Chapter 1

Introduction

In 1947, Bardeen, Brattain, and Shockly invent the first semiconductor transistor, a germanium transistor. Afterward, many kinds of semiconductor devices such as the integrated circuit (IC), the light-emitting diode (LED) and the solar cell have been invented. It goes without saying that the development of the ICs benefits a lot of industries. Therefore the ICs have been drastically developed over the past few decades. Nowadays the ICs consist of more than millions tiny transistors, most of which are metal-oxide semiconductor field effect transistors (MOSFET). MOSFETs are very important since each of them plays a role of a switch which represent a bit with the state "0" as current flow or the state "1" as no current flow.

For a successful design of the MOSFET, engineers are mainly concerned with the electron current, generated by the applied voltage, through the devices. Several mathematical models have been proposed to numerically simulate the current. These models, such as a hydrodynamic, an energy-transport and a drift-diffusion models, are often used for the device simulation with the suitable choice, depending on the purpose of the device usage. In fact, the hydrodynamic and the energy-transport models are especially important in analyzing the temperature change to study the hot carrier problem. The hydrodynamic model is more accurate in a physical point of view, and makes more detailed simulations possible than the energy-transport model. However, the hydrodynamic model contains hyperbolic equations, which may cause discontinuity of a solution. On the other hand, the energy-transport model consists of parabolic and elliptic equations only, which is easier to analyze numerically and theoretically. For example, engineers usually use the energy-transport model although they lose some accuracy. Furthermore, in cases in which the change of temperature is not significant, they may use the drift-diffusion model instead because it is obtained from the energy-transport model for the constant temperature. Hence, it is important not only in engineering but also in mathematics to study a model hierarchy, relations among these models, and to prove the validity of the use of simpler models rigorously.

The model hierarchy is formally understood by letting the momentum relaxation time τ_m and the energy relaxation time τ_e tend to zero. The relaxation times are the physical parameters in the hydrodynamic model, corresponding to the average time between collisions in kinetic theory. The energy-transport model is derived from the hydrodynamic models by letting the momentum relaxation time τ_m tend to zero. Furthermore, the drift-diffusion model is also formally derived from the energy-transport model by making the energy relaxation time τ_e zero (see Figure 2.1). These limit procedures are called relaxation time limits or the relaxation limits in short. The main purpose of the present paper is to study the mathematical justifications of the relaxation limits. Precisely, we show that the solution for the hydrodynamic model converges to that for the energy-transport model as the relaxation time τ_m tends to zero. It is also proven that the solution for the energy-transport model converges to that for the drift-diffusion model as the energy relaxation time τ_e tends to zero. In these justifications, the initial layers are given rise to the solution because the initial data is not necessarily in "momentum equilibrium" nor "energy equilibrium". However, we show that the layers decay exponentially fast as the relaxation times τ_m and τ_e tend to zero. They also decay exponentially as time t tends to infinity. For the results, no smallness of the initial data is assumed even though the hydrodynamic model contains nonlinear hyperbolic equations. Instead of the initial smallness assumption, we suppose that the relaxation times are sufficiently small. This assumption allows us to handle the large initial data since the drift-diffusion model, a limited system, consists of a uniformly parabolic equation and the Poisson equation.

Outline of paper. Before discussing our mathematical results, we give a brief description of the structure of semiconductor devices in Section 1.1. The derivations of semiconductor models are discussed in Section 1.2. Section 1.3 is devoted to a review of mathematical previous results. In Section 2, we formulate our mathematical problems and state the main results. In Section 3, we begin the detailed proofs with studying the unique existence and the relaxation limits of the stationary solutions to the hydrodynamic and the energy-transport models. It is shown in Section 4 that the time global solution for the energy-transport model exists and converges to that for the drift-diffusion model as the energy relaxation time τ_e tends to zero. Section 5 is devoted to showing the time global solvability of the hydrodynamic model and justifying the momentum and energy relaxation limits.

1.1 Structure of semiconductor device

We first explain the fundamental properties of semiconductors (refer to texts [16, 23, 34] for the more details). Semiconductors are the materials of which electrical conductivity intermediates between that of a conductor and an insulator. Precisely, the electrical conductivity of the semiconductors is less than $10^6\ \Omega^{-1}\mathrm{m}^{-1}$ and greater than $10^{-6}\ \Omega^{-1}\mathrm{m}^{-1}$. They typ-

ically consists of Group IV atoms such as silicon and germanium. Group IV atoms have four valence electrons and make four covalent bonds, which are very stable at low temperature. At high temperature, a lot of valence electrons become conduction electrons (free electrons). These electrons leave "holes", of which number equals that of the electrons, in the crystal lattices. These holes can be regarded as the positively charged carriers. The movement of these conduction electrons and holes causes electrical conductions. Therefore the temperature influences the electrical conductivity.

The electrical properties of the semiconductors is modified by implantation of impurity atoms into pure semiconductor crystals. This implantation process and the implanted atoms are called "doping" and "dopants", respectively. Dopants are classified as either a "donor" and an "acceptor". A donor produces conduction electrons and an acceptor accepts electrons to produce holes. The typical donors are Group V atoms; acceptors are Group III atoms. An N-type semiconductor, of which main carriers are conduction electrons, is produced by the implantation of donors. Typically the conduction electron density of the semiconductor raises to $10^{20}\mathrm{cm}^{-3}$ from $10^{10}\mathrm{cm}^{-3}$ by the doping. In the N-type semiconductor, the donors atoms liberate conduction electrons and thus are regarded as fixed positive ions. On the other hand, a P-type semiconductor, of which main carriers are holes, is produced by acceptors. Here we regard the acceptors as fixed negative ions. These two kinds of semiconductors are used in making transistors and diodes. The performance of these semiconductor devices are mainly determined by the distribution of dopants, which are called "doping profile".

We explain a structure of a simplest semiconductor device called a pn-junction diode. Joining P-type and N-type semiconductors yields this diode. Here "junction" means the boundary between the P-type and the N-type semiconductors (see Figure 1.1). If the applied voltage ϕ_b is negative, a current flows through this diode. On the contrary, if positive voltage ϕ_b is applied, the current is almost zero. This diode allow a flow of electricity in one direction from left to right but not in the opposite direction. This property is called rectification. The MOSFET, which we are interested in, has a structure of multiple pn-junctions.

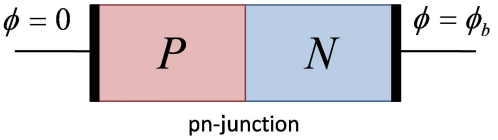

Figure 1.1: pn-junction diode

The MOSFET is a three layer device with two highly doped N^+-regions, called source and drain, and a P-region (see Figure 1.2). A gate made of metal electrode is separated from the semiconductor materials by a thin oxide, which is an insulator. If no voltage ϕ_b is applied to the gate, the rectification of pn-junction do not allow electrons to flow between source and

drain. On the other hand, when the voltage ϕ_g is applied, the electrons, which exist a few even in P-region, are concentrated just below the oxide. Then a N-type inversion layer, called channel, is formed under the oxide surface. This channel carries the electron current from source to drain if positive voltage ϕ_b is applied to drain. Therefore the MOSFET is a unipolar device, meaning that charge transport is only due the movement of electrons. Hence, the behavior of the electrons in the MOSFET can be idealized as the motion of the electrons in N^+-N-N^+ diodes (see Figure 1.3). In the present paper, we analyze one-dimensional electron flow through the N^+-N-N^+ diode.

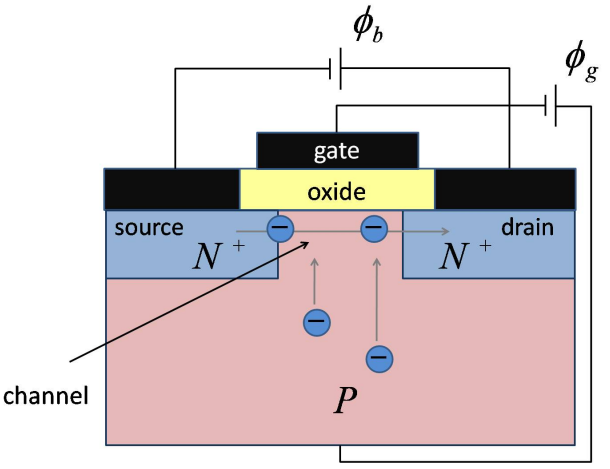

Figure 1.2: MOSFET

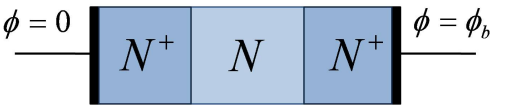

Figure 1.3: N^+-N-N^+ diode

1.2 Mathematical models

The semiconductor models are classified into two groups: macroscopic models and microscopic models. One of the microscopic models is a semiclassical Boltzmann equation (for details, see [17, 25]). The hydrodynamic, the energy-transport and the drift-diffusion models are categorized as macroscopic models. For the derivation of these models, readers are referred to textbooks [15,16,17,24]. Also see the review paper [13].

From the physical point of view, the semiclassical Boltzmann equation is considered to be more accurate than that of the macroscopic models. However, the computational costs for the device simulation using the semi-classical Boltzmann equation is much higher than that of the macroscopic models. On the other hand, the hydrodynamic and the energy-transport models seem reasonable in terms of both cost and physical accuracy. The hydrodynamic model is a perturbed conservation low system, which is built up from charge, momentum and energy exchange. This model contains the hyperbolic equations, which may cause discontinuities. These discontinuities apparently make the device simulation truly difficult. The energy-transport model, which is obtained by taking the relaxation time limit in the hydrodynamic model, consists of the parabolic and the elliptic equations. Hence, this model is used without fear of the discontinuities. Both models are frequently studied in the analysis of a hot carrier problem since these two models account the temperature change. By taking the energy relaxation limit or assuming the temperature is constant, the drift-diffusion model can be obtained from the energy-transport model. Since the drift-diffusion model is simplest among the macroscopic models, engineers use this model to save the computation cost. In this section, we briefly introduce the derivation of these macroscopic models. Hereafter we only treat the unipolar models describing the behavior of the electrons as our concern is the N^+-N-N^+ diode.

Hydrodynamic model. Bløtekjær in [4] derives the hydrodynamic model, consisting of four equations corresponding to a conservation law of mass, a balance law of momentum and energy as well as the Poisson equation,

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1.1a}$$

$$(m^*\rho u)_t + \operatorname{div}(m^*\rho u \otimes u) + \nabla p = e\rho\nabla\phi - \frac{m^*\rho u}{\tau_m}, \tag{1.1b}$$

$$E_t + \operatorname{div}(uE + up + q) = e\rho(u \cdot \nabla\phi) - \frac{E - E_0}{\tau_e}, \tag{1.1c}$$

$$\operatorname{div}(\varepsilon\nabla\phi) = e(\rho - D), \tag{1.1d}$$

where ρ, u, E and ϕ stand for the electron density, the electron velocity, the total energy and the electrostatic potential, respectively. The doping profile D determines the performance of the semiconductor devices. The pressure p, the total energy E and the heat flux q are

supposed to be given by

$$p := k_B \rho \theta, \quad E := \rho \left(\frac{m^*}{2} |u|^2 + \frac{3}{2} k_B \theta \right), \quad q := -\kappa \nabla \theta,$$

where θ stands for the electron temperature. Here E_0 means the total energy in the thermodynamic equilibrium, that is, $E_0 = \frac{3}{2} \rho k_B \bar{\vartheta}$. The physical meaning of coefficients are as follows. m^* is the effective electron mass; e is the electron charge modulus; τ_m is the momentum relaxation time; τ_e is the energy relaxation time; $\bar{\vartheta}$ is the ambient device temperature; ε is the permittivity; k_B is the Boltzmann constant; κ is the thermal conductivity. Note that κ is exactly given by the Wiedemann-Franz law as

$$\kappa = \left(\frac{5}{2} - c \right) \frac{\tau_m k_B^2}{m^*} \rho \theta,$$

where c is a correction factor.

We derive the hydrodynamic model (1.1) by following the traditional derivation of the fluid equations with taking the influence of electrostatics into account. It is an independent method from the original derivation in [4]. In both derivations, we regard the electrons as the gas particles. The electrostatic potential ϕ satisfies the Poisson equation (1.1d) from the Gauss law. Hence we discuss the derivation of (1.1a)–(1.1c) only. For this purpose, we firstly introduce a general conservation principle. Let v be the conserved quantity, w be the corresponding flux, and f be source term. Then the total changes of the amount of v over an arbitrary domain Ω, with a boundary $\partial\Omega$, is given by the sum of the amount which is lost (or gained) through the boundary $\partial\Omega$, and the amount which is created by sources inside of Ω. Namely, it holds that

$$\frac{d}{dt} \iiint_\Omega v \, dV = \iint_{\partial\Omega} w \, dS + \iiint_\Omega f \, dV. \tag{1.2}$$

Owing to the divergence theorem, we have

$$\iiint_\Omega v_t + \mathrm{div} w - f \; dV = 0.$$

Since the domain Ω is arbitrary,

$$v_t + \mathrm{div} w = f. \tag{1.3}$$

To derive the conservation law of mass (1.1a), we take the conserved quantity v in (1.3) as the electron density ρ. Then the corresponding flux w is given by

$$w := \rho u.$$

Hence, the equation (1.3) together with $f = 0$ gives (1.1a). Similarly the balance law of energy (1.1c) is obtained as follows. Let the conserved quantity v be the total energy E. The energy flux w consists of the adiabatic and the heat dissipative components, that is,

$$w := u(E + p) - \kappa \nabla \theta. \tag{1.4}$$

The source f is given by the electrical (Joule heating) term and the damping term:

$$f := e\rho(u \cdot \nabla \phi) - \frac{E - E_0}{\tau_e}. \tag{1.5}$$

Then substituting $v = E$, (1.4) and (1.5) in (1.3) leads to the balance low of energy (1.1c).

We derive the balance low of momentum (1.1b) by using (1.2). Let the conserved quantity v be the momentum $m^*\rho u$. The momentum flux w is given by

$$w := m^*\rho u(u \cdot n) + pn, \tag{1.6}$$

where n is an unit normal vector of the boundary $\partial\Omega$. In this case, f corresponds to external force. Precisely, the external force consists of the electrical and the resistive force. Namely,

$$f := e\rho\nabla\phi - \frac{m^*\rho u}{\tau_m}. \tag{1.7}$$

Then substituting $v = \rho u$, (1.6) and (1.7) in (1.2) and applying the Gauss Theorem yield (1.1b) as the domain Ω is arbitrary. Consequently, the hydrodynamic model (1.1) have been obtained. The model is originally derived from the semiclassical Boltzmann equation by a moment method in [4]. Readers are also referred to [25] for this derivation.

We make the hydrodynamic model (1.1) nondimensional. Precisely, we rewrite the equations (1.1) with the nondimensional variables $(x_s, t_s, \rho_s, u_s, \phi_s, \theta_s)$ as

$$x = Lx_s, \quad t = \sqrt{\frac{m^*L^2}{k_B\bar{\vartheta}}}t_s, \quad \rho = D_M\rho_s, \quad u = L\sqrt{\frac{k_B\bar{\vartheta}}{m^*L^2}}u_s, \quad \phi = \frac{k_B\bar{\vartheta}}{e}\phi_s, \quad \theta = \bar{\vartheta}\theta_s$$

and the current density

$$j_s := \rho_s u_s.$$

The reference length L is the device diameter; the reference density D_M is the maximal value of the doping profile D. Substituting the above variables and omitting the suffix "s",

we obtain

$$\rho_t + \mathrm{div} j = 0, \tag{1.8a}$$

$$j_t + \mathrm{div}\left(\frac{j\otimes j}{\rho}\right) + \nabla(\rho\theta) = \rho\nabla\phi - \frac{j}{\tau_m'}, \tag{1.8b}$$

$$\left(\frac{|j|^2}{2\rho} + \frac{3}{2}\rho\theta\right)_t + \mathrm{div}\left\{\left(\frac{|j|^2}{2\rho} + \frac{3}{2}\rho\theta\right)\frac{j}{\rho} + \theta j - \kappa_0\tau_m'\nabla\theta\right\} = j\cdot\nabla\phi - \frac{|j|^2}{2\tau_e'\rho} - \frac{3}{2\tau_e'}\rho(\theta-1), \tag{1.8c}$$

$$\mathrm{div}(\varepsilon'\nabla\phi) = \rho - \frac{D}{D_M}, \tag{1.8d}$$

where τ_m', τ_e' and ε' are given by

$$\tau_m' = \sqrt{\frac{k_B\bar{\vartheta}}{m^*L^2}}\tau_m, \quad \tau_e' = \sqrt{\frac{k_B\bar{\vartheta}}{m^*L^2}}\tau_e, \quad \kappa_0 = \left(\frac{5}{2} - c\right)\rho\theta, \quad \varepsilon' = \frac{\varepsilon k_B\bar{\vartheta}}{D_M L^2 e^2}. \tag{1.9}$$

We study the initial boundary value problem over a one-dimensional bounded domain for (2.1c), which is obtained by assuming the physical coefficients in (1.9) are positive constants and $\varepsilon' = 1$.

Energy-transport and drift-diffusion models. The energy-transport and the drift-diffusion models are obtained from the hydrodynamic model by taking the relaxation time limits. Note that the Poisson equation (1.8d) is unchanged in the following rescaling. We employ the scaled variables

$$t_s = t\tau_m', \quad j_s = \frac{j}{\tau_m'}, \quad \varepsilon = \tau_m'^2, \quad \zeta = \tau_m'\tau_e'$$

and substitute them in (1.8). These computations give a system (omitting the suffix "s")

$$\rho_t + \mathrm{div} j = 0, \tag{1.10a}$$

$$\varepsilon j_t + \varepsilon\mathrm{div}\left(\frac{j\otimes j}{\rho}\right) + \nabla(\rho\theta) = \rho\nabla\phi - j, \tag{1.10b}$$

$$\left(\frac{\varepsilon\,|j|^2}{2\rho} + \frac{3}{2}\rho\theta\right)_t + \mathrm{div}\left\{\left(\frac{\varepsilon\,|j|^2}{2\rho} + \frac{3}{2}\rho\theta\right)\frac{j}{\rho} + \theta j - \kappa_0\nabla\theta\right\} = j\cdot\nabla\phi - \frac{\varepsilon\,|j|^2}{2\zeta\rho} - \frac{3}{2\zeta}\rho(\theta-1), \tag{1.10c}$$

$$\mathrm{div}(\varepsilon'\nabla\phi) = \rho - \frac{D}{D_M}. \tag{1.10d}$$

In the real devices, τ_m is smaller than τ_e. For example, in the N^+-N-N^+ diode, $\varepsilon/\zeta = \tau_m/\tau_e$ is of order 10^{-2}. By letting ε tend to zero with ζ kept constant, we have the energy-transport model

$$\rho_t + \mathrm{div} j = 0, \tag{1.11a}$$

$$j = \rho\nabla\phi - \nabla(\rho\theta), \tag{1.11b}$$

$$\left(\frac{3}{2}\rho\theta\right)_t + \mathrm{div}\left\{\frac{5}{2}\theta j - \kappa_0\nabla\theta\right\} = j\cdot\nabla\phi - \frac{3}{2\zeta}\rho(\theta-1), \tag{1.11c}$$

$$\mathrm{div}(\varepsilon'\nabla\phi) = \rho - \frac{D}{D_M}. \tag{1.11d}$$

Moreover letting ζ tend zero in (1.11) yields the drift-diffusion model

$$\rho_t + \mathrm{div} j = 0, \tag{1.12a}$$

$$j = \rho\nabla\phi - \nabla\rho, \tag{1.12b}$$

$$\mathrm{div}(\varepsilon'\nabla\phi) = \rho - \frac{D}{D_M}. \tag{1.12c}$$

The above formal computations reveal the relations among the macroscopic models. The original derivations of the energy-transport and the drift-diffusion models are given in [36, 38].

Provided that the device diameter L is relatively large or the ambient temperature $\bar{\vartheta}$ is relatively low, the constants τ_m' and τ_e' are small and thus the hydrodynamic model (1.10) is well approximated by both of the simpler models (1.11) and (1.12). Hence the accuracy of the device simulation using (1.11) or (1.12) is ensured at least formally. Our mathematical results in Theorems 2.3 and 2.4 as well as Corollary 2.5 give rigorous support of these formal computation.

1.3 Review of previous results

In this section, we mention several results on the macroscopic models, of which the time global solvability and the model hierarchy are main concern of the present paper.

The drift-diffusion model is the simplest model, which is derived by Roosbroeck in [36]. The first mathematical is given by Mock in [27], which establishes the existence of the stationary solution over the multi-dimensional bounded domain. He also shows the asymptotic stability of the stationary solution in [28]. In these researches, he adopts a special boundary condition, which does not allow any electron flow through the boundary. The research [11] by Gajewski and Gröger extends the result in [28] to more general boundary conditions, which covers the case that electric current permeates the boundary. However, the paper only studies a special stationary solution, of which current density is zero.

The energy-transport model is derived by Stratton in [38] (also see [2] and [3]). The pioneering mathematical work is done by Degond, Génieys and Jüngel in [7, 8]. They study the parabolic system coupled with the Poisson equation, which is generalized from the energy-transport model, over the multi-dimensional bounded domain with the Dirichlet-Neumann mixed boundary conditions. However, they assume the existence of the (strictly positive) energy form, and the uniform parabolicity. In the present paper, we prove the same results without these assumptions for the one-dimensional problem. For the multi-dimensional case, Chen, Hsiao and Li in [5] discuss this model with a special boundary condition, which does not allow any electron flow through the boundary as in [28], and obtain the stability theorem for the small initial disturbance without assuming uniform parabolicity and the existence of the energy form.

Recently the researchers have more interest in the hydrodynamic model, introduced in [4], as the devices become truly minute in these days. The isothermal and the isentropic hydrodynamic models are often studied. Degond and Markowich in [9] establish the unique existence of the stationary solutions to these models. The stability of the stationary solutions are also shown by Li, Markowich and Mei in [24] under the assumption that the doping profiles are flat. Guo and Strauss in [14] extend the result [24] to cover the non-flat doping profile. In their researches, it is assumed that the current in the stationary solution is sufficiently small although this fact should be derived from the smallness of a voltage on boundary, from physical point of view. The authors' paper [30] derive the estimate of the current in terms of the voltage. This estimate allows to show the stability theorem for the small voltage. Also see [26] for the research on the periodic boundary condition.

In designing real devices, the analysis on the heat convection is very important to avoid a hot carrier problem and make devices stable. For this purpose, the hydrodynamic model with heat conduction, rather than the isothermal or the isentropic hydrodynamic model, is suitable. The authors recently prove in [32] the unique existence and the asymptotic stability of the stationary solution to the heat conductive model with the non-flat doping profile. Here the Dirichlet boundary condition is adopted for the electron temperature in place of the Neumann boundary condition, studied in the present paper. The above researches in [14, 24, 26, 30, 32] require the smallness of the initial disturbance. In the present research, we succeed in removing this assumption in place of assuming the smallness of the relaxation times.

Nowadays, engneers study the models with quantum correction since carriers tunnel through potential barriers in naoscale MOSFETs. To minimize leakage, new structure in devices are invented such as double gate MOSFETs and FinFETs. The quantum models are especially important for simulation on new innovative devices. For a quantum hydrodynamic model, Jüngel and Li in [18, 19] prove the existence and the stability of the stationary solution with flat doping profile by assuming the smallness of the stationary current. The authors in [31] obtain the same results with the non-flat doping profile under the smallness assumption

of the voltage on the boundary. It is also proven in [31] that the time global solution for the quantum hydrodynamic model converges to that for the hydrodynamic model without quantum correction as the Planck constant tends to zero. Similar results are shown in [29] for a quantum drift-diffusion model.

The relaxation limits of the hydrodynamic model is also intensively studied. For example, the limit of the isothermal model is investigated by many mathematicians (see [6, 22, 33, 40]). In particular, the authors' previous paper [33] shows that the time global solution for the model approaches to that for the drift-diffusion model as the relaxation time tends to zero. Here the initial layer is also shown to decay exponentially fast as the relaxation time tends to zero and/or the time variable tends to infinite. It is the first result which resolves the initial layer problem for the semiconductor model. In this result, they do not have to assume the smallness of the initial data. For the hydrodynamic model with heat-conduction, Ali, Bini and Rionero in [1] justify the relaxation limit of the hydrodynamic model to the energy-transport model over the full space $\mathbb{R}$ provided that the initial data is close enough to the special stationary solution with no current. On the other hand, the relaxation limit of the heat-conductive hydrodynamic model to the drift-diffusion model is justified over the one-dimensional bounded domain by Chen, Jerome and Zhang in [6]. They treat the flat doping profile and the Dirichlet zero boundary condition on the electric current, which makes no current in the stationary solution. These relaxation limits, however, should be discussed over the bounded domain with physically admissible boundary conditions, which allows flow of the electric current, since devices are minute and boundary conditions influence the behavior of the current.

The main purpose of the present paper is to extend the result on the isothermal hydrodynamic model in [33] to the hydrodynamic model with heat-conduction. Namely, we justify the relaxation limits of the hydrodynamic model to the energy-transport and the drift-diffusion models under the physically admissible conditions. To this end, we also prove the time global solvability for the hydrodynamic and the energy-transport models and the asymptotic stability of the stationary solutions. Here we study these problems with the large initial data.

1.4 Notation

For a nonnegative integer $l \geq 0$, $H^l(\Omega)$ denotes the l-th order Sobolev space in the L^2 sense, equipped with the norm $\|\cdot\|_l$. We note $H^0 = L^2$ and $\|\cdot\| := \|\cdot\|_0$. $C^k([0,T]; H^l(\Omega))$ denotes the space of the k-times continuously differentiable functions on the interval $[0,T]$ with values in $H^l(\Omega)$. $H^k(0,T; H^l(\Omega))$ is the space of H^k–functions on $(0,T)$ with values in

$H^l(\Omega)$. Moreover, $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ denote the function spaces

$$\mathfrak{X}_i^j([0,T]) := \bigcap_{k=0}^{i} C^k([0,T]; H^{j+i-k}(\Omega)), \quad \mathfrak{X}_i([0,T]) := \mathfrak{X}_i^0([0,T]),$$
$$\mathfrak{Y}([0,T]) := C^1([0,T]; L^2(\Omega)) \cap C^0([0,T]; H^2(\Omega)) \cap H^1(0,T; H^1(\Omega)),$$
$$\mathfrak{Z}([0,T]) := C([0,T]; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega))$$

for nonnegative integers $i, j \geq 0$. For a nonnegative integer $k \geq 0$, $\mathcal{B}^k(\overline{\Omega})$ denotes the space of the functions whose derivatives up to k-th order are continuous and bounded over $\overline{\Omega}$, equipped with the norm

$$|f|_k := \sum_{i=0}^{k} \sup_{x \in \overline{\Omega}} |\partial_x^i f(x)|.$$

Lastly C and c denotes a generic constant and $C[\alpha, \beta, \cdots]$ and $c[\alpha, \beta, \cdots]$ denotes a positive constant depending on $\alpha, \beta, \cdots$.