Chapter 3

Stationary solutions

3.1 Unique existence of stationary solutions

This section is devoted to discussion about the unique existence of stationery solutions for the hydrodynamic, the energy-transport and the drift-diffusion models. We write the solution for the hydrodynamic model by $(\tilde{\rho}^{\varepsilon}_{\zeta}, \tilde{j}^{\varepsilon}_{\zeta}, \tilde{\theta}^{\varepsilon}_{\zeta}, \tilde{\phi}^{\varepsilon}_{\zeta})$ for the clarity of its dependence on ε and ζ. Namely,

$$
\begin{gathered}
(\tilde{j}^{\varepsilon}_{\zeta})_x = 0, \\
S[\tilde{\rho}^{\varepsilon}_{\zeta}, \tilde{j}^{\varepsilon}_{\zeta}, \tilde{\theta}^{\varepsilon}_{\zeta}](\tilde{\rho}^{\varepsilon}_{\zeta})_x + \tilde{\rho}^{\varepsilon}_{\zeta}(\tilde{\theta}^{\varepsilon}_{\zeta})_x = \tilde{\rho}^{\varepsilon}_{\zeta}(\tilde{\phi}^{\varepsilon}_{\zeta})_x - \tilde{j}^{\varepsilon}_{\zeta}, \\
\tilde{j}^{\varepsilon}_{\zeta}(\tilde{\theta}^{\varepsilon}_{\zeta})_x - \frac{2}{3}\tilde{j}^{\varepsilon}_{\zeta}\tilde{\theta}^{\varepsilon}_{\zeta}\left(\log \tilde{\rho}^{\varepsilon}_{\zeta}\right)_x - \frac{2}{3}\kappa_0(\tilde{\theta}^{\varepsilon}_{\zeta})_{xx} = \left(\frac{2}{3} - \frac{\varepsilon}{3\zeta}\right)\frac{(\tilde{j}^{\varepsilon}_{\zeta})^2}{\tilde{\rho}^{\varepsilon}_{\zeta}} - \frac{\tilde{\rho}^{\varepsilon}_{\zeta}}{\zeta}(\tilde{\theta}^{\varepsilon}_{\zeta} - 1), \\
(\tilde{\phi}^{\varepsilon}_{\zeta})_{xx} = \tilde{\rho}^{\varepsilon}_{\zeta} - D.
\end{gathered}
$$

The stationary solution for the energy-transport model is written by $(\tilde{\rho}^{0}_{\zeta}, \tilde{j}^{0}_{\zeta}, \tilde{\theta}^{0}_{\zeta}, \tilde{\phi}^{0}_{\zeta})$ and satisfies

$$
(\tilde{j}^{0}_{\zeta})_x = 0, \tag{3.1a}
$$

$$
\tilde{j}^{0}_{\zeta}(\tilde{\theta}^{0}_{\zeta})_x - \frac{2}{3}\tilde{j}^{0}_{\zeta}\tilde{\theta}^{0}_{\zeta}\left(\log \tilde{\rho}^{0}_{\zeta}\right)_x - \frac{2}{3}\kappa_0(\tilde{\theta}^{0}_{\zeta})_{xx} = \frac{2}{3}\frac{(\tilde{j}^{0})^2}{\tilde{\rho}^{0}} - \frac{\tilde{\rho}^{0}_{\zeta}}{\zeta}(\tilde{\theta}^{0}_{\zeta} - 1), \tag{3.1b}
$$

$$
(\tilde{\phi}^{0}_{\zeta})_{xx} = \tilde{\rho}^{0}_{\zeta} - D, \tag{3.1c}
$$

$$
\tilde{j}^{0}_{\zeta} = -(\tilde{\theta}^{0}_{\zeta}\tilde{\rho}^{0}_{\zeta})_x + \tilde{\rho}^{0}_{\zeta}(\tilde{\phi}^{0}_{\zeta})_x \tag{3.1d}
$$

with the boundary condition (2.18)–(2.20). Moreover, the stationary solution $(\tilde{\rho}_0^0, \tilde{j}_0^0, \tilde{\phi}_0^0)$ of the drift-diffusion model satisfies

$$(\tilde{j}_0^0)_x = 0, \tag{3.2a}$$

$$(\tilde{\phi}_0^0)_{xx} = \tilde{\rho}_0^0 - D, \tag{3.2b}$$

$$\tilde{j}_0^0 = -\tilde{\rho}_{0x}^0 + \tilde{\rho}_0^0 (\tilde{\phi}_0^0)_x \tag{3.2c}$$

with the boundary condition (2.18) and (2.20). The unique existence of the stationary solution for the drift-diffusion model is proven in the author's previous paper [33], which reads

Lemma 3.1. *Let the doping profile and the boundary data satisfy conditions* (2.2)*,* (2.4) *and* (2.6)*. Then the boundary value problem* (3.2)*,* (2.18) *and* (2.20) *has a unique solution* $(\tilde{\rho}_0^0, \tilde{j}_0^0, \tilde{\phi}_0^0) \in \mathcal{B}^2(\overline{\Omega})$ *satisfying the conditions* (2.10a)*. Moreover, it verifies the estimates*

$$\min\left\{\rho_l,\ \rho_r,\ \inf_{x\in\Omega} D(x)\right\} \le \tilde{\rho}_0^0 \le \max\left\{\rho_l,\ \rho_r,\ \sup_{x\in\Omega} D(x)\right\}, \tag{3.3}$$

$$|(\tilde{\rho}_0^0, \tilde{\phi}_0^0)|_2 \le C, \tag{3.4}$$

$$|\tilde{j}_0^0| \le C\delta, \tag{3.5}$$

where C is a positive constant independent of ε, and $\tilde{j}_0^0$ is a constant given by a formula

$$\tilde{j}_0^0 = (\phi_r - \log\rho_r + \log\rho_l)\left(\int_0^1 \frac{1}{\tilde{\rho}_0^0}\,dx\right)^{-1}. \tag{3.6}$$

Hence, we show the unique existence of the stationary solution for the hydrodynamic and the energy-transport models. Here the boundary value problem (2.17)–(2.20) for the hydrodynamic model covers the problem (3.1) and (2.18)–(2.20) for the energy-transport model as a special case $\varepsilon = 0$. Thus it suffices to solve the problem (2.17)–(2.20) for $\varepsilon \ge 0$ to construct the stationary solutions for both models. We begin detailed discussions with deriving several equalities and formulae. Divide (2.17b) by $\tilde{\rho}_\zeta^\varepsilon$ and differentiate the resultant equality in x to get

$$\left(\frac{1}{\tilde{\rho}_\zeta^\varepsilon} S[\tilde{\rho}_\zeta^\varepsilon, \tilde{j}_\zeta^\varepsilon, \tilde{\theta}_\zeta^\varepsilon](\tilde{\rho}_\zeta^\varepsilon)_x\right)_x + (\tilde{\theta}_\zeta^\varepsilon)_{xx} - \frac{\tilde{j}_\zeta^\varepsilon}{(\tilde{\rho}_\zeta^\varepsilon)^2}(\tilde{\rho}_\zeta^\varepsilon)_x - \tilde{\rho}_\zeta^\varepsilon = -D. \tag{3.7}$$

By dividing (2.17b) by $\tilde{\rho}_\zeta^\varepsilon$ and integrating the resultant over the domain Ω, we have the current-voltage relationship

$$\phi_r = \frac{\varepsilon}{2}\left(\frac{1}{\rho_r^2} - \frac{1}{\rho_l^2}\right)(\tilde{j}_\zeta^\varepsilon)^2 + \left(\int_0^1 \frac{1}{\tilde{\rho}_\zeta^\varepsilon}\,dx\right)\tilde{j}_\zeta^\varepsilon$$
$$+ \tilde{\theta}_\zeta^\varepsilon(1) - \tilde{\theta}_\zeta^\varepsilon(0) + \tilde{\theta}_\zeta^\varepsilon(1)\log\rho_r - \tilde{\theta}_\zeta^\varepsilon(0)\log\rho_l - \int_0^1 (\tilde{\theta}_\zeta^\varepsilon)_x \log\tilde{\rho}_\zeta^\varepsilon dx. \quad (3.8)$$

Furthermore, solving the equation (3.8) with respect to $\tilde{j}_\zeta^\varepsilon$ gives the formula of the electric current

$$\tilde{j}_\zeta^\varepsilon = \mathcal{J}[\tilde{\rho}_\zeta^\varepsilon, \tilde{\theta}_\zeta^\varepsilon] := 2\left(\mathcal{L}[\tilde{\theta}_\zeta^\varepsilon] + \int_0^1 (\tilde{\theta}_\zeta^\varepsilon)_x \log\tilde{\rho}_\zeta^\varepsilon\,dx\right)\mathcal{K}[\tilde{\rho}_\zeta^\varepsilon, \tilde{\theta}_\zeta^\varepsilon]^{-1}, \quad (3.9a)$$

$$\mathcal{K}[\rho,\theta] := \int_0^1 \rho^{-1}\,dx + \sqrt{\left(\int_0^1 \rho^{-1}\,dx\right)^2 + 2\varepsilon\left(\mathcal{L}[\theta] + \int_0^1 \theta_x \log\rho\,dx\right)\left(\rho_r^{-2} - \rho_l^{-2}\right)}, \quad (3.9b)$$

$$\mathcal{L}[\theta] := \phi_r - \theta(1) + \theta(0) - \theta(1)\log\rho_r + \theta(0)\log\rho_l. \quad (3.9c)$$

Although the equality (3.8) is the quadratic equation in $\tilde{j}_\zeta^\varepsilon$ and has two distinct roots if $\varepsilon \neq 0$ and $\rho_l \neq \rho_r$, we see that the other solution violates the subsonic condition (2.13) for sufficiently small δ. Hence, the admissible quantity of the electric current is given by (3.9a). Moreover, the formula of potential

$$\tilde{\phi}_\zeta^\varepsilon(x) = \Phi[\tilde{\rho}_\zeta^\varepsilon] = \int_0^x\int_0^y (\tilde{\rho}_\zeta^\varepsilon - D)(z)\,dzdy + \left(\phi_r - \int_0^1\int_0^y (\tilde{\rho}_\zeta^\varepsilon - D)(z)\,dzdy\right)x \quad (3.10)$$

follows from the same computation as the derivation of (2.8). Here and hereafter, we frequently use the constants and the function

$$B_m := \min\left\{\rho_l,\ \rho_r,\ \inf_{x\in\Omega} D(x)\right\}, \quad B_M := \max\left\{\rho_l,\ \rho_r,\ \sup_{x\in\Omega} D(x)\right\},$$
$$\Gamma(x) := \rho_l(1-x) + \rho_r x.$$

The existences of the stationary solutions $(\tilde{\rho}_\zeta^\varepsilon, \tilde{j}_\zeta^\varepsilon, \tilde{\phi}_\zeta^\varepsilon, \tilde{\theta}_\zeta^\varepsilon)$ for the hydrodynamic model, and $(\tilde{\rho}_\zeta^0, \tilde{j}_\zeta^0, \tilde{\phi}_\zeta^0, \tilde{\theta}_\zeta^0)$ for the energy-transport model are summarized in the next lemma, where the latter is covered as the special case $\varepsilon = 0$.

Lemma 3.2. *Let the doping profile and the boundary data satisfy conditions* (2.2)*,* (2.4) *and* (2.6)*. For arbitrary ρ_l, there exist positive constants δ_0 and ζ_0 such that if $\delta \leq \delta_0$ and $0 \leq \varepsilon < \zeta \leq \zeta_0$, then the boundary value problem* (2.17)*–*(2.20) *has a solution* $(\tilde{\rho}_\zeta^\varepsilon, \tilde{j}_\zeta^\varepsilon, \tilde{\theta}_\zeta^\varepsilon, \tilde{\phi}_\zeta^\varepsilon) \in$

$\mathcal{B}^2(\overline{\Omega}) \times \mathcal{B}^2(\overline{\Omega}) \times \mathcal{B}^3(\overline{\Omega}) \times \mathcal{B}^2(\overline{\Omega})$ *satisfying the conditions*

$$\frac{1}{2}B_m \leq \tilde{\rho}^\varepsilon_\zeta \leq 2B_M, \tag{3.11a}$$

$$|\tilde{\theta}^\varepsilon_\zeta - 1|_0 \leq C\delta \tag{3.11b}$$

and (2.13)*, where the positive constant* C *is independent of* ε*,* ζ *and* δ*.*

Proof. The proof is given by the following procedures, which is divided into three parts. To construct the stationary solutions by the Schauder fixed-point theorem, we define a mapping $T : (r, q) \mapsto (R, Q)$ over

$$W[N_0, N_1, N_2, N_3] := \left\{ (f, g) \in H^2(\Omega) \;\middle|\; \begin{array}{c} \frac{1}{2}B_m \leq f \leq 2B_M, \\ \|f_x\| \leq N_0, \quad \|f_{xx}\| \leq N_1, \\ \|g-1\|_1 \leq \delta N_2, \quad \|g_{xx}\| \leq N_3 \end{array} \right\},$$

where N_0, N_1, N_2 and N_3 are positive constants to be determined latter. For given (r, q), define Q by solving the linearized problem

$$JQ_x - \frac{2}{3}Jq\,(\log r)_x - \frac{2}{3}\kappa_0 Q_{xx} = \left(\frac{2}{3} - \frac{\varepsilon}{3\zeta}\right)\frac{J^2}{r} - \frac{r}{\zeta}(Q-1), \tag{3.12a}$$

$$Q_x(0) = Q_x(1) = 0, \tag{3.12b}$$

$$J := \mathcal{J}[r, q], \tag{3.12c}$$

where $\mathcal{J}$ is defined in (3.9a). Next R is defined by solving

$$\left(\frac{1}{r}S[r, J, Q]R_x\right)_x + Q_{xx} - \frac{J}{r^2}R_x - R = -D, \tag{3.13a}$$

$$R(0) = \rho_l, \quad R(1) = \rho_r, \tag{3.13b}$$

where (3.13a) is the linearization of (3.7). We show the solvability of the problems (3.12) and (3.13) in First step. In Second step, it is proven that the mapping T has a fixed-point. The desired solution to (2.17)–(2.20) is constructed from the fixed-point of the mapping T in Third step.

First step. Since the equation (3.12a) is uniformly elliptic, the solvability of the problem (3.12) in $\mathcal{B}^2(\overline{\Omega})$ immediately follows from the standard theory for the linear elliptic equations. Hence, Q is determined from (r, q). We show the unique existence of solution R in $\mathcal{B}^2(\overline{\Omega})$ to the problem (3.13). For this purpose, it is sufficient to show $S[r, J, Q] \geq c > 0$ which means (3.13a) is uniformly elliptic, thanks to the standard theory again. Multiply the equation

(3.12a) by $Q-1$ and integrate the result by part over the domain Ω to obtain

$$\begin{aligned}\int_0^1 \frac{2}{3}\kappa_0 Q_x^2 + \frac{r}{\zeta}(Q-1)^2\,dx &= -\int_0^1 \left\{ JQ_x - \frac{2}{3}Jq(\log r)_x - \left(\frac{2}{3} - \frac{\varepsilon}{3\zeta}\right)\frac{J^2}{r}\right\}(Q-1)\,dx \\ &\le \delta^2 + C[B_m, B_M, N_2]\delta\|Q_x\|^2 + C[B_m, B_M, N_0, N_2]\|Q-1\|^2. \end{aligned} \tag{3.14}$$

In deriving the above inequality, we have used the Schwartz inequality and the estimate

$$|J| \le C[B_m, B_M, N_2]\delta, \tag{3.15}$$

which follows from (3.9). Then taking δ and ζ sufficiently small in (3.14) yields

$$\|Q_x\| + \frac{1}{\sqrt{\zeta}}\|Q-1\| \le \overline{C}_2[B_m, B_M]\delta, \tag{3.16}$$

where $\overline{C}_2$ is a positive constant independent of N_0, N_1, N_2 and N_3. Thus we see from (3.15), (3.16) and the smallness of δ that there exists a certain positive constant $\bar{c}$, independent of N_0, N_1, N_2 and N_3, such that

$$S[r, J_*, Q] \ge \bar{c} > 0. \tag{3.17}$$

Consequently, the mapping T is defined over $W[N_0, N_1, N_2, N_3] \to \mathcal{B}^2(\overline{\Omega})$.

Second step. In order to apply the Schauder fixed-point theorem, we determine the constants N_0, N_1, N_2, N_3 so that the image of T is contained in $W[N_0, N_1, N_2, N_3]$. Firstly, we derive the estimate of R_x as follows. Multiplying (3.13a) by $-R+\Gamma$ and integrating the resultant equality by part over Ω give

$$\begin{aligned}&\int_0^1 \frac{1}{r}S[r,J,Q](R-\Gamma)_x^2 + (R-\Gamma)^2\,dx \\ &\quad = -\int_0^1 \left\{\Gamma - D + \frac{J}{r^2}(R-\Gamma+\Gamma)_x\right\}(R-\Gamma) + \left(\frac{1}{r}S[r,J,Q]\Gamma_x + Q_x\right)(R-\Gamma)_x\,dx \\ &\quad \le (\mu+\delta)\|R-\Gamma\|_1^2 + C[\mu, B_m, B_M, \overline{C}_2, N_2], \end{aligned} \tag{3.18}$$

where μ is an arbitrary positive constant to be determined. In deriving the above inequality, we have used (3.15), (3.16) and the Schwarz inequality. Then let μ and δ in (3.18) so small that

$$\|R_x\| \le \overline{C}_0[B_m, B_M, \overline{C}_2, N_2], \tag{3.19}$$

where $\overline{C}_0$ is a positive constant independent of N_0, N_1 and N_3.

Secondly, we show the estimates of Q_{xx} and R_{xx}. Multiplying (3.12a) by $-Q_{xx}/r$ and integrating the resultant equality by part over Ω yield

$$\begin{aligned}\int_0^1 \frac{2\kappa_0}{3r}Q_{xx}^2 + \frac{1}{\zeta}Q_x^2\,dx &= \int_0^1 \left\{ JQ_x - \frac{2}{3}Jq\,(\log r)_x - \left(\frac{2}{3} - \frac{\varepsilon}{3\zeta}\right)\frac{J^2}{r} \right\}\frac{Q_{xx}}{r}\,dx \\ &\le (\mu + C[B_m, B_M, N_2]\delta)\,\|Q_x\|_1^2 + C[\mu, B_m, B_M, N_0, N_2]\delta^2,\end{aligned}$$

where we have also used (3.15) and the Schwarz inequality. Then, making μ and δ sufficiently small, we obtain

$$\|Q_{xx}\| + \frac{1}{\sqrt{\zeta}}\|Q_x\| \le \overline{C}_3[B_m, B_M, N_0, N_2]\delta, \tag{3.20}$$

where $\overline{C}_0$ is a positive constant independent of N_1 and N_3. Solve (3.13a) with respect to R_{xx}, take the L^2–norm, and then estimate the result by using (3.15), (3.17), (3.19) and (3.20). These computations give

$$\|R_{xx}\| \le \overline{C}_1[B_m, B_M, N_0, N_2, \overline{C}_0, \overline{C}_2], \tag{3.21}$$

where $\overline{C}_1$ is a positive constant independent of N_1 and N_3.

Thirdly, we prove the estimate

$$\frac{1}{2}B_m \le R \le 2B_M. \tag{3.22}$$

Divide the equation (3.12a) by r and differentiate the resultant equation. Multiply the result by $-Q_{xxx}$, integrate it by part over Ω and then use (3.15) and (3.20) as well as the Schwarz and the Sobolev inequalities. The resulting inequality is

$$\begin{aligned}\int_0^1 \frac{2\kappa_0}{3r}Q_{xxx}^2 + \frac{1}{\zeta}Q_{xx}^2\,dx &= \int_0^1 \left\{ \frac{2\kappa_0 r_x}{3r^2}Q_{xx} + \left(\frac{J}{\rho}Q_x - \frac{2}{3}Jq\frac{r_x}{r^2} - \left(\frac{2}{3} - \frac{\varepsilon}{3\zeta}\right)\frac{J^2}{r^2}\right)_x \right\}\frac{Q_{xxx}}{r}\,dx \\ &\le \mu\|Q_{xxx}\|_1^2 + C[\mu, B_m, B_M, \overline{C}_3, N_0, N_1, N_2]\delta^2.\end{aligned}$$

Then, letting μ small enough, we have

$$\|Q_{xxx}\| + \frac{1}{\sqrt{\zeta}}\|Q_{xx}\| \le C[B_m, B_M, \overline{C}_3, N_0, N_1, N_2]\delta. \tag{3.23}$$

Applying the maximal principle to the elliptic equation (3.13a) with (3.17), we have

$$\inf_{x\in\Omega}\left(D(x) + Q_{xx}(x)\right) \le R \le \sup_{x\in\Omega}\left(D(x) + Q_{xx}(x)\right). \tag{3.24}$$

Taking δ sufficiently small in (3.20) and (3.23) with the aid of the Sobolev inequality, we can make $|Q_{xx}|_0$ so small that the inequality (3.24) means the estimate (3.22).

Now we determine the constants N_0, N_1, N_3 and N_4 by letting $N_2 := \overline{C}_2$, $N_0 =: \overline{C}_0$ $N_3 := \overline{C}_3$ and $N_1 = \overline{C}_1$ in this order. Then the estimates (3.16), (3.19) and (3.20)–(3.22) show that the image of the mapping T is contained in $W[\,\cdot\,] := W[N_0, N_1, N_2, N_3]$. Here the set $W[\,\cdot\,]$ is a compact convex subset in $\mathcal{B}^1(\overline{\Omega})$. We also easily confirm that the mapping $T : W[\,\cdot\,] \to W[\,\cdot\,]$ is continuous in $\mathcal{B}^1(\overline{\Omega})$-norm. Hence, the mapping T has a fixed-point

$$(\tilde{\rho}^{\varepsilon}_{\zeta}, \tilde{\theta}^{\varepsilon}_{\zeta}) = T[(\tilde{\rho}^{\varepsilon}_{\zeta}, \tilde{\theta}^{\varepsilon}_{\zeta})] \in W[N_0, N_1, N_2, N_3],$$

owing to the Schauder fixed-point theorem (see Theorem 11.1 in [12] for example).

<u>*Third step.*</u> Finally we construct the solution to the boundary value problem (2.17)–(2.20) from the fixed-point $(\tilde{\rho}^{\varepsilon}_{\zeta}, \tilde{\theta}^{\varepsilon}_{\zeta})$. We define a constant $\tilde{j}^{\varepsilon}_{\zeta} := \mathcal{J}[\tilde{\rho}^{\varepsilon}_{\zeta}, \tilde{\theta}^{\varepsilon}_{\zeta}]$ and a function $\tilde{\phi}^{\varepsilon}_{\zeta} := \Phi[\tilde{\rho}^{\varepsilon}_{\zeta}]$, where $\mathcal{J}$ and Φ are given in (3.9a) and (3.10). Then we see that

$$(\tilde{\rho}^{\varepsilon}_{\zeta}, \tilde{j}^{\varepsilon}_{\zeta}, \tilde{\theta}^{\varepsilon}_{\zeta}, \tilde{\phi}^{\varepsilon}_{\zeta}) \in \mathcal{B}^2(\overline{\Omega}) \times \mathcal{B}^2(\overline{\Omega}) \times \mathcal{B}^3(\overline{\Omega}) \times \mathcal{B}^2(\overline{\Omega})$$

is the desired solution to (2.17)–(2.20). It satisfies the subsonic condition (2.13), owing to (3.17). Moreover, the estimates (3.11a) and (3.11b) follow from (3.22) and (3.16), respectively. □

Consequently, we have shown the existence of the stationary solutions for the hydrodynamic and the energy-transport models. Before proving their uniqueness, we derive several estimates.

Lemma 3.3. *Let* $(\tilde{\rho}^{\varepsilon}_{\zeta}, \tilde{j}^{\varepsilon}_{\zeta}, \tilde{\theta}^{\varepsilon}_{\zeta}, \tilde{\phi}^{\varepsilon}_{\zeta})$ *be a solution in* $\mathcal{B}^2(\overline{\Omega}) \times \mathcal{B}^2(\overline{\Omega}) \times \mathcal{B}^3(\overline{\Omega}) \times \mathcal{B}^2(\overline{\Omega})$ *to the boundary value problem* (2.17)–(2.20) *satisfying the conditions* (2.13) *and* (3.11a). *For arbitrary constant* ρ_l, *there exist positive constants* δ_0, ζ_0 *and* η_0 *such that if* $\delta \le \delta_0$, $0 \le \varepsilon < \zeta \le \zeta_0$ *and* $|\tilde{\theta} - 1|_0 \le \eta_0$, *the solution* $(\tilde{\rho}^{\varepsilon}_{\zeta}, \tilde{j}^{\varepsilon}_{\zeta}, \tilde{\theta}^{\varepsilon}_{\zeta}, \tilde{\phi}^{\varepsilon}_{\zeta})$ *verifies the formula* (3.9a) *and the estimates*

$$|\tilde{j}^{\varepsilon}_{\zeta}| \le C\delta, \tag{3.25a}$$

$$S[\tilde{\rho}^{\varepsilon}_{\zeta}, \tilde{j}^{\varepsilon}_{\zeta}, \tilde{\theta}^{\varepsilon}_{\zeta}] \ge c, \tag{3.25b}$$

$$|(\tilde{\rho}^{\varepsilon}_{\zeta}, \tilde{\phi}^{\varepsilon}_{\zeta})|_2 \le C, \tag{3.25c}$$

$$\frac{1}{\zeta}\|\tilde{\theta}^{\varepsilon}_{\zeta} - 1\|_1 + \frac{1}{\sqrt{\zeta}}\|(\tilde{\theta}^{\varepsilon}_{\zeta})_{xx}\| + \|(\tilde{\theta}^{\varepsilon}_{\zeta})_{xxx}\| \le C\delta, \tag{3.25d}$$

where C *and* c *are positive constants independent of* ε, ζ *and* δ.

Proof. Note that the current $\tilde{j}$ must be given by a solution to the quadratic equation (3.8). As it is also required to satisfy the subsonic condition (2.13), the current is explicitly given by the formula (3.9a) for sufficiently small δ. By estimating (3.9a), we show (3.25a) as follows.

Estimating (3.9a) by using (3.11a) and $|\tilde{\theta}^{\varepsilon} - 1|_0 \le \eta_0$ gives the estimate

$$|\tilde{j}^{\varepsilon}_{\zeta}| = |\mathcal{J}[\tilde{\rho}^{\varepsilon}_{\zeta}, \tilde{\theta}^{\varepsilon}_{\zeta}]| \le C[B_m, B_M](\delta + \|(\tilde{\theta}^{\varepsilon}_{\zeta})_x\|). \tag{3.26}$$

Multiply the equation (2.17c) by $\theta^{\varepsilon}_{\zeta} - 1$ and integrate the resulting equality by part over the domain Ω to obtain

$$\begin{aligned}\int_0^1 \frac{2}{3}\kappa_0(\tilde{\theta}^{\varepsilon}_{\zeta})^2_x + \frac{\tilde{\rho}^{\varepsilon}_{\zeta}}{\zeta}(\tilde{\theta}^{\varepsilon}_{\zeta} - 1)^2\, dx &= \frac{2}{3}\tilde{j}^{\varepsilon}_{\zeta}\int_0^1 (\log\tilde{\rho}^{\varepsilon}_{\zeta})_x(\tilde{\theta}^{\varepsilon}_{\zeta} - 1)\, dx \\ + \frac{2}{3}\tilde{j}^{\varepsilon}_{\zeta}\int_0^1 (\log\tilde{\rho}^{\varepsilon}_{\zeta})_x(\tilde{\theta}^{\varepsilon}_{\zeta} - 1)^2\, dx &- \int_0^1 \left\{\tilde{j}_{\zeta}(\tilde{\theta}^{\varepsilon}_{\zeta})_x - \left(\frac{2}{3} - \frac{\varepsilon}{3\zeta}\right)\frac{(\tilde{j}^{\varepsilon}_{\zeta})^2}{\tilde{\rho}^{\varepsilon}_{\zeta}}\right\}(\tilde{\theta}^{\varepsilon}_{\zeta} - 1)\, dx. \end{aligned} \tag{3.27}$$

We estimate each term in the right hand side of (3.27) one by one. Substituting (3.9a) in the first term in the right hand side of (3.27) and applying the integration by part, the Sobolev and the Young inequalities, we have

$$\begin{aligned}(\text{First term}) &= \frac{4}{3}\mathcal{K}^{-1}[\tilde{\rho}^{\varepsilon}_{\zeta}, \tilde{\theta}^{\varepsilon}_{\zeta}]\left(\mathcal{L}[\tilde{\theta}^{\varepsilon}_{\zeta}] + \int_0^1 (\tilde{\theta}^{\varepsilon}_{\zeta})_x \log\tilde{\rho}^{\varepsilon}_{\zeta}\, dx\right) \\ &\quad\times\left((\tilde{\theta}^{\varepsilon}_{\zeta}(1) - 1)\log\rho_r - (\tilde{\theta}^{\varepsilon}_{\zeta}(0) - 1)\log\rho_l - \int_0^1 (\tilde{\theta}^{\varepsilon}_{\zeta})_x \log\tilde{\rho}^{\varepsilon}_{\zeta}\, dx\right) \\ &\le \frac{4}{3}\mathcal{K}^{-1}[\tilde{\rho}^{\varepsilon}_{\zeta}, \tilde{\theta}^{\varepsilon}_{\zeta}]\left((\tilde{\theta}^{\varepsilon}_{\zeta}(1) - 1)\log\rho_r - (\tilde{\theta}^{\varepsilon}_{\zeta}(0) - 1)\log\rho_l\right)\int_0^1 (\tilde{\theta}^{\varepsilon}_{\zeta})_x \log\tilde{\rho}^{\varepsilon}_{\zeta}\, dx \\ &\quad + \frac{4}{3}\mathcal{K}^{-1}[\tilde{\rho}^{\varepsilon}_{\zeta}, \tilde{\theta}^{\varepsilon}_{\zeta}]\mathcal{L}[\tilde{\theta}^{\varepsilon}_{\zeta}]\left((\tilde{\theta}^{\varepsilon}_{\zeta}(1) - 1)\log\rho_r - (\tilde{\theta}^{\varepsilon}_{\zeta}(1) - 1)\log\rho_l\right) \\ &\quad + \frac{4}{3}\mathcal{K}^{-1}[\tilde{\rho}^{\varepsilon}_{\zeta}, \tilde{\theta}^{\varepsilon}_{\zeta}]\mathcal{L}[\tilde{\theta}^{\varepsilon}_{\zeta}]\int_0^1 (\tilde{\theta}^{\varepsilon}_{\zeta})_x \log\tilde{\rho}^{\varepsilon}_{\zeta}\, dx \\ &\le C[B_m, B_M]\left(|\tilde{\theta}^{\varepsilon}_{\zeta} - 1|_0 + \delta\right)\left(|\tilde{\theta}^{\varepsilon}_{\zeta} - 1|_0 + \delta + \|(\tilde{\theta}^{\varepsilon}_{\zeta})_x\|\right) \\ &\le \mu\|(\tilde{\theta}^{\varepsilon}_{\zeta})_x\|^2 + C[\mu, B_m, B_M](\|\tilde{\theta}^{\varepsilon}_{\zeta} - 1\|^2 + \delta^2), \end{aligned} \tag{3.28}$$

where μ is an arbitrary positive constant. In the first inequality above, we have also used $\mathcal{K}^{-1}[\tilde{\rho}^{\varepsilon}_{\zeta}, \tilde{\theta}^{\varepsilon}_{\zeta}](\int_0^1 (\tilde{\theta}^{\varepsilon}_{\zeta})_x \log\tilde{\rho}^{\varepsilon}_{\zeta}\, dx)^2 \ge 0$. By the integration by part, the Sobolev inequality and the estimate (3.26), the second term is estimated as

$$\begin{aligned}(\text{Second term}) &= \frac{2}{3}\tilde{j}^{\varepsilon}_{\zeta}\left\{(\tilde{\theta}^{\varepsilon}_{\zeta}(1) - 1)^2\log\rho_l - (\tilde{\theta}^{\varepsilon}_{\zeta}(0) - 1)^2\log\rho_r\right\} \\ &\quad - \frac{2}{3}\tilde{j}^{\varepsilon}_{\zeta}\int_0^1 \left\{(\tilde{\theta}^{\varepsilon}_{\zeta} - 1)^2\right\}_x \log\tilde{\rho}^{\varepsilon}_{\zeta}\, dx \\ &\le C[B_m, B_M]\left(\delta + \|\tilde{\theta}^{\varepsilon}_{\zeta} - 1\|\right)\|\tilde{\theta}^{\varepsilon}_{\zeta} - 1\|_1^2. \end{aligned} \tag{3.29}$$

Furthermore, the straightforward computation gives

$$\text{(Third term)} \le C[B_m, B_M] \left\{ \left(\delta + \|\tilde{\theta}_\zeta^\varepsilon - 1\| \right) \|\tilde{\theta}_\zeta^\varepsilon - 1\|_1^2 + \|\tilde{\theta}_\zeta^\varepsilon - 1\|^2 + \delta^2 \right\}. \tag{3.30}$$

Substituting (3.28)–(3.30) in (3.27) and successively making μ, δ, ζ and η_0 sufficiently small yield

$$\|(\tilde{\theta}_\zeta^\varepsilon)_x\| + \frac{1}{\sqrt{\zeta}} \|\tilde{\theta}_\zeta^\varepsilon - 1\| \le C[B_m, B_M]\delta. \tag{3.31}$$

Combining (3.31) and (3.26), we have the desired estimate (3.25a), which together with the estimates (3.11a) and (3.31) means that (3.25b) holds for sufficiently small δ.

We show the estimates (3.25c) and (3.25d). The inequality $|\tilde{\phi}_\zeta^\varepsilon|_2 \le C$ in (3.25c) immediately follows from the formula (3.10) and the inequality (3.11a). The estimates of $\tilde{\rho}_\zeta^\varepsilon$ and $\tilde{\theta}_\zeta^\varepsilon$ in (3.25c) and (3.25d) are derived as follows. By a similar computation as the derivation of (3.19), we have the estimate $\|(\tilde{\rho}_\zeta^\varepsilon)_x\| \le C$. Moreover, similar computations as the derivation of (3.20) and (3.21) give the estimates $\|(\tilde{\theta}_\zeta^\varepsilon)_{xx}\| \le C\delta$ and $\|(\tilde{\rho}_\zeta^\varepsilon)_{xx}\| \le C$, respectively. The estimate $\|(\tilde{\theta}_\zeta^\varepsilon)_{xxx}\| + \|(\tilde{\theta}_\zeta^\varepsilon)_{xx}\|/\sqrt{\zeta} \le C\delta$ is also shown similarly as (3.23). Furthermore, solve the equation (3.7) with respect to $(\tilde{\rho}_\zeta^\varepsilon)_{xx}$ and then take $\mathcal{B}^0$-norm of the result to obtain the estimate $|(\tilde{\rho}_\zeta^\varepsilon)_{xx}|_0 \le C$. Similarly, with using (2.17c), $\|\tilde{\theta}_\zeta^\varepsilon - 1\|_1/\zeta \le C$. Hence, we have the desired estimates (3.25c) and (3.25d). □

Now we are at a position to show the uniqueness of the stationary solutions for the hydrodynamic and the energy-transport models.

Lemma 3.4. *Under the same conditions in Lemma 3.3, the solution* $(\tilde{\rho}_\zeta^\varepsilon, \tilde{j}_\zeta^\varepsilon, \tilde{\theta}_\zeta^\varepsilon, \tilde{\phi}_\zeta^\varepsilon) \in \mathcal{B}^2(\overline{\Omega}) \times \mathcal{B}^2(\overline{\Omega}) \times \mathcal{B}^3(\overline{\Omega}) \times \mathcal{B}^2(\overline{\Omega})$ *is unique.*

Proof. Let $(\tilde{\rho}_1, \tilde{j}_1, \tilde{\theta}_1, \tilde{\phi}_1)$ and $(\tilde{\rho}_2, \tilde{j}_2, \tilde{\theta}_2, \tilde{\phi}_2)$ be solutions to the stationary problem (2.17)–(2.20). By using the estimates (3.11a) and (3.25), the mean value theorem and the Poincaré inequality, we estimate the difference between $\tilde{j}_1 = \mathcal{J}[\tilde{\rho}_1, \tilde{\theta}_1]$ and $\tilde{j}_2 = \mathcal{J}[\tilde{\rho}_2, \tilde{\theta}_2]$ as

$$|\tilde{j}_1 - \tilde{j}_2| \le C\left(\delta\|\bar{r}_x\| + \|\bar{q}\|_1\right), \tag{3.32}$$
$$\bar{r} := \log\tilde{\rho}_1 - \log\tilde{\rho}_2, \quad \bar{q} := \tilde{\theta}_1 - \tilde{\theta}_2,$$

where C is a positive constant independent of ε, ζ and δ.

Notice that the function $\bar{q} = \tilde{\theta}_1 - \tilde{\theta}_2$ verifies the equation

$$\begin{aligned} &\tilde{j}_1\tilde{\theta}_{1x} - \tilde{j}_2\tilde{\theta}_{2x} - \frac{2}{3}\tilde{j}_1(\log\tilde{\rho}_1)_x\tilde{\theta}_1 + \frac{2}{3}\tilde{j}_2(\log\tilde{\rho}_2)_x\tilde{\theta}_2 - \frac{2}{3}\kappa_0\bar{q}_{xx} \\ &\qquad = \left(\frac{2}{3} - \frac{\varepsilon}{3\zeta}\right)\left(\frac{\tilde{j}_1^2}{\tilde{\rho}_1} - \frac{\tilde{j}_2^2}{\tilde{\rho}_2}\right) - \frac{\tilde{\rho}_1}{\zeta}\bar{q} - \frac{\tilde{\rho}_1 - \tilde{\rho}_2}{\zeta}(\tilde{\theta}_2 - 1), \end{aligned} \tag{3.33}$$

owing to the equation (2.17c). Multiplying the equation (3.33) by $\bar{q}$, integrating the resultant equality by part over Ω and then using the boundary condition $\bar{q}_x(0) = \bar{q}_x(1) = 0$, we have

$$\begin{aligned}\int_0^1 \frac{2}{3}\kappa_0 \bar{q}_x^2 + \frac{\tilde{\rho}_1}{\zeta}\bar{q}^2\,dx &= \int_0^1 \left\{\frac{2}{3}\tilde{j}_1(\log\tilde{\rho}_1)_x\tilde{\theta}_1 - \frac{2}{3}\tilde{j}_2(\log\tilde{\rho}_2)_x\tilde{\theta}_2 - \tilde{j}_1\tilde{\theta}_{1x} + \tilde{j}_2\tilde{\theta}_{2x}\right\}\bar{q} \\ &\quad + \left\{\left(\frac{2}{3} - \frac{\varepsilon}{3\zeta}\right)\left(\frac{\tilde{j}_1^2}{\tilde{\rho}_1} - \frac{\tilde{j}_2^2}{\tilde{\rho}_2}\right) - \frac{\tilde{\rho}_1 - \tilde{\rho}_2}{\zeta}(\tilde{\theta}_2 - 1)\right\}\bar{q}\,dx \\ &\le \mu\|\bar{q}_x\|^2 + C\delta(\|\bar{r}_x\|^2 + \|\bar{q}_x\|^2) + C[\mu]\|\bar{q}\|^2,\end{aligned}$$

where we have also used the mean value theorem, the Schwarz and the Poincaré inequalities as well as (3.25) and (3.32). Thus, letting μ, δ and ζ small enough leads to

$$\|\bar{q}\|_1 \le \delta C\|\bar{r}_x\|. \tag{3.34}$$

We see from (2.17a) that the function $\bar{r} = \log\tilde{\rho}_1 - \log\tilde{\rho}_2$ satisfies

$$\begin{aligned}S[\tilde{\rho}_1, \tilde{j}_1, \tilde{\theta}_1]\bar{r}_x + \Big(S[\tilde{\rho}_1, \tilde{j}_1, \tilde{\theta}_1] - S[\tilde{\rho}_2, \tilde{j}_2, \tilde{\theta}_2]\Big)(\log\tilde{\rho}_2)_x + \bar{q}_x \\ = (\tilde{\phi}_1 - \tilde{\phi}_2)_x - \left(\frac{\tilde{j}_1}{\tilde{\rho}_1} - \frac{\tilde{j}_2}{\tilde{\rho}_2}\right).\end{aligned} \tag{3.35}$$

Multiply (3.35) by $\bar{r}_x$, integrate the resultant equality by parts over Ω using the boundary condition $\bar{r}(0) = \bar{r}(1) = 0$ and the equation (2.17d) to get

$$\begin{aligned}&\int_0^1 S[\tilde{\rho}_1, \tilde{j}_1, \tilde{\theta}_1]\bar{r}_x^2 + (\tilde{\rho}_1 - \tilde{\rho}_2)\bar{r}\,dx \\ &\quad = -\int_0^1 \left\{\Big(S[\tilde{\rho}_1, \tilde{j}_1, \tilde{\theta}_1] - S[\tilde{\rho}_2, \tilde{j}_2, \tilde{\theta}_2]\Big)(\log\tilde{\rho}_2)_x + \bar{q}_x + \left(\frac{\tilde{j}_1}{\tilde{\rho}_1} - \frac{\tilde{j}_2}{\tilde{\rho}_2}\right)\right\}\bar{r}_x\,dx.\end{aligned} \tag{3.36}$$

The right hand side of the equality (3.36) is estimated by $\delta C\|\bar{r}_x\|^2$ by the mean value theorem and the Poincaré inequality as well as the estimates (3.25), (3.32) and (3.34). Taking δ small enough, we have $\|\bar{r}_x\|^2 \le 0$ with aid of $(\tilde{\rho}_1 - \tilde{\rho}_2)\bar{r} \ge 0$ and (3.25b). Consequently, we see $\tilde{\rho}_1 \equiv \tilde{\rho}_2$ holds, which immediately shows $\tilde{\theta}_1 \equiv \tilde{\theta}_2$, $\tilde{j}_1 \equiv \tilde{j}_2$ and $\tilde{\phi}_1 \equiv \tilde{\phi}_2$ thanks to (3.10). (3.32) and (3.34). □

Lemmas 3.2 and 3.4 immediately give the unique existence of the stationary solution to both the hydrodynamic and the energy-transport models.

Theorem 3.5. *Let the doping profile and the boundary data satisfy conditions* (2.2), (2.4) *and* (2.6). *For arbitrary* ρ_l, *there exist positive constants* δ_0, ζ_0 *and* η_0 *such that if* $\delta \le \delta_0$ *and* $0 \le \varepsilon < \zeta \le \zeta_0$, *then the boundary value problem* (2.17)–(2.20) *has a unique solution* $(\tilde{\rho}^\varepsilon_\zeta, \tilde{j}^\varepsilon_\zeta, \tilde{\theta}^\varepsilon_\zeta, \tilde{\phi}^\varepsilon_\zeta) \in \mathcal{B}^2(\overline{\Omega}) \times \mathcal{B}^2(\overline{\Omega}) \times \mathcal{B}^3(\overline{\Omega}) \times \mathcal{B}^2(\overline{\Omega})$ *satisfying the conditions* (2.13), (3.11a) *and* $|\tilde{\theta}^\varepsilon_\zeta - 1|_0 \le \eta_0$.

3.2 Relaxation limits of stationary solutions

In this section, we justify the relaxation limits of stationary solutions. Firstly, it is shown that the stationary solution for the hydrodynamic model converges to that for the energy-transport model as the parameter ε tends to zero. Secondly, we prove that the stationary solution for the energy-transport model converges to that for the drift-diffusion model as the parameter ζ tends to zero. These results also give the justification of relaxation limit from the hydrodynamic model to the drift-diffusion model.

Lemma 3.6. *Under the same assumptions as in Theorem 3.5, the stationary solution* $(\tilde{\rho}^\varepsilon_\zeta, \tilde{j}^\varepsilon_\zeta, \tilde{\theta}^\varepsilon_\zeta, \tilde{\phi}^\varepsilon_\zeta)$ *for* (2.17) *converges to the stationary solution* $(\tilde{\rho}^0_\zeta, \tilde{j}^0_\zeta, \tilde{\theta}^0_\zeta, \tilde{\phi}^0_\zeta)$ *for* (3.1) *as* ε *tends to zero. Precisely, there exists a positive constant* ζ_0 *such that, for* $\varepsilon < \zeta \le \zeta_0$*,*

$$\frac{1}{\sqrt{\zeta}}\|\tilde{\theta}^\varepsilon_\zeta - \tilde{\theta}^0_\zeta\| + \|(\tilde{\rho}^\varepsilon_\zeta - \tilde{\rho}^0_\zeta, \tilde{\theta}^\varepsilon_\zeta - \tilde{\theta}^0_\zeta)\|_1 + |\tilde{j}^\varepsilon_\zeta - \tilde{j}^0_\zeta| \le C\delta^2\left(\varepsilon + \frac{\varepsilon}{\zeta}\right), \tag{3.37}$$

$$\frac{1}{\sqrt{\zeta}}\|(\tilde{\theta}^\varepsilon_\zeta - \tilde{\theta}^0_\zeta)_x\| + \left\|\left(\left\{\tilde{\rho}^\varepsilon_\zeta - \tilde{\rho}^0_\zeta\right\}_{xx}, \left\{\tilde{\theta}^\varepsilon_\zeta - \tilde{\theta}^0_\zeta\right\}_{xx}\right)\right\| + \|\tilde{\phi}^\varepsilon_\zeta - \tilde{\phi}^0_\zeta\|_4 \le C\delta^2\left(\varepsilon + \frac{\varepsilon}{\zeta}\right), \tag{3.38}$$

where the positive constant C *is independent of* ε*,* ζ *and* δ*.*

Proof. In this proof, we omit the suffix ζ to express the solution for simplicity as $(\tilde{\rho}^\varepsilon_\zeta, \tilde{j}^\varepsilon_\zeta, \tilde{\theta}^\varepsilon_\zeta, \tilde{\phi}^\varepsilon_\zeta)$ is denoted by $(\tilde{\rho}^\varepsilon, \tilde{j}^\varepsilon, \tilde{\theta}^\varepsilon, \tilde{\phi}^\varepsilon)$, and $(\tilde{\rho}^0_\zeta, \tilde{j}^0_\zeta, \tilde{\theta}^0_\zeta, \tilde{\phi}^0_\zeta)$ by $(\tilde{\rho}^0, \tilde{j}^0, \tilde{\theta}^0, \tilde{\phi}^0)$. By virtue of the formula of the current densities $\tilde{j}^\varepsilon$ and $\tilde{j}^0$, we have the estimate

$$\begin{aligned}
&|\tilde{j}^\varepsilon - \tilde{j}^0| \le C\left(\delta\|r^\varepsilon_x\| + \|q^\varepsilon\|_1 + \delta^2\varepsilon\right), \\
&r^\varepsilon := \log\tilde{\rho}^\varepsilon - \log\tilde{\rho}^0, \quad q^\varepsilon := \tilde{\theta}^\varepsilon - \tilde{\theta}^0
\end{aligned} \tag{3.39}$$

owing to the mean value theorem and the Poincaré inequality with (3.11a) and (3.25d).

Subtracting the equation (2.17c) from the equation (3.1b) gives

$$\begin{aligned}
&-\frac{2}{3}\kappa_0 q^\varepsilon_{xx} + \frac{\tilde{\rho}^\varepsilon}{\zeta}q^\varepsilon = F, \\
&F := -\tilde{j}^\varepsilon\tilde{\theta}^\varepsilon_x + \tilde{j}^0\tilde{\theta}^0_x + \frac{2}{3}\tilde{j}^\varepsilon(\log\tilde{\rho}^\varepsilon)_x\tilde{\theta}^\varepsilon - \frac{2}{3}\tilde{j}^0(\log\tilde{\rho}^0)_x\tilde{\theta}^0 \\
&\qquad + \frac{2}{3}\left(\frac{(\tilde{j}^\varepsilon)^2}{\tilde{\rho}^\varepsilon} - \frac{(\tilde{j}^0)^2}{\tilde{\rho}^0}\right) - \frac{\varepsilon}{3\zeta}\frac{(\tilde{j}^\varepsilon)^2}{\tilde{\rho}^\varepsilon} - \frac{\tilde{\rho}^\varepsilon - \tilde{\rho}^0}{\zeta}(\tilde{\theta}^0 - 1).
\end{aligned} \tag{3.40}$$

Note that the L^2-norm of F in (3.40) is estimated as

$$\|F\| \le C\delta\|r^\varepsilon_x\| + C\|q^\varepsilon\|_1 + C\delta^2\left(\varepsilon + \frac{\varepsilon}{\zeta}\right) \tag{3.41}$$

by the mean value theorem, the Poincaré and the Sobolev inequalities as well as (3.11a), (3.25) and (3.39). Multiply the equation (3.40) by q^ε, integrate the result over the domain Ω by part and utilize the boundary conditions $q^\varepsilon_x(0) = q^\varepsilon_x(1) = 0$. Then estimate the resulting equality with using (3.41). These calculations lead to

$$\int_0^1 \frac{2}{3}\kappa_0 (q^\varepsilon_x)^2 + \frac{\tilde{\rho}^\varepsilon}{\zeta}(q^\varepsilon)^2 \, dx = \int_0^1 F q^\varepsilon \, dx$$
$$\leq (\mu + C\delta)(\|r^\varepsilon_x\|^2 + \|q^\varepsilon_x\|^2) + C[\mu]\|q^\varepsilon\|^2 + C\delta^4 \left(\varepsilon^2 + \frac{\varepsilon^2}{\zeta^2}\right),$$

where μ is an arbitrary positive constant. Successively, making μ, δ and ζ small enough, we have

$$\|q^\varepsilon_x\|^2 + \frac{1}{\zeta}\|q^\varepsilon\|^2 \leq C\delta\|r^\varepsilon_x\|^2 + C\delta^4 \left(\varepsilon^2 + \frac{\varepsilon^2}{\zeta^2}\right). \tag{3.42}$$

Divide (2.17b) by $\tilde{\rho}^\varepsilon$ and (3.1d) by $\tilde{\rho}^0$, respectively. Then take the difference between the two resultant equations to obtain

$$\tilde{\theta}^\varepsilon r^\varepsilon_x + (\log \rho^0)_x q^\varepsilon + q^\varepsilon_x + (\tilde{\phi}^\varepsilon - \tilde{\phi}^0)_x - \varepsilon \left(\frac{\tilde{j}^\varepsilon}{\tilde{\rho}^\varepsilon}\right)^2 (\log \tilde{\rho}^\varepsilon)_x + \frac{\tilde{j}^\varepsilon}{\tilde{\rho}^\varepsilon} - \frac{\tilde{j}^0}{\tilde{\rho}^0} = 0. \tag{3.43}$$

Multiplying (3.43) by r^ε_x and then integrating the resulting equality over the domain Ω by part with using $r^\varepsilon(0) = r^\varepsilon(1) = 0$, (2.17d) and (3.1c), we have

$$\begin{aligned}
&\int_0^1 \tilde{\theta}^\varepsilon (r^\varepsilon_x)^2 + (\tilde{\rho}^\varepsilon - \tilde{\rho}^0) r^\varepsilon \, dx \\
&= \int_0^1 \left\{ -q^\varepsilon (\log \tilde{\rho}^0)_x - q^\varepsilon_x - \frac{\tilde{j}^\varepsilon}{\tilde{\rho}^\varepsilon} + \frac{\tilde{j}^0}{\tilde{\rho}^0} + \varepsilon \left(\frac{\tilde{j}^\varepsilon}{\tilde{\rho}^\varepsilon}\right)^2 (\log \tilde{\rho}^\varepsilon)_x \right\} r^\varepsilon_x \, dx \\
&\leq (\mu + C\delta)\|r^\varepsilon_x\|^2 + C[\mu]\delta^4 \left(\varepsilon^2 + \frac{\varepsilon^2}{\zeta^2}\right),
\end{aligned} \tag{3.44}$$

where we have also used (3.39) and (3.42). Since $(\tilde{\rho}^\varepsilon - \tilde{\rho}^0) r^\varepsilon \geq 0$, making μ and δ small enough in (3.44) yields the estimate $\|r^\varepsilon_x\|^2 \leq C\delta^4 (\varepsilon^2 + \varepsilon^2/\zeta^2)$, which means

$$\|r^\varepsilon\|_1^2 \leq C\delta^4 \left(\varepsilon^2 + \frac{\varepsilon^2}{\zeta^2}\right) \tag{3.45}$$

with aid of the Poincaré inequality. The inequality (3.45) together with (3.11a) yields the estimate of $(\tilde{\rho}^\varepsilon - \tilde{\rho}^0)$ in (3.37). Estimating (3.45) with (3.42), we have the estimate of $(\tilde{\theta}^\varepsilon - \tilde{\theta}^0)$

in (3.37). Substituting this estimate and (3.45) in (3.39) gives the estimate of $(\tilde{j}^{\varepsilon} - \tilde{j}^{0})$ in (3.37), which completes the derivation of the desired estimate (3.37).

To completion of the proof, it suffices to show the estimate (3.38). Multiplying (3.40) by $-q^{\varepsilon}_{xx}/\tilde{\rho}^{\varepsilon}$ and integrating the resulting equality over the domain Ω by part gives

$$\int_0^1 \frac{2\kappa_0}{3\tilde{\rho}^{\varepsilon}}(q^{\varepsilon}_{xx})^2 + \frac{1}{\zeta}(q^{\varepsilon}_{x})^2\,dx = \int_0^1 \frac{q^{\varepsilon}_{xx}}{\tilde{\rho}^{\varepsilon}} F\,dx \le \mu\|q^{\varepsilon}_{xx}\|^2 + C[\mu]\delta^4\left(\varepsilon^2 + \frac{\varepsilon^2}{\zeta^2}\right), \tag{3.46}$$

where μ is an arbitrary constant. In deriving the above inequality, we have also used the estimates (3.37) and (3.41). Then take μ sufficiently small in (3.46) to get

$$\|q^{\varepsilon}_{xx}\|^2 + \frac{1}{\zeta}\|q^{\varepsilon}_{x}\|^2 \le C\delta^4\left(\varepsilon^2 + \frac{\varepsilon^2}{\zeta^2}\right). \tag{3.47}$$

Differentiate the equation (3.43), solve the resultant equation with respect to r^{ε}_{xx}, take the L^2-norm of the result. Then applying (3.37) and (3.47), we obtain the estimate

$$\|r^{\varepsilon}\|_2^2 \le C\delta^4\left(\varepsilon^2 + \frac{\varepsilon^2}{\zeta^2}\right), \tag{3.48}$$

which also shows the estimate of $(\tilde{\rho}^{\varepsilon} - \tilde{\rho}^{0})_{xx}$ and $\tilde{\phi}^{\varepsilon} - \tilde{\phi}^{0}$ in (3.38) with aid of (3.10) and (3.11a). Consequently, the proof is completed. □

Lemma 3.7. *Under the same assumptions as in Lemmas 3.1–3.3, the stationary solution* $(\tilde{\rho}^{0}_{\zeta}, \tilde{j}^{0}_{\zeta}, \tilde{\theta}^{0}_{\zeta}, \tilde{\phi}^{0}_{\zeta})$ *for* (2.17) *converges to the stationary solution* $(\tilde{\rho}^{0}_{0}, \tilde{j}^{0}_{0}, \tilde{\phi}^{0}_{0})$ *for* (3.2) *as* ζ *tends to zero. Precisely, there exists a positive constant* ζ_0 *such that, for* $\zeta \le \zeta_0$,

$$\|(\tilde{\rho}^{0}_{\zeta} - \tilde{\rho}^{0}_{0}, \tilde{\theta}^{0}_{\zeta} - 1)\|_1 + |\tilde{j}^{0}_{\zeta} - \tilde{j}^{0}_{0}| + \|\tilde{\phi}^{0}_{\zeta} - \tilde{\phi}^{0}_{0}\|_3 \le C\delta\zeta, \tag{3.49}$$

$$\|(\{\tilde{\rho}^{0}_{\zeta} - \tilde{\rho}^{0}_{0}\}_{xx}, \{\tilde{\theta}^{0}_{\zeta} - 1\}_{xx})\| \le C\delta\sqrt{\zeta}, \tag{3.50}$$

where the positive constant C *is independent of* ζ *and* δ.

Proof. For simplicity, we abbreviate $(\tilde{\rho}^{0}_{\zeta}, \tilde{j}^{0}_{\zeta}, \tilde{\theta}^{0}_{\zeta}, \tilde{\phi}^{0}_{\zeta})$ as $(\tilde{\rho}_{\zeta}, \tilde{j}_{\zeta}, \tilde{\theta}_{\zeta}, \tilde{\phi}_{\zeta})$ and $(\tilde{\rho}^{0}_{0}, \tilde{j}^{0}_{0}, \tilde{\phi}^{0}_{0})$ as $(\tilde{\rho}_{0}, \tilde{j}_{0}, \tilde{\phi}_{0})$ in this proof. The estimates of the difference $\tilde{\theta}_{\zeta} - 1$ in (3.49) and (3.50) have been already shown in (3.25d). In the similar way as in the derivation of (3.39), the difference between $\tilde{j}_{\zeta}$ and $\tilde{j}_{0}$ are estimated as

$$|\tilde{j}_{\zeta} - \tilde{j}_{0}| \le C\delta\left(\|(r_{\zeta})_x\| + \zeta\right), \quad r_{\zeta} := \log\tilde{\rho}_{\zeta} - \log\tilde{\rho}_{0}, \tag{3.51}$$

thanks to the formula (3.6) and (3.9a). Once the estimate of $\tilde{\rho}_{\zeta} - \tilde{\rho}_{0}$ in (3.49) is shown, the estimate of $\tilde{j}_{\zeta} - \tilde{j}_{0}$ in (3.49) immediately follows. Hence, it suffices to derive the estimates of the r_ζ. Divide (3.1d) by $\tilde{\rho}_\zeta$ and (3.2c) by $\tilde{\rho}_0$, take the difference of the results and then multiply the difference by r_x^ε. Integrate the result over the domain Ω by part and estimate the resultant equality by using (3.25d) and (3.51). These calculations lead to

$$\begin{aligned}\int_0^1 (r_\zeta)_x^2 + (\tilde{\rho}_\zeta - \tilde{\rho}_0) r_\zeta \, dx &= -\int_0^1 \left\{ \frac{1}{\tilde{\rho}^\varepsilon} \{(\tilde{\theta}_\zeta - 1)\tilde{\rho}_\zeta\}_x + \frac{\tilde{j}_\zeta}{\tilde{\rho}_\zeta} - \frac{\tilde{j}_0}{\tilde{\rho}_0} \right\} (r_\zeta)_x \, dx \\ &\le (\mu + C\delta) \|(r_\zeta)_x\|^2 + C[\mu]\delta^2\zeta^2,\end{aligned}$$

where μ is an arbitrary constant. Take μ and δ so small that $\mu + C\delta < 1$ and then use the Poincaré inequality to obtain

$$\|r_\zeta\|_1 \le C\delta\zeta, \tag{3.52}$$

since $(\tilde{\rho}_\zeta - \tilde{\rho}_0) r_\zeta \ge 0$. Finally, the similar manner as in the estimation of (3.48) gives

$$\|r_\zeta\|_2 \le C\delta\sqrt{\zeta}. \tag{3.53}$$

Combining the estimates (3.51), (3.52) and (3.53) completes the proof. □

Lemmas 3.6 and 3.7 immediately give the next corollary, concerning the relaxation limit from the hydrodynamic to the drift-diffusion model.

Corollary 3.8. *Under the same assumptions as in Lemmas 3.1–3.3, the stationary solution* $(\tilde{\rho}_\zeta^\varepsilon, \tilde{j}_\zeta^\varepsilon, \tilde{\theta}_\zeta^\varepsilon, \tilde{\phi}_\zeta^\varepsilon)$ *for* (2.17) *converges to the stationary solution* $(\tilde{\rho}_0^0, \tilde{j}_0^0, \tilde{\phi}_0^0)$ *for* (3.2) *as* ε *and* ζ *tend to zero. Precisely, there exists a positive constant* ζ_0 *such that, for* $\varepsilon < \zeta \le \zeta_0$,

$$\|(\tilde{\rho}_\zeta^\varepsilon - \tilde{\rho}_0^\varepsilon, \tilde{\theta}_\zeta^\varepsilon - 1)\|_1 + |\tilde{j}_\zeta^\varepsilon - \tilde{j}_0^0| + \|\tilde{\phi}_\zeta^\varepsilon - \tilde{\phi}_0^0\|_3 \le C\delta \left(\varepsilon + \frac{\varepsilon}{\zeta} + \zeta \right), \tag{3.54}$$

$$\|(\{\tilde{\rho}_\zeta^\varepsilon - \tilde{\rho}_0^0\}_{xx}, \{\tilde{\theta}_\zeta^\varepsilon - 1\}_{xx})\| \le C\delta \left(\varepsilon + \frac{\varepsilon}{\zeta} + \sqrt{\zeta} \right), \tag{3.55}$$

where the positive constant C *is independent of* ε, ζ *and* δ.