Chapter 1

Introduction

One aim of mathematics is to explore many objects purely defined and created out of imaginations in the hope that they would explain many unknown and unsolved phenomena in mathematics and other fields. As one knows, the manifold theory enjoyed a great deal of attention in the 20th century mathematics involving many talented mathematicians. Perhaps, mathematicians should develop more abstract theories that can accommodate many things that we promised to unravel in the earlier part of 20th century. The theory of orbifolds might be a small step in the right direction as orbifolds have all the notions of the manifold theory easily generalized as discovered by Satake and developed by Thurston. In fact, orbifolds have most notions developed from the manifold theory carried over to them although perhaps in an indirect manner, using the language of the category theory. Indeed, to make the orbifold theory most rigorously understood, only the category theory provides the natural settings.

Orbifolds provide a natural setting to study discrete group actions on manifolds, and orbifolds can be more useful than manifolds in many ways involving in the classification of knots, the graph embeddings, theoretical physics and so on. At least in two- or three-dimensions, orbifolds are much easier to produce and classifiable using Thurston's geometrization program. (See for example the program "Orb" by Heard and Hodgson (2007).) In particular, one obtains many examples with ease to experiment with. The subject of higher dimensional orbifolds is still very mysterious where many mathematicians and theoretical physicists are working on. In fact, the common notion that orbifolds are almost always covered by manifolds is not entirely relevant particularly for the higer-dimensional orbifolds. For example, these kinds of orbifolds might exist in abundance and might prove to be very useful. It is thought that orbifolds are integral part of theoretical physics such as the string theory, and they have natural generalizations in algebraic geometry as stacks.

For 2-manifolds, it was known from the classical times that the geometry provides a sharp insight into the topology of surfaces and their groups of automorphisms as observed by Dehn and others. In late 1970s, Thurston proposed a program to generalize these kinds of insights to the 3-manifold theory. This program is now completed by Perelman's proof of the Geometrization conjecture as is well-known.

The computer programs such as Snappea initiated by Thurston and completed mainly by Weeks, Hodgson, and so on, now compute most topological properties of 3-manifolds completely given the 3-manifold topological data.

It seems that the direction of the research in the low-dimensional manifold theory currently is perhaps to complete the understanding of 3-manifolds by volume ordering, arithmetic properties, and group theoretical properties. Perhaps, we should start to move to higher-dimensional manifolds and to more applied areas.

One area which can be of possible interest is to study the projectively flat, affinely flat, or conformally flat structures on 3-manifolds. This will complete the understanding of all classical geometric properties of 3-manifolds. This aspect is related to understanding all representations of the fundamental groups of 3-manifolds into Lie groups where many interesting questions still remain, upon which we mention that we are yet to understand fully the 2-orbifold or surface fundamental group representations into Lie groups.

This book introduces 2-orbifolds and geometric structures on them for senior undergraduates and the beginning graduate students. Some background in topology, the manifold theory, differential geometry, and particularly the category theory would be helpful.

The covering space theory is explained using both the fiber-product approach of Thurston and the path-approach of Haefliger. In fact, these form a very satisfying direct generalization of the classical covering space theory of Poincaré. The main part of the book is the geometric structures on orbifolds. We define the deformation space of geometric structures on orbifolds and state the local homeomorphism theorem that the deformation spaces are locally homeomorphic to the representation spaces of the fundamental groups. The main emphases are on studying geometric structures and ways to cut and paste the geometric structures on 2-orbifolds. These form a main topic of this book and will hopefully aid the reader in studying many possible geometric structures on orbifolds including affine, projective, and so on. Also, these other types of geometries seem to be of use in the Mirror symmetry and so on.

We will learn the orbifold theory and the geometric structures on orbifolds. We will cover some of the background materials such as the Lie group theory, principal bundles, and connections. The theory of orbifolds has much to do with discrete subgroups of Lie groups but has more topological flavors. We discuss the topology of orbifolds including covering spaces and orbifold-fundamental groups. The fundamental groups of orbifolds include many interesting infinite groups. We obtain the understanding of the deformation space of hyperbolic structures on a 2-orbifold, which is the space of conjugacy classes of discrete faithful $\mathbb{PSL}(2, \mathbb{R})$-representations of the 2-orbifold fundamental group. Finally, we survey the deformation spaces of real projective structures on 2-orbifolds, which correspond to the Hitchin-Teichmüller components of the spaces of conjugacy classes of $\mathbb{PGL}(3, \mathbb{R})$-representations of the fundamental groups.

This book has three parts. In the first part consisting of Chapters 2 and 3, we review the manifold theory with $\mathcal{G}$-structures. In the second part consisting of Chapters 4 and 5, we present the topological theory of orbifolds. In the third part, consisting of Chapters 6, 7, and 8, we present the theory of geometric structures of orbifolds.

In Chapter 2, manifolds and $\mathcal{G}$-structures, we review smooth structures on manifolds starting from topological constructions, homotopy groups and covering spaces, and simplicial manifolds including examples of surfaces. Then we move onto pseudogroups and $\mathcal{G}$-structures. Finally, we review Lie groups and the principal bundle theory in terms of the smooth manifold theory.

In Chapter 3, geometry and discrete groups, we first review Euclidean, spherical, affine, projective, and conformal geometry centering on their properties under the Lie group actions. Next, we go over to hyperbolic geometry. We begin from the Lorentzian hyperboloid model and move onto the Beltrami-Klein model, the conformal model and the upper half-space model. Hyperbolic triangle laws are studied also and the isometry group of hyperbolic spaces is introduced. We also discuss the discrete group actions on manifolds using the Poincaré fundamental polyhedron theorem and discuss Coxeter groups, triangle groups, and crystallographic groups.

In Chapter 4, topology of orbifolds, we start reviewing compact group actions on manifolds. We talk about the orbit spaces and tubes, smooth actions, and equivariant triangulations. Next, we introduce orbifolds from the classical definition by Satake using atlases of charts. We define singular sets and suborbifolds. We also present orbifolds as Lie groupoids from the category theory as was initiated by Haefliger. We present differentiable structures on orbifolds, bundles over orbifolds, the Gauss-Bonnet theorem, and smooth triangulations. To find the universal covers of orbifolds, we start from defining covering spaces of orbifolds and discuss how to obtain a fiber-product of many covering orbifolds. This leads us to the universal covering orbifolds and deck transformation groups and their properties such as uniqueness. We also present the path-approach to the universal covering orbifolds of Haefliger. Hence, we define the fundamental groups of orbifolds.

In Chapter 5, topology of 2-orbifolds, we present how to compute the Euler characteristics of 2-orbifolds using the Riemann-Hurwitz formula. We show how to topologically construct 2-orbifolds from other 2-orbifolds using cutting and sewing methods. This is reinterpreted in two other manners.

In Chapter 6, geometry of orbifolds, we define (G, X)-structures on orbifold using the method of atlases of charts, the method of developing maps and holonomy homomorphisms, and the method of cross-sections to bundles. We show that these definitions are equivalent. We also show that orbifolds admitting geometric structures are always good; that is, they are covered by manifolds. Here, we define the deformation spaces of (G, X)-structures on orbifolds and discuss about the local homeomorphism from the deformation space of (G, X)-structures on an orbifold to the space of representations of the fundamental groups of the orbifold to the Lie group G.

In Chapter 7, the deformation spaces of hyperbolic structures on 2-orbifolds, i.e., the Teichmüller space, we first define the Teichmüller space and present geometric cutting and pasting constructions of hyperbolic structures on 2-orbifolds. We show that any 2-orbifolds decompose into elementary orbifolds. We show how to compute the Teichmüller spaces of elementary orbifolds using hyperbolic trigonometry and piece these together to understand the Teichmüller space of the 2-orbifold following Thurston.

Finally in Chapter 8, we introduce the deformation spaces of real projective structures on 2-orbifolds. We first give some examples. Next, we sketch some history on this subject, including the classification result, Hitchin's conjecture and its solution, and the discrete groups and the representation theory aspects. We go over to the deformation spaces where we use the method very similar to the above chapter. We decompose 2-orbifolds into elementary 2-orbifolds and determine the deformation spaces there and reassemble. Here, we merely indicate proofs. In this chapter, we do many computations for elementary 2-orbifolds.

Our principal source for this lecture note is Chapter 5 of the book [Thurston (1977)]. However, we do not go into his generalization of the Andreev theorem. (Also, the book [Thurston (1997)] is a good source of many materials here.)

We shall maintain some computations files related to graphics in this book. Mathematica™files designated $***$.nb are files that we wrote and maintain in our homepages.

There are many standard textbooks giving us preliminary viewpoint and alternative viewpoints of the foundational material for this book. The book [Kobayashi and Nomizu (1997)] provides us a good introduction to connections on principal bundles and the books [Sharpe (1997); Ivey and Landsberg (2003)] give us more differential geometric viewpoint of geometric structures. The book [Bredon (1972)] is a good source for understanding the local orbifold group actions. Finally, the book [Berger (2009)] provides us with the knowledge of geometry that is probably most prevalently used in this book.