Chapter 2

Manifolds and $\mathcal{G}$-structures

In this chapter, we review many notions in the manifold theory that can be generalized to the orbifold theory.

We begin by reviewing manifolds and simplicial manifolds beginning with cell-complexes and the homotopy and covering theory. The following theories for manifolds will be transferred to orbifolds. We briefly mention them here as a "review" and develop them for orbifolds later (mostly for 2-dimensional orbifolds):

- Lie groups and group actions
- Pseudo-groups and $\mathcal{G}$-structures
- Differential geometry: Riemanian manifolds, principal bundles, connections, and flat connections

We follow a coordinate-free approach to differential geometry. We do not need to actually compute curvatures and so on.

Some of these are standard materials in a differentiable manifold course. We will not give proofs in Chapters 2 and 3 but will indicate one when necessary.

2.1 The review of topology

We present a review of the manifold topology. We will find that many of these directly can be generalized into the orbifold theory later.

2.1.1 *Manifolds*

The useful methods of topology come from taking equivalence classes and finding quotient topology. Given a topological space X with an equivalence relation, we give the quotient topology on $X/\sim$ so that for any function $f : X \to Y$ inducing a well-defined function $f' : X/\sim \to Y$, f' is continuous if and only if f is continuous. This translates to the fact that a subset U of $X/\sim$ is open if and only if $p^{-1}(U)$ is open in X for the quotient map $p : X \to X/\sim$.

A *cell* is a topological space homeomorphic to an n-dimensional open convex domain defined in $\mathbb{R}^n$ for $n \geq 0$. We will mostly use cell-complexes (Hatcher, 2002). A *cell-complex* is a topological space that is a union of n-skeletons X^n defined inductively. A 0-skeleton is a discrete set of points. Let $I = \bigcup_{n\geq 0} I_n$ be the collection of cells. An $(n+1)$-skeleton X^{n+1} is obtained from the n-skeleton X^n as a quotient space of $X^n \cup \bigcup_{\alpha \in I_{n+1}} D_\alpha^{n+1}$ for a collection of $(n+1)$-dimensional balls D_α^{n+1} for $\alpha \in I_{n+1}$ with a collection of functions $f_\alpha : \partial D_\alpha^{n+1} \to X^n$ so that the equivalence relation is given by $x \sim f_\alpha(x)$ for $x \in \partial D_\alpha^{n+1}$. To obtain the topology of X, we use the weak topology that a subset U of X is open if and only if $U \cap X^n$ is open for every n. Most of the times, cell-complexes will be *finite* ones, i.e., have finitely many cells.

A topological n-dimensional manifold (n-manifold) is a Hausdorff space with a countable basis and charts to a Euclidean space E^n; e.g curves, surfaces, and 3-manifolds. The charts could also go to a positive half-space H^n defined by $x_0 \geq 0$ in $\mathbb{R}^n$ for a coordinate function x_0 of $\mathbb{R}^n$. Then the set of points mapping to $\{0\} \times \mathbb{R}^{n-1}$ under charts is well-defined and is said to be the boundary of the manifold. By the invariance of domain theorem, we see that this is a well-defined notion.

For example, $\mathbb{R}^n$ and H^n themselves or open subsets of $\mathbb{R}^n$ or H^n are manifolds of dimension n.

The unit sphere $\mathbf{S}^n$ in $\mathbb{R}^{n+1}$ is a standard example. The quotient space $\mathbb{R}^{n+1} - \{O\}$ by the relation $v \sim w$ for $v, w \in \mathbb{R}^{n+1}$ if $v = sw$ for $s \in \mathbb{R} - \{O\}$ is called the real projective space $\mathbb{RP}^n$ and is another example.

An n-ball is a manifold with boundary. The boundary is the unit sphere $\mathbf{S}^{n-1}$.

Given two manifolds M_1 and M_2 of dimensions m and n respectively, we obtain the product space $M_1 \times M_2$ a manifold of dimension $m + n$.

An annulus is a disk removed with the interior of a smaller disk. It is also homeomorphic to a circle times a closed interval.

A manifold M is a smooth manifold if it has an atlas of charts of form (U, ϕ) where U is an open subset of M and ϕ is a homeomorphism from U to an open subset of $\mathbb{R}^n$ or H^n and transition functions between charts are all smooth.

A *smooth map* $f : M \to N$ for two smooth manifolds M and N is a map represented by smooth maps under coordinate systems of M and N; i.e., $\phi \circ f \circ \psi^{-1}$ is a smooth map from an open subset of Euclidean space or a half-space to another Euclidean space for coordinate charts ϕ of N and ψ of M. A *diffeomorphism* $f : M \to N$ of two smooth manifolds M and N is a smooth map with a smooth inverse map f^{-1}.

Example 2.1. The n-dimensional torus T^n is homeomorphic to the product of n circles $\mathbf{S}^1$. (For 2-torus, see `http://en.wikipedia.org/wiki/Torus` for its embeddings in $\mathbb{R}^3$ and so on.)

A group G *acts* on a manifold M if there is a differentiable map $k : G \times M \to M$ so that $k(g, k(h, x)) = k(gh, x)$ and $k(\mathrm{I}, x) = x$ for $x \in M$ and the identity $\mathrm{I} \in G$. Given an action of G on a manifold, one obtains a homomorphism $G \to \mathbf{Diff}(M)$ so that an element $g \in G$ goes to a diffeomorphism $g' : M \to M$ sending x to $k(g, x)$

where $\mathbf{Diff}(M)$ is the group of diffeomorphisms of M.

Given a group G acting on a manifold M, we obtain the quotient space $M/\sim$ where $\sim$ is given by $x \sim y$ if and only if $x = g(y), g \in G$, which is denoted by M/G. Let $e_i, i = 1, 2, \ldots, n$, denote the standard unit vectors in $\mathbb{R}^n$. Let T_n be a group of translations generated by $T_i : x \mapsto x + e_i$ for each $i = 1, 2, \ldots, n$. Then $\mathbb{R}^n/T_n$ is homeomorphic to T^n.

Example 2.2. We define the *connected sum* of two n-manifolds M_1 and M_2. Remove the interior of the union of two disjointly and tamely embedded closed balls from M_i for each i. Then each M_i has a boundary component homeomorphic to $\mathbf{S}^{n-1}$. We identify the spheres.

Take many 2-dimensional tori or projective planes and do connected sums. Also remove the interiors of some disks. We can obtain all compact surfaces in this way. (See `http://en.wikipedia.org/wiki/Surface`.)

2.1.2 *Some homotopy theory*

We will assume that our topological spaces here are path-connected, locally path-connected and semi-locally simply connected unless we mention otherwise. Here, the maps are always assumed to be continuous.

Let X and Y be topological spaces. A *homotopy* is a map $F : X \times I \to Y$ for an interval I. Two maps f and $g : X \to Y$ are *homotopic* by a homotopy F if $f(x) = F(x, 0)$ and $g(x) = F(x, 1)$ for all x. The homotopic property is an equivalence relation on the set of maps $X \to Y$. A *homotopy equivalence* of two spaces X and Y is a map $f : X \to Y$ with a map $g : Y \to X$ so that $f \circ g$ and $g \circ f$ are homotopic to I_Y and I_X respectively. (See the book [Hatcher (2002)] for details of the homotopy theory presented here.)

The fundamental group of a topological space X is defined as follows: A *path* is a map $f : I \to X$ for an interval $I = [a, b]$ in $\mathbb{R}$. We will normally use $I = [0, 1]$. An endpoint of the path is $f(0)$ and $f(1)$.

Any two paths $f, g : I \to \mathbb{R}^n$ are homotopic by a linear homotopy that is given by $F(t, s) = (1 - s)f(t) + sg(t)$ for $(t, s) \in [0, 1]^2$.

A *homotopy class* is an equivalence class of homotopic maps relative to endpoints.

The *fundamental group* $\pi_1(X, x_0)$ at the *base point* x_0 is the set of homotopy class of paths with both endpoints x_0.

The product in the fundamental group is defined by joining. That is, given two paths $f, g : I \to X$ with endpoints x_0, we define a path $f * g$ with endpoints x_0 by setting $f * g(t) = f(2t)$ if $t \in [0, 1/2]$ and $f * g(t) = g(2t - 1)$ if $t \in [1/2, 1]$. This induces a product $[f] * [g] = [f * g]$, which we need to verify to be well-defined with respect to the equivalence relation of homotopy. The constant path c_0 given by setting $c_0(t) = x_0$ for all t satisfies $[c_0] * [f] = [f] = [f] * [c_0]$. We denote $[c_0]$ by I_{x_0}. Given a path f, we define an inverse path $f^{-1} : I \to X$ by setting $f^{-1}(t) = f(1 - t)$. We also obtain $[f^{-1}] * [f] = \mathrm{I}_{x_0} = [f] * [f^{-1}]$. By verifying

$[f] * ([g] * [h]) = ([f] * [g]) * [h]$ for three paths with endpoints x_0, we see that the fundamental group is a group.

If we change the base to another point y_0 which is in the same path-component of X, we obtain an isomorphic fundamental group $\pi_1(X, y_0)$. Let γ be a path from x_0 to y_0. Then we define $\gamma^* : [f] \in \pi_1(X, x_0) \mapsto [\gamma^{-1} * f * \gamma]$ which is an isomorphism. The inverse is given by $\gamma^{-1,*}$. This isomorphism does depend on γ and hence cannot produce a canonical identification.

Theorem 2.1.1. *The fundamental group of a circle is isomorphic to* $\mathbb{Z}$.

This has a well-known corollary, the Brouwer-fixed-point theorem, that a self-map of a disk to itself always has a fixed point.

Given a map $f : X \to Y$ with $f(x_0) = y_0$, we define a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $f_*([h]) = [f \circ h]$ for any path h in X with endpoints x_0.

Theorem 2.1.2 (Van Kampen). *We are given a path-connected space X covered by open path-connected subsets A_i, $i \in I$, containing a common point x_0 for an index set I and such that every intersection of any two or three members is a nonempty path-connected set. Then $\pi_1(X, x_0)$ is a quotient group of the free product $*_{i \in I} \pi_1(A_i, x_0)$. The kernel is the normal subgroup generated by $i_j^*(a) i_k^*(a)^{-1}$ for all a in $\pi_1(A_j \cap A_k, x_0)$.*

A *bouquet of n circles* is the quotient space of a union of n circles with one point from each identified with one another. Then the fundamental group at a basepoint x_0 is isomorphic to a free group of rank n.

For cell-complexes, this theorem is useful for computing the fundamental group: If a space Y is obtained from X by attaching the 2-cells, then $\pi_1(Y, y_0)$ is isomorphic to $\pi_1(X, y_0)/N$ where N is the normal subgroup generated by "boundary curves" of the attaching maps where y_0 is a basepoint in Y.

We will later compute the fundamental groups of surfaces using this method.

2.1.3 *Covering spaces and discrete group actions*

Given a manifold M, we define a *covering map* $p : \tilde{M} \to M$ from another manifold $\tilde{M}$ to be a surjective map such that each point of M has a neighborhood O such that $p|p^{-1}(O) : p^{-1}(O) \to O$ is a homeomorphism for each component of $p^{-1}(O)$. Normally $\tilde{M}$ is assumed to be connected. (See Chapter 5 of the book [Massey (1987)].)

Consider $\mathbf{S}^1$ as the set of unit length complex numbers. The coverings of a circle $\mathbf{S}^1$ are given by $f : \mathbf{S}^1 \to \mathbf{S}^1$ defined by $x \mapsto x^n$. These are finite to one covering maps. Define $\mathbb{R} \to \mathbf{S}^1$ by $t \mapsto \exp(2\pi t i)$. Then this is an infinite covering.

Example 2.3 (Standard Example). Consider a closed disk with interiors of a finite number of disjoint smaller disks removed. Then draw mutually disjoint arcs from the boundary of the disk to all the boundary curves of the smaller disks. We remove mutually disjoint open regular neighborhoods of the disjoint arcs. Call these strips. Let $D, I_1, I_2, \ldots, I_n$ denote the closures of the complement of the union of the strips and the strips themselves. Let α_i^+, α_i^- the two boundary arcs of the strip I_i parallel to the arcs in the counter-clock wise direction. We take a product with a discrete countable set F and label them by $D^i, I_1^i, l_2^i, \ldots, I_n^i$ for $i \in F$. Then we select a permutation $k_j : F \to F$ for each $j = 1, 2, \ldots, n$. For each i, we glue D^i with I_j^i over the arc α_i^+ and then we glue $D^{k_j(i)}$ with I_j^i over α_i^-. We do this for all arcs. Suppose that we obtain a connected space. By sending $D^i \to D, I_j^i \to I_j$ by projections, we obtain a covering.

Another good example is the join of two circles: See pages 56-58 of the book [Hatcher (2002)].

An important property of homotopy with respect to the covering space is the homotopy lifting property: Let $\tilde{M}$ be a covering of M. Given two homotopic maps f and g from a space X to M, we find that if f lifts to $\tilde{M}$, then so does g. If we let $F : X \times I \to M$ be the homotopy, the map lifts to $\tilde{F} : X \times I \to \tilde{M}$. This is completely determined if the lift of f is specified.

For example, one can consider a path to be a homotopy for X equal to a point. Any path in M lifts to a unique path in $\tilde{M}$ once the initial point is assigned.

Moreover, if two paths f and g are homotopic relative to endpoints, and their initial point $\tilde{f}(0)$ and $\tilde{g}(0)$ of the lifts $\tilde{f}$ and $\tilde{g}$ are the same, then $\tilde{f}(1) = \tilde{g}(1)$. Using this idea, we prove:

Theorem 2.1.3. *Let M be a manifold. Let $p : \tilde{M} \to M$ be a covering map and x_0 a base point of M. Given a map $f : Y \to M$ with $f(y_0) = x_0$, and a point $\tilde{x}_0 \in p^{-1}(x_0)$, we can uniquely lift f to $\tilde{f} : Y \to \tilde{M}$ so that $\tilde{f}(y_0) = \tilde{x}_0$ if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{M}, \tilde{x}_0))$.*

An *isomorphism* of two covering spaces X_1 with a covering map $p_1 : X_1 \to X$ and X_2 with $p_2 : X_2 \to X$ is a homeomorphism $f : X_1 \to X_2$ so that $p_2 \circ f = p_1$. The *automorphism group* of a covering map $p : M' \to M$ is a group of homeomorphisms $f : M' \to M'$ so that $p \circ f = p$. We also use the term the *deck transformation group*. Each element is a *deck transformation* or a *covering automorphism*.

Let x_0 be a base point of M. Let $p : \tilde{M} \to M$ be a covering map. The fundamental group $\pi_1(M, x_0)$ acts on $\tilde{M}$ on the right by path-liftings: we choose an inverse image $\tilde{x}_0$ in $\tilde{M}$. For a path γ in M with endpoints x_0, we define $\tilde{x}_0 \cdot \gamma = \tilde{\gamma}(1)$ for the lift $\tilde{\gamma}$ of γ with initial point $\tilde{\gamma}(0) = \tilde{x}_0$. This gives us a right-action $\pi_1(M, x_0) \times p^{-1}(x_0) \to p^{-1}(x_0)$, called a *monodromy action*, since we have $\tilde{x}_0 \cdot (\gamma * \delta) = (\tilde{x}_0 \cdot \gamma) \cdot \delta$.

A covering $p : M' \to M$ is *regular* if the covering map $p : M' \to M$ is a quotient map under the action of a discrete group Γ acting properly discontinuously and freely. Here M is homeomorphic to M'/Γ.

Given a covering map $p : \tilde{M} \to M$, we obtain a subgroup $p_*(\pi_1(\tilde{M}, \tilde{x}_0)) \subset \pi_1(M, x_0)$. Conversely, given a subgroup G of $\pi_1(M, x_0)$, we can construct a covering $\tilde{M}$ containing a point $\hat{x}_0$ and a covering map $p : \tilde{M} \to M$ so that $p_*(\pi_1(\tilde{M}, \hat{x}_0)) = G$ and $p(\hat{x}_0) = x_0$.

One classifies covering spaces of M by the subgroups of $\pi_1(M, x_0)$. That is, two coverings M_1 with basepoint m_1 and the covering map p_1 and M_2 with basepoint m_2 and covering map p_2 of M with $p_1(m_1) = p_2(m_2) = x_0$ are isomorphic with a map sending m_1 to m_2 if we have $p_{1*}(\pi_1(M_1, m_1)) = p_{2*}(\pi_1(M_2, m_2))$. Thus, the set of covering spaces is ordered by inclusion relations of the subgroups. If the subgroup is normal, the corresponding covering is regular.

A manifold has a *universal covering*; a covering space whose fundamental group is trivial. A universal cover covers every other covering of a given manifold.

The universal covering $\tilde{M}$ of a manifold M has the covering automorphism group Γ isomorphic to $\pi_1(M, x_0)$. A manifold M is homeomorphic to $\tilde{M}/\Gamma$ for its universal cover $\tilde{M}$ where Γ is the deck transformation group.

For example, let $\tilde{M}$ be $\mathbb{R}^2$ and T^2 be a torus. Then there is a map $p : \mathbb{R}^2 \to T^2$ sending (x, y) to $([x], [y])$ where $[x] = x \mod 2\pi$ and $[y] = y \mod 2\pi$.

Let M be a surface of genus 2. $\tilde{M}$ is homeomorphic to a disk, identified with a hyperbolic plane. The deck transformation group can be realized as isometries of a hyperbolic plane. We will see this in more details later.

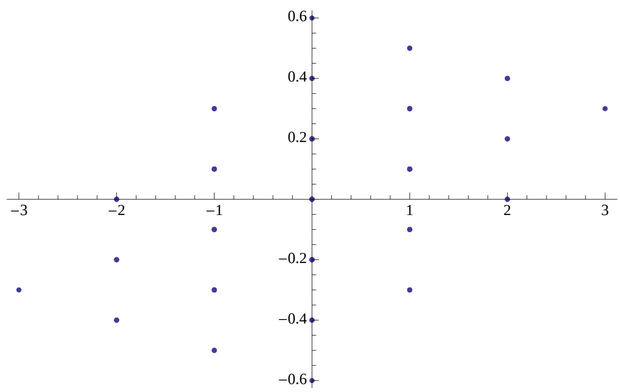

Fig. 2.1 Some orbit points of a translation group of rank two

2.1.4 *Simplicial manifolds*

In this section, we will try to realize manifolds as simplicial sets.

An affine space A^n is a vector space $\mathbb{R}^n$ where we do not remember the origin. More formally, A^n equals $\mathbb{R}^n$ as a set but has an operation $\mathbb{R}^n \times A^n$ given by sending $(a, b) \mapsto a + b$ for $a \in \mathbb{R}^n$ and $b \in A^n$ and satisfies $(a + (b + c)) = (a + b) + c$ for $a, b \in \mathbb{R}^n$ and $c \in A^n$. We can define the difference $b - a$ of two affine vectors a, b

by setting $b - a$ to equal $c \in \mathbb{R}^n$ such that $c + a = b$.

If one takes a point p as the origin, we can make A^n into a vector space $\mathbb{R}^n$ by a map $a \mapsto a - p$ for all $a \in A^n$.

A set of $n+1$ points $v_1, v_2, \ldots, v_{n+1}$ in $\mathbb{R}^n$ is *affinely independent* if the set $v_i - v_1$ for $i = 2, \ldots, n+1$ is linearly independent as vectors. An n-simplex is a convex hull of the set of affinely independent $(n+1)$-points. An n-simplex is homeomorphic to a closed unit ball B^n in $\mathbb{R}^n$.

A *simplicial complex* is a locally finite collection S of simplices so that any face of a simplex is a simplex in S and the intersection of two elements of S, if not empty, is a face of the both. The union is a topological set, which is said to be a *polyhedron*. We can define barycentric subdivisions by taking a barycentric subdivision for each simplex. A link of a simplex σ is the simplicial complex made up of simplicies disjoint from σ in a simplex containing σ.

An n-manifold X can be constructed by gluing n-simplices by face-identifications: Suppose that X is an n-dimensional triangulated space. If the link of every p-simplex is homeomorphic to a sphere of dimension $(n - p - 1)$, then X is an n-manifold. If X is a simplicial n-manifold, we say that X is *orientable* if we can give an orientation on each n-simplex so that over the common faces the orientations extend one another.

2.1.4.1 *Surfaces*

We begin with a construction of a compact surface. Given a polygon with even number of sides, we assign an identification pattern by labeling by $a_1, a_2, \ldots, a_g, a_1^{-1}, a_2^{-1}, \ldots, a_g^{-1}$ so that a_i means an edge labelled by a_i oriented counter-clockwise and a_i^{-1} means an edge labelled by a_i oriented clockwise, and if a pair a_i and a_i or a_i^{-1} occurs, then we identify them respecting the orientations.

- We begin with a bigon. We divide the boundary into two edges and identify by labels a, a^{-1}. Then the result is a surface homeomorphic to a 2-sphere.
- We divide the boundary into two edges and identify by labels a, a. Then the result is homeomorphic to a projective plane.
- Suppose now that we have a quadrilateral with labels a, b, a^{-1}, b^{-1}. We identify the top segment with the bottom one and the right side with the left side. The result is homeomorphic to a 2-torus.

Any closed surface can be represented in this manner.

Let us be given a $4n$-gon. We label edges

$$a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}, \ldots, a_n, b_n, a_n^{-1}, b_n^{-1}.$$

The result is a connected sum of n tori and is orientable. The genus of such a surface is n.

Suppose that we are given a $2n$-gon. We label edges $a_1 a_1 a_2 a_2 \ldots a_n a_n$. The result is a connected sum of n projective planes and is not orientable. The genus of

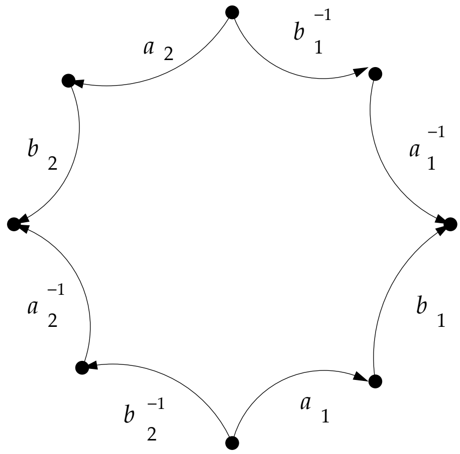

Fig. 2.2 A genus 2 surface as a quotient space of a disk

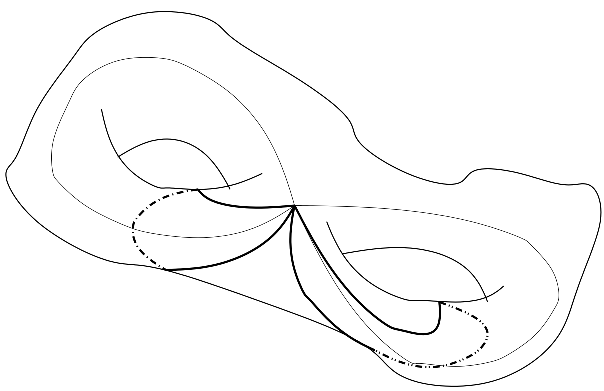

Fig. 2.3 A genus 2-surface patched up

such a surface is n.

We can remove the interiors of disjoint closed balls from the surfaces. The results are surfaces with boundary.

By the Van Kampen theorem, we compute the fundamental group of a surface using this identification. A surface is a cell complex starting from a 1-complex which is a bouquet of circles and attached with a cell. Therefore, we have the fundamental group $\pi_1(S, x)$ for a basepoint x is presented as

$$\langle a_1, b_1, \ldots, a_g, b_g | [a_1, b_1][a_2, b_2] \ldots [a_g, b_g] \rangle$$

for an orientable surface S of genus g, $g \geq 1$, where the notation implies that the group is isomorphic to a free group generated by $a_1, b_1, \ldots a_g, b_g$ quotient by the normal subgroup generated by the word $[a_1, b_1][a_2, b_2] \ldots [a_g, b_g]$.

The Euler characteristic of a 2-dimensional simplicial complex is given by $F - E + V$ where F denotes the number of 2-dimensional cells, E the number of 1-dimensional cells, and V the number of 0-dimensional cells. This is a topological invariant. We count from the above identification picture that the Euler characteristic of an orientable compact surface of genus g with n boundary components is $2 - 2g - n$.

By a *simple curve* in a surface, we mean an embedded interval. A *simple closed curve* in a surface is an embedded circle. They play important roles in studying surfaces as Dehn and Nielson first discovered.

Let a 2-sphere be given a triangulation. A *pair-of-pants* is the topological space homeomorphic to the complement of the interior of the union of three disjoint closed simplicial 2-cells in the sphere. It has three boundary components homeomorphic to circles. Moreover, a pair-of-pants is obtained by identifying two hexagons in their alternating segments in pairs.

In fact, a closed orientable surface of genus g, $g > 1$, contains $3g - 3$ disjoint simple closed curves so that the complement of its union is a disjoint union of open pairs-of-pants, i.e., spheres with three holes. Hence, the surface can be obtained by identifying boundary components of the pairs-of-pants.

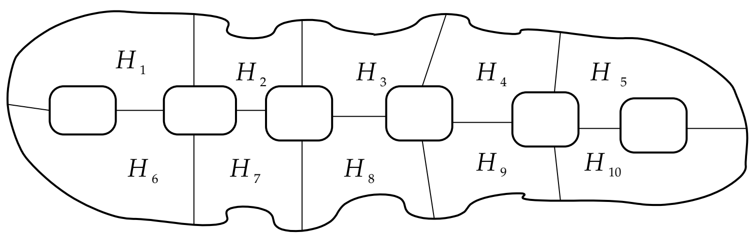

Fig. 2.4 A genus n surface is obtained by doubling a planar surface. That is, we take two copies of this but identify the boundary of the planar surfaces indicated as thick closed arcs. The planar surface is divided into hexagons denoted by H_i by thin lines. Then the doubled hexagons correspond to pairs-of-pants. This process is actually doubling of orbifolds if the boundary is silvered here. (See Section 4.6.1.2 for details on doubling.)

A pair-of-pants P can have a simple closed curve embedded in it but such a circle, if not homotopically trivial, always bounds an annulus containing a boundary component of P. Hence, a pair-of-pants can be built from a pair-of-pants and annuli by identification along circles. One cannot but build a pair-of-pants from a surface other than annuli and a single pair-of-pants. Therefore, a pair-of-pants is an "elementary" surface in that any closed surface can be built from these types of pieces by identifying boundary components where we regard annuli as being trivial elements of the constructions.

2.2 Lie groups

2.2.1 *Manifolds and tangent spaces*

We regard any manifold as being smoothly embedded in some Euclidean space. A *tangent vector* to a manifold M is a vector tangent to a point of M. The *tangent space* $T_x(M)$ at a point x of M is the vector space of vectors tangent to M at x. The *tangent bundle* of M is the space $\{(x, v)|x \in M, v \in T_x(M)\}$ with natural topology. For example, if M is an open subspace of $\mathbb{R}^n$, the tangent vectors are ordinary vectors based at a point of M and the tangent bundle is diffeomorphic to $M \times \mathbb{R}^n$.

At the moment this notion depends on the embedding of M; however, there are definitions showing that these spaces are well-defined.

A smooth map $f : M \to N$ induces a smooth map $Df : T(M) \to T(N)$ restricting to a linear map $D_x f : T_x(M) \to T_{f(x)}(N)$ of the vector spaces at each $x \in M$ defined by

$$\frac{d}{dt}(f \circ \alpha)\bigg|_{t=t_0} = Df_{x_0}\left(\frac{d}{dt}\alpha(t)\bigg|_{t=t_0}\right)$$

for $x_0 = \alpha(t_0)$. Df is said to be a *differential* of f.

2.2.2 *Lie groups*

A Lie group can be thought of as a space of symmetries of some space. More formally, a *Lie group* is a manifold with a group operation $\circ : G \times G \to G$ that satisfies

- $\circ$ is smooth.
- the inverse $\iota : G \to G$ is smooth also.

From $\circ$, we form a homomorphism $G \to \mathbf{Diff}(G)$ given by $g \mapsto L_g$ and $L_g : G \to G$ is a diffeomorphism given by a left-multiplication $L_g(h) = gh$. Since we have $L_{gh} = L_g \circ L_h$, this is a homomorphism.

As examples, we have:

- The permutation group of a finite set forms a 0-dimensional Lie group, which is a finite set, and a countable infinite group with the discrete topology is a 0-dimensional Lie group.
- $\mathbb{R}$ or $\mathbb{C}$ with $+$ as an operation. ($\mathbb{R}^+$ with $+$ is merely a Lie semigroup.)
- $\mathbb{R} - \{O\}$ or $\mathbb{C} - \{O\}$ with $\times$ as an operation.
- $T^n = \mathbb{R}^n/\Gamma$ with $+$ as an operation and O as the equivalence class of $(0, 0, \ldots, 0)$ and Γ is a group of translations by integral vectors. (The last three are abelian ones.)

We go to the noncommutative groups.

- The general linear group is given by
$$\mathbb{GL}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det(A) \neq 0\} :$$
Here, $\mathbb{GL}(n, \mathbb{R})$ is an open subset of $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$. The multiplication is smooth since the coordinate product has a polynomial expression.
- The special linear group is given as
$$\mathbb{SL}(n, \mathbb{R}) = \{A \in \mathbb{GL}(n, \mathbb{R}) | \det(A) = 1\} :$$
The restriction by a system of polynomial equations gives us a smooth submanifold of $\mathbb{GL}(n, \mathbb{R})$. The multiplication is also a restriction.
- The orthogonal group is given by
$$\mathbb{O}(n, \mathbb{R}) = \{A \in \mathbb{GL}(n, \mathbb{R}) | A^T A = \mathrm{I}\}.$$
This is another submanifold formed by a system of polynomial equations.
- The Euclidean isometry group is given by
$$\mathbf{Isom}(\mathbb{R}^n) = \{T : \mathbb{R}^n \to \mathbb{R}^n | T(x) = Ax + b \text{ for } A \in \mathbb{O}(n, \mathbb{R}), b \in \mathbb{R}^n\}.$$

Let us state some needed facts.

- A product of Lie groups forms a Lie group where the product operation is obviously defined.
- A covering space of a connected Lie group forms a Lie group. Here, we need to specify which element of the inverse image of the identity is the identity element.
- A *Lie subgroup* of a Lie group is a closed subgroup that is a Lie group with the induced operation and is a submanifold. For example, consider
  - $\mathbb{SO}(n, \mathbb{R}) \subset \mathbb{SL}(n, \mathbb{R}) \subset \mathbb{GL}(n, \mathbb{R})$.
  - $\mathbb{O}(n, \mathbb{R}) \subset \mathbf{Isom}(\mathbb{R}^n)$.

A *homomorphism* $f : G \to H$ of two Lie groups G and H is a smooth map that is a group homomorphism. The above inclusion maps are homomorphisms. The kernel of a homomorphism is a closed normal subgroup. Hence it is a Lie subgroup also. f induces the unique homomorphism of the Lie algebra of G to that of H which equals the differential $D_e f$ of f at the identity e of G and conversely. (See Subsection 2.2.3 for the definition of the Lie algebras and their homomorphsms.)

2.2.3 *Lie algebras*

A *Lie algebra* is a real or complex vector space V with a bilinear operation $[,] : V \times V \to V$ that satisfies:

- $[x, x] = 0$ for every $x \in V$ (thus, $[x, y] = -[y, x]$),
- and the Jacobi identity: $[x, [y, z]]+[z, [x, y]]+[y, [z, x]] = 0$ for all $x, y, z \in V$.

As examples, we have:

- Sending $V \times V$ to the zero-element O form a Lie algebra. This is defined to be the *abelian Lie algebras.*
- The direct sum of Lie algebras is a Lie algebra.
- A *subalgebra* is a subspace closed under the bracket $[,]$.
- An *ideal* K of V is a subalgebra such that $[x, y] \in K$ for $x \in K$ and $y \in V$.

A homomorphism of a Lie algebra is a linear map preserving $[,]$. The kernel of a homomorphism is an ideal.

2.2.4 *Lie groups and Lie algebras*

Let G be a Lie group. For an element $g \in G$, a *left translation* $L_g : G \to G$ is given by $x \mapsto g(x)$. A *left-invariant vector field* of G is a vector field X so that the left translation leaves it invariant, i.e., $DL_g(X(h)) = X(gh)$ for $g, h \in G$.

- The set of left-invariant vector fields forms a vector space under addition and scalar multiplication and is a vector space isomorphic to the tangent space at I. Moreover, the bracket $[,]$ is defined as vector-fields brackets. This forms a Lie algebra.
- The Lie algebra of G is the Lie algebra of the left-invariant vector fields on G.

A Lie algebra of an abelian Lie group is abelian.

The Lie algebra η of a Lie subgroup H is clearly a Lie subalgebra of the Lie algebra of G: A vector tangent to H at a point h_0 is realizable as a path in H passing h_0. A left-invariant vector field tangent to H at some point is always tangent to H at every point of H since H is closed under left-multiplications by elements of H. The Lie bracket operation is viewed as the derivative of the commutator of two flows generated by two left-invariant vector fields. Therefore, the Lie bracket is a closed operation for any tangent left-invariant vector fields of H.

Let $\mathfrak{gl}(n, \mathbb{R})$ denote the $M_n(\mathbb{R})$ with $[,] : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R})$ given by $[A, B] = AB - BA$ for $A, B \in M_n(\mathbb{R})$. The Lie algebra of $\mathbb{GL}(n, \mathbb{R})$ is isomorphic to $\mathfrak{gl}(n, \mathbb{R})$:

- For X in the Lie algebra of $\mathbb{GL}(n, \mathbb{R})$, we can define a flow generated by X and a path $X(t)$ along it where $X(0) = \mathrm{I}$ for the identity I.
- We obtain an element of $\mathfrak{gl}(n, \mathbb{R})$ by taking the derivative of $X(t)$ at 0 seen as a matrix.
- Now, we show that the brackets are preserved. That is, a vector-field bracket becomes a matrix bracket by the above map. (See the book [Bishop and Crittendon (2002)] for these computations.)

Thus, for any Lie algebra of any finite quotient Lie group of a Lie subgroup of $\mathbb{GL}(n, \mathbb{R})$, the bracket is computed by matrix brackets.

Given X in the Lie algebra $\mathfrak{g}$ of G, we find an integral curve $X(t)$ through I. We define the *exponential map* $\exp : \mathfrak{g} \to G$ by sending X to $X(1)$. The exponential map is defined everywhere, smooth, and is a diffeomorphism near O. With some work, we can show that the matrix exponential defined by

$$A \mapsto e^A = \sum_{i=0}^{\infty} \frac{1}{k!} A^k$$

is the exponential map $\exp : \mathfrak{gl}(n, \mathbb{R}) \to \mathbb{GL}(n, \mathbb{R})$ from the computation

$$\frac{d}{dt}\left(e^{tA}\right)\Big|_{t=1} = e^A A = L_{e^A}(A) = D(L_{e^A})(A)$$

for $A \in \mathfrak{gl}(n, \mathbb{R})$. Hence, this holds for any Lie subgroup of $\mathbb{GL}(n, \mathbb{R})$ and corresponding Lie subalgebra.

(See for example the books [Warner (1983); Bishop and Crittendon (2002)].)

2.2.5 *Lie group actions*

A *left* Lie group *G-action* on a smooth manifold X is given by a smooth map $k : G \times X \to X$ so that $k(e, x) \mapsto x$ and $k(gh, x) = k(g, k(h, x))$. Normally, $k(g, x)$ is simply written $g(x)$. In other words, denoting by $\mathbf{Diff}(X)$ the group of diffeomorphisms of X, k gives us a homomorphism $k'\ G \to \mathbf{Diff}(X)$ so that $k'(gh)(x) = k'(g)(k'(h)(x))$ and $k'(I) = I_X$. This is said to be the *left-action.* (We will not use notations k and k' in most cases.)

- A *right action* satisfies $(x)(gh) = ((x)g)h$ or more precisely, $(gh)(x) = (h(g(x))$.
- Define $\chi(X)$ to be the real vector space of vector fields on X. Each Lie algebra element corresponds to a vector field on X by a homomorphism $\chi_{(X,G)} : \mathfrak{g} \to \chi(X)$ defined by $\chi_{(X,G)}(\eta) = \vec{v}$ satisfying
$$\frac{d}{dt}\bigg|_{t=0} k(\exp(t\eta), x) = \vec{v}(x) \text{ for all } x \in X.$$
- The action is *faithful* if $g(x) = x$ for all x, then g is the identity element of G. This means that only e corresponds to the identity on X. (If this is true in particular, then the correspondence $\chi_{(X,G)}$ is injective.)
- The action is *transitive* if given two points $x, y \in X$, there is $g \in G$ such that $g(x) = y$.

As examples, consider

- $\mathbb{GL}(n, \mathbb{R})$ acting on $\mathbb{R}^n - \{O\}$ faithfully and transitively.
- $\mathbb{PGL}(n+1, \mathbb{R})$ acting on $\mathbb{RP}^n$ faithfully and transitively.

2.3 Pseudo-groups and $\mathcal{G}$-structures

In this section, we introduce pseudo-groups. Topological manifolds and its submanifolds are very wild and complicated objects to study as the topologist in 1950s and 1960s found out. The pseudo-groups will be used to put "calming" structures on manifolds.

Often the structures will be modeled on some geometries. We are mainly interested in classical geometries. We will be concerned with a Lie group G acting on a manifold M. Most obvious ones are Euclidean geometry where G is the group of rigid motions acting on the Euclidean space $\mathbb{R}^n$. The spherical geometry is one where G is the group $\mathbb{O}(n+1)$ of orthogonal transformations acting on the unit sphere $\mathbf{S}^n$.

Topological manifolds form too large a category to understand sufficiently. To restrict the local property, we introduce *pseudo-groups*. A *pseudo-group* $\mathcal{G}$ on a topological space X is the set of homeomorphisms between open sets of X so that the following statements hold:

- The domains of $g \in \mathcal{G}$ cover X.
- The restriction of $g \in \mathcal{G}$ to an open subset of its domain is also in $\mathcal{G}$.
- The composition of two elements of $\mathcal{G}$ when defined is in $\mathcal{G}$.
- The inverse of an element of $\mathcal{G}$ is in $\mathcal{G}$.
- If $g : U \to V$ is a homeomorphism for open subsets U, V of X, and if U is a union of open sets U_α for $\alpha \in I$ for some index set I such that $g|U_\alpha$ is in $\mathcal{G}$ for each α, then g is in $\mathcal{G}$.

Let us give some examples:

- The *trivial pseudo-group* is one where every element is a restriction of the identity $X \to X$ to an open subset.
- Any pseudo-group contains a trivial pseudo-group.
- The maximal pseudo-group of $\mathbb{R}^n$ is TOP formed from the set of all homeomorphisms between open subsets of $\mathbb{R}^n$.
- The pseudo-group C^r, $r \geq 1$, is formed from the set of C^r-diffeomorphisms between open subsets of $\mathbb{R}^n$.
- The pseudo-group PL is formed from the set of piecewise linear homeomorphisms between open subsets of $\mathbb{R}^n$.
- A (G, X)-*pseudo-group* is defined as follows. Let G be a Lie group acting on a manifold X faithfully and transitively. Then we define the pseudo-group as the set of all pairs $(g|U, U)$ for $g \in G$ where U is an open subset of X.
- The group $\mathbf{Isom}(\mathbb{R}^n)$ of rigid motions acting on $\mathbb{R}^n$ or the orthogonal group $\mathbb{O}(n+1, \mathbb{R})$ acting on $\mathbf{S}^n$ gives examples.

2.3.1 $\mathcal{G}$-manifolds

A $\mathcal{G}$-manifold is obtained as a manifold with special type of gluing only in $\mathcal{G}$: Let $\mathcal{G}$ be a pseudo-group on a manifold X. A *$\mathcal{G}$-manifold* is an n-manifold M with a maximal $\mathcal{G}$-atlas.

A *$\mathcal{G}$-atlas* is a collection of charts (embeddings) $\phi : U \to X$ where U is an open subset of M such that whose domains cover M and any two charts are $\mathcal{G}$-compatible.

- Two charts $(U, \phi), (V, \psi)$ are *$\mathcal{G}$-compatible* if the transition map satisfies
$$\gamma = \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) \in \mathcal{G}.$$

A set of $\mathcal{G}$-atlases is a partially ordered set under the ordering given by the inclusion relation. Two $\mathcal{G}$-atlases are *compatible* if any two charts in the atlases are $\mathcal{G}$-compatible. In this case, the union is another $\mathcal{G}$-atlas. One can choose a locally finite $\mathcal{G}$-atlas from a given maximal one and conversely. We obtain that the set of compatible $\mathcal{G}$-atlases has a unique maximal $\mathcal{G}$-atlas.

Under the compatibility relation, we obtain that the set of all $\mathcal{G}$-structures is partitioned into equivalence classes. We define the *$\mathcal{G}$-structure* on M as a maximal $\mathcal{G}$-atlas or as an equivalence class in the above partition.

The manifold X is trivially a $\mathcal{G}$-manifold if $\mathcal{G}$ is a pseudo-group on X. A topological manifold has a TOP-structure. A *C^r-manifold* is a manifold with a C^r-structure. A *differentiable manifold* is a manifold with a C^∞-structure. A *PL-manifold* is a manifold with a PL-structure.

A $\mathcal{G}$-map $f : M \to N$ for two $\mathcal{G}$-manifolds is a local homeomorphism or even an immersion so that if f sends a domain of a chart ϕ into a domain of a chart ψ, then
$$\psi \circ f \circ \phi^{-1} \in \mathcal{G}.$$
That is, f is an element of $\mathcal{G}$ locally up to charts.

Given two manifolds M and N, let $f : M \to N$ be a local homeomorphism. If N has a $\mathcal{G}$-structure, then so does M so that the map is a $\mathcal{G}$-map. A $\mathcal{G}$-atlas is given on M by taking open sets so that they map into open sets with charts in N under f and then use the induced charts. This $\mathcal{G}$-structure is said to be the *induced $\mathcal{G}$-structure.*

Suppose that M has a $\mathcal{G}$-structure. Let Γ be a discrete group of $\mathcal{G}$-homeomorphisms of M acting properly and freely. Then M/Γ has a $\mathcal{G}$-structure. The charts will be the charts of the lifted open sets. The $\mathcal{G}$-structure here is said to be the *quotient $\mathcal{G}$-structure.* (Sullivan and Thurston (1983) explain a class of such examples such as θ-annuli and π-annuli that arise in the study of complex projective and real projective surfaces.)

The torus T^n has a C^r-structure and a PL-structure since so does $\mathbb{R}^n$ and the each element of the group of translations all preserves these structures.

Given a pair (G, X) of Lie group G acting on a manifold X, we define a *(G, X)-structure* as a $\mathcal{G}$-structure and a *(G, X)-manifold* as a $\mathcal{G}$-manifold where $\mathcal{G}$ is the (G, X)-pseudo-group.

A *Euclidean manifold* is an $(\mathbf{Isom}(\mathbb{R}^n), \mathbb{R}^n)$-manifold.
A *spherical manifold* is an $(\mathbb{O}(n+1), \mathbf{S}^n)$-manifold.

2.4 Differential geometry

We wish to understand geometric structures in terms of differential geometry, i.e., methods of bundles, connections, and so on, since such an understanding helps us to see the issues in different ways. Actually, this is not central to the book. However, we should try to relate to the traditional fields where our subject can be studied in another way.

2.4.1 *Riemannian manifolds*

A differentiable manifold has a Riemannian metric, i.e., an inner-product at each tangent space that is smooth with respect coordinate charts. Such a manifold is said to be a Riemannian manifold.

An *isometric immersion* (embedding) of a Riemannian manifold to another one is a (one-to-one) map that preserves the Riemannian metric. We will be concerned with isometric embeddings of M into itself usually. A length of an arc is the value of an integral of the norm of tangent vectors to the arc. This gives us a metric on a manifold. An isometric embedding of M into itself is always an isometry. A geodesic is an arc minimizing length locally.

The *sectional curvature* $K(p)$ of a Riemannian metric along a 2-plane at a point p is given as the rate of area growth of r-balls on a disk $D(p)$ composed of geodesics from p tangent to a 2-plane:

$$K(p) = \lim_{r \to 0_+} 12 \frac{\pi r^2 - A(r)}{\pi r^4}$$

where $A(r)$ is the area of the r-ball centered at p in $D(p)$ with the induced metric. (See Page 133 of the book [Do Carmo (1992)]. This is the Bertrand-Diquet-Puiseux theorem.)

A constant curvature manifold is one whose sectional curvature is identical to a constant in every planar direction at every point.

- A Euclidean space E^n with the standard norm metric of a constant curvature $= 0$.
- A sphere $\mathbf{S}^n$ with the standard induced metric from $\mathbb{R}^{n+1}$ has a constant curvature $= 1$.
- Given a discrete torsion-free subgroup Γ of the isometry group of E^n (resp. $\mathbf{S}^n$. we obtain E^n/Γ (resp. $\mathbf{S}^n/\Gamma$) a manifold with a constant curvature $= 0$ (resp. 1).

2.4.2 *Principal bundles and connections: flat connections*

Let M be a manifold and G a Lie group. A *principal fiber bundle* P over M with a group G is the object satisfying

- P is a manifold.
- G acts freely on P on the right given by a smooth map $P \times G \to P$.
- $M = P/G$ and the map $\pi : P \to M$ is differentiable.
- P is locally trivial. That is, there is a diffeomorphism $\phi : \pi^{-1}(U) \to U \times G$ for at least one neighborhood U of any point of M.

We say that P is the *bundle space*, M is the *base space*, and $\pi^{-1}(x)$ is a *fiber* which also equals $\pi^{-1}(x) = \{ug | g \in G\}$ for any choice of $u \in \pi^{-1}(x)$. G is said to be the *structure group*.

As an example, consider $L(M)$: the set of all frames of the tangent bundle $T(M)$. One can give a topology on $L(M)$ so that sending a frame to its base point is the smooth quotient map $L(M) \to M$. $\mathbb{GL}(n, \mathbb{R})$ acts freely on $L(M)$. We can verify that $\pi : L(M) \to M$ is a principal bundle.

Given a collection of open subsets U_α covering M, we construct a bundle by a collection of mappings

$$\{\phi_{\beta,\alpha} : U_\alpha \cap U_\beta \to G\}$$

satisfying

$$\phi_{\gamma,\alpha} = \phi_{\gamma,\beta} \circ \phi_{\beta,\alpha}, \phi_{\alpha,\alpha} = \mathrm{I}$$

for any triple $U_\alpha, U_\beta, U_\gamma$. Then form $U_\alpha \times G$ for each α. For any pair $U_\alpha \times G$ and $U_\beta \times G$, identify by $\tilde{\phi}_{\beta,\alpha} : U_\alpha \times G \to U_\beta \times G$ given by $(x, g) \mapsto (x, \phi_{\beta,\alpha}(x)(g))$. The quotient space is a principal bundle over M.

A principal bundle over M with the structure group G is often denoted by $P(G, M)$. Given two Lie groups G and G', and a monomorphism $f : G' \to G$, we call a map $f : P(G', M) \to P(G, M)$ inducing identity $M \to M$ a *reduction* of the structure group G to G'. There may be many reductions for given G' and G. We say that $P(G, M)$ is reducible to $P(G', M)$ if and only if $\phi_{\alpha,\beta}$ can be taken to be in G'. (See the books [Kobayashi and Nomizu (1997); Bishop and Crittendon (2002)] for details.)

2.4.2.1 *Associated bundles*

Let F be a manifold with a left-action of G. G acts on $P \times F$ on the right by

$$g : (u, x) \to (ug, g^{-1}(x)), g \in G, u \in P, x \in F.$$

Form the quotient space $E = P \times_G F$ with a map $\pi_E : E \to M$ induced from the projection $\pi : P \times F \to M$ and we can verify that $\pi_E^{-1}(U)$ is identifiable with $U \times F$ up to making some choices of sections on U to P. The space E is said to be the *associated bundle* over M with M as the *base space*. The *structure group* is the same

G. The induced quotient map $\pi_E : E \to M$ has a fiber $\pi_E^{-1}(x)$ diffeomorphic to F for any $x \in M$.

Here E can also be built from a cover U_α of M by taking $U_\alpha \times F$ and pasting by appropriate diffeomorphisms of F induced by elements of G as above.

The tangent bundle $T(M)$ is an example. $\mathbb{GL}(n, \mathbb{R})$ acts on $\mathbb{R}^n$ on the left. Let $F = \mathbb{R}^n$. We obtain $T(M)$ as $L(M) \times_{\mathbb{GL}(n,\mathbb{R})} \mathbb{R}^n$. A tensor bundle $T^r_s(M)$ is another example. $\mathbb{GL}(n, \mathbb{R})$ acts on the space of (r, s)-tensors $T^r_s(\mathbb{R}^n)$, and let F be $T^r_s(\mathbb{R})$. Then we obtain $T^r_s(M)$ as $L(M) \times_{\mathbb{GL}(n,\mathbb{R})} T^r_s(\mathbb{R}^n)$.

2.4.2.2 *Connections*

Let $P(M, G)$ be a principal bundle. A *connection* is a decomposition of each $T_u(P)$ for each $u \in P$ so that the following statements hold:

- $T_u(P) = G_u \oplus Q_u$ where G_u is a subspace tangent to the fiber. (G_u is said to be the *vertical space* and Q_u the *horizontal space*.)
- $Q_{ug} = R^*_g(Q_u)$ for each $g \in G$ and $u \in P$.
- Q_u depends smoothly on u.

Let $\mathfrak{g}$ denote the Lie algebra of G. More formally, we define a connection as a $\mathfrak{g}$-valued form ω on P is given as $T_u(P) \to G_u$ obtained by taking the vertical component of each tangent vector of P: We could define a *connection* as a smooth $\mathfrak{g}$-valued form ω.

- $\omega(A^*) = A$ for every $A \in \mathfrak{g}$ and A^* the fundamental vector field on P generated by A, i.e., the vector field tangent to the one parameter group of diffeomorphisms on P generated by the action of $\exp(tA) \in G$ at $t = 0$.
- $(R_g)^*\omega = Ad(g^{-1})\omega$.

A *horizontal* lift of a piecewise-smooth path τ on M is a piecewise-smooth path τ' lifting τ so that the tangent vectors are all horizontal. A horizontal lift is determined once the initial point is given.

- Given a curve on M with two endpoints, we find that the lifts of the curve define a parallel displacement between fibers above the two endpoints (commuting with the right G-actions).
- Fixing a point x_0 on M, these parallelisms along closed loops with endpoints at x_0 form a holonomy group that is identifiable with a subgroup of G acting on the left on the fiber at x_0.
- The curvature of a connection is a measure of how much the horizontal lift of a small loop in M differs from a loop in P. A connection is *flat* if the curvatures are zero identically.
- For the flat connections, we can lift homotopically trivial loops in M to loops in P. Thus, the flatness is equivalent to local lifting of small coordinate charts of M to horizontal sections in P.

- A flat connection on P gives us a smooth foliation of dimension n transversal to the fibers where n is the dimension of M. A *flat bundle* is a bundle with a flat connection.

The associated bundle E also inherits a *connection*, i.e., a splitting of the tangent space of E into vertical space and horizontal space. Here again, the vertical spaces are obtained as tangent spaces to fibers. Again given a curve on M, horizontal liftings and parallel displacements between fibers in E make sense . The flatness is also equivalent to the local lifting property, and the flat connection on E gives us a smooth foliation of dimension n transversal to the fibers.

An *affine frame* in a vector (or affine) space $\mathbb{R}^n$ is a set of $n+1$ points $a_0, a_1, \ldots, a_n$ so that $a_1 - a_0, a_2 - a_0, \ldots, a_n - a_0$ form a linear frame. These assignments give us the canonical map from the space of affine frames $A(\mathbb{R}^n)$ to linear frames $L(\mathbb{R}^n)$. An affine group $\mathbb{A}(n, \mathbb{R})$ acts on $A(\mathbb{R}^n)$ also by sending $(a_0, a_1, \ldots, a_n)$ to $(L(a_0), L(a_1), \ldots, L(a_n))$ for an affine automorphism $L : \mathbb{R}^n \to \mathbb{R}^n$.

An *affine connection* on a manifold M is defined as follows. An affine frame over M is an affine frame on a tangent space of a point of M, treating as an affine space. The set of all affine frames over a manifold forms a manifold of higher dimension. Let $A(M)$ be the space of affine frames over M with the affine group $\mathbb{A}(n, \mathbb{R})$ acting on it fiberwise on the left.

- The Lie algebra $\mathfrak{a}(n, \mathbb{R})$ is a semi-direct product of $M_n(\mathbb{R})$ and $\mathbb{R}^n$.
- There is a natural map $A(M) \to L(M)$ where $L(M)$ is the set of linear frames over M and is given by the natural map $A(\mathbb{R}^n) \to L(\mathbb{R}^n)$.
- An affine connection on M is a linear connection plus the canonical $\mathbb{R}^n$-valued 1-form. The curvature of the affine connection is the sum of the curvature of the linear connection and the torsion.

A nice example is when M is a 1-manifold, say an open interval I. Then P is $I \times G$, and the associated bundle is $I \times X$. A connection is simply given as an infinitesimal way to connect each fiber by a left multiplication by an element of G. In this case, a connection is flat always and $I \times G$ and $I \times X$ are fibered by open intervals transversal to the fibers.

If M is a circle, then P becomes a mapping circle with fiber X and E a mapping circle with fiber E:

$$\begin{array}{c} G \to P \\ \downarrow \\ \mathbf{S}^1. \end{array}$$

Now, such spaces are classified by a map $\pi_1(\mathbf{S}^1) \to G$.

For the affine connections, let M be an interval I, and let $G = \mathbb{A}(1, \mathbb{R})$ and $X = \mathbb{R}$. Then E is now a strip $I \times \mathbb{R}$. An affine connection gives a foliation on the strip transversal to $\mathbb{R}$ and is invariant under translation in the $\mathbb{R}$-direction.

Even for higher-dimensional manifolds, we can think of a connection as the collection of 1-dimensional ones over each path. The local dependence on paths is measured by the curvature.

2.4.2.3 *The principal bundles and (G, X)-structures.*

Given a manifold M of dimension n and a Lie group G acting on a manifold X of dimension n, we form a principal bundle P over a manifold M and the associated bundle E fibered by X with a flat connection. Suppose that we can choose a section $f : M \to E$ which is transverse everywhere to the foliation given by the flat connection. This gives us a (G, X)-structure. The main reason is that the section f sends an open set of M to a transversal submanifold to the foliation. Locally, the foliation gives us a projection to X. The composition gives us charts. The charts are compatible since E has a left-action.

Conversely a (G, X)-structure gives us a principal bundle P, the associated bundle E, the flat connection and a transverse section f.

We will elaborate this later when we are studying orbifolds and geometric structures in Chapter 6.

2.5 Notes

Chapter 0 and 1 of the book [Hatcher (2002)] and the books [Munkres (1991); Warner (1983)] are good source of preliminary knowledges here. The books [Do Carmo (1992); Kobayashi and Nomizu (1997); Bishop and Crittendon (2002)] give us good knowledge of connections, curvature, and Riemannian geometry. Also, the book [Thurston (1997)] is a source for studying (G, X)-structures and pseudo-groups as well as geometry and Lie groups presented here. Goldman's book [Goldman (1988)] treats materials here also in a more abstract manner.