regular neighborhood and replacing an annulus or a silvered strip.

Proposition 5.2.3. *The Euler characteristic of a 2-orbifold before and after splitting or sewing remains unchanged.*

Proof. Form regular neighborhoods of the involved boundary components of the split 2-orbifold and those of the original 2-orbifold. They have zero Euler characteristics. Since their boundary 1-orbifolds have zero Euler characteristics, the lemma follows by the additivity formula (5.1). □

5.2.4 *Identification interpretations of splitting and sewing*

The sewing can be understood as follows: The pasting map f is defined on open neighborhood U of the union of the associated boundary components in an ambient open 2-orbifold S' where f satisfies the equation $\tilde{f} \circ \vartheta = \vartheta' \circ \tilde{f}$ where $\tilde{f}$ is a lift of f defined on $\tilde{U}$ the inverse image and ϑ and ϑ' are corresponding deck transformations acting on components of the inverse images in $\tilde{S}'$ of boundary components of f to be pasted by f.

In the following, we describe the topological identification process of the underlying space involved in these six types of sewings. The orbifold structure on the sewed orbifold should be clear.

Let a 2-orbifold Σ have a boundary component b. (Σ is not necessarily connected.) b is either a simple closed curve or a full 1-orbifold. We find a 2-orbifold Σ'' constructed from Σ by sewing along b or another component of Σ. (We also need the notation here for the later purposes.)

(A) Suppose that b is diffeomorphic to a circle; that is, b is a closed curve. Let Σ' be a component of the 2-orbifold Σ with boundary component b'. Suppose that there is a diffeomorphism $f : b \to b'$. Then we obtain a bigger 2-orbifold Σ'' glued along b and b' topologically.

  (I) The construction so that Σ'' does not create any more singular point results in a 2-orbifold Σ'' so that

$$\Sigma'' - (\Sigma - b \cup b')$$

  is a circle with a neighborhood either diffeomorphic to an annulus or a Möbius band.

  (1) In the first case, we have $b \neq b'$ (pasting).
  (2) In the second case, we have $b = b'$, and $\langle f \rangle$ is of order two without fixed points (cross-capping).

(II) When $b = b'$, the construction so that Σ'' does introduce more singular points to occur in a 2-orbifold Σ'' so that

$$\Sigma'' - (\Sigma - b)$$

is a circle of mirror points or is a full 1-orbifold with endpoints in cone-points of order two depending on whether $f : b \to b$

(1) is the identity map (silvering), or
(2) is of order two and has exactly two fixed points (folding).

(B) Consider when b is a full 1-orbifold with endpoints mirror points.

(I) Let Σ' be a component (possibly the same as one containing b) with boundary full 1-orbifold b' with endpoints mirror points where $b \neq b'$. We obtain a bigger 2-orbifold Σ'' by gluing b and b' by a diffeomorphism $f : b \to b'$. This does not create new singular points (pasting).
(II) Suppose that $b = b'$. Let $f : b \to b$ be the attaching map. Then

(1) if f is the identity, then b is silvered and the end points are changed into corner-reflectors of order two (silvering).
(2) If f is of order two, then Σ'' has a new cone-point of order two and has one-boundary component orbifold removed away. b corresponds to a 1-orbifold in Σ' (folding). This creates just one more singularity of cone-type of order 2.

We can easily put the obvious orbifold structure on Σ'' using the previous descriptions by regular neighborhoods above.

5.3 Notes

Low-dimensional orbifolds were first studied by Thurston (1977) who was to put much emphasis on cut and paste operations, a point of view not attempted before then. There was a nice exposition of the 2-orbifold theory in the article [Scott (1983)], which we followed here. Topological operations with orbifolds are widely used in many papers. They include the papers [Matsumoto and Montesinos-Amilibia (1991); Dunbar (1988); Takeuchi (1989, 1996); Choi and Goldman (2005)]. This topic is not so well treated in groupoid theoretical approach to orbifolds.