CHAPTER 5

Relations with the Residue Theory

It is pointed out by Hurder [46] that every known example of transversely holomorphic foliation with non-vanishing secondary classes is related to the residue theory by Heitsch [41]. The examples considered in Chapter 3 can be also related to the residue.

We begin by recalling the notion of the residue for transversely holomorphic foliations by Heitsch [41]. The original paper deals with real foliations. Here we formulate the residue theory for transversely holomorphic foliations after straightforward modifications.

Let $\mathcal{F}$ be a transversely holomorphic foliation on M and let $T_{\mathbb{C}}M$, E and $Q(\mathcal{F})$ be as in Definition 1.1.4. Sections of $T_{\mathbb{C}}M$ are nothing but $\mathbb{C}$-valued vector fields on M.

Definition 5.1. A $\mathbb{C}$-valued vector field X is called a *Γ-vector field* for $\mathcal{F}$ if $[E, X] \subset E$, namely, if $[s, X]$ is a local section of E for any local section s of E.

Definition 5.2. Let X be a Γ-vector field. The *singular set* of X is the union of points x of M where $X(x) \in E_x$ and denoted by $\operatorname{Sing} X$.

A Γ-vector field is locally of the form $X = \displaystyle\sum_{i=1}^{q} f_i \frac{\partial}{\partial z_i}$ modulo sections of E, where f_i's are locally constant along the leaves and holomorphic in the transversal direction. The singular set $\operatorname{Sing} X$ is saturated, namely, it is a union of leaves of $\mathcal{F}$.

Let E and X be as above. Then they span an integrable subbundle of $T_{\mathbb{C}}M$ on $M \setminus \operatorname{Sing} X$. The induced foliation $\mathcal{F}_X$ is transversely holomorphic and of complex codimension $q-1$, where q is the complex codimension of $\mathcal{F}$.

Definition 5.3. Let X be a Γ-vector field for $\mathcal{F}$ and let U be an open neighborhood of $\operatorname{Sing} X$. A Bott connection ∇ for $\mathcal{F}$ is called a *basic X-connection supported off U* if $\nabla_X Y = \mathcal{L}_X Y$ for any section Y of $Q(\mathcal{F})$ on a neighborhood of $M \setminus U$, where $\mathcal{L}_X Y$ denotes the Lie derivative of Y with respect to X.

Remark 5.4. Basic X-connections are Bott connections for $\mathcal{F}_X$ on the neighborhood of $M \setminus U$ in Definition 5.3.

When residues are considered, the complex normal bundle $Q(\mathcal{F})$ is often assumed to be trivial. But it suffices to assume the triviality of the canonical bundle $\bigwedge^q Q(\mathcal{F})^*$ for our purpose. The residue of the Bott class is constructed as follows. If $\bigwedge^q Q(\mathcal{F})^*$ is trivial, then the Bott class $\mathrm{Bott}_q(\mathcal{F}) = u_1 v_1^q(\mathcal{F})$ is well-defined as an element of $H^{2q+1}(M;\mathbb{C})$. More precisely, the Bott class is calculated by choosing a Bott connection ∇^b and a flat connection ∇^s of $\bigwedge^q Q(\mathcal{F})$ (see Chapter 1). By using these connections, a well-defined 1-form $u_1(\nabla^b, \nabla^s)$ and a 2-form $v_1(\nabla^b)$ such that $du_1(\nabla^b, \nabla^s) = v_1(\nabla^b)$ are obtained. Let X be a Γ-vector field for $\mathcal{F}$ and choose a basic X-connection supported off U as a Bott connection ∇^b. Then the support of the differential form $u_1(\nabla^b, \nabla^s) v_1(\nabla^b)^q$ is contained in U because $v_1(\nabla^b)^q = 0$ by the Bott vanishing theorem for $\mathcal{F}_X$. Thus an element of $H_c^{2q+1}(U;\mathbb{C})$ is obtained, where $H_c^*(U;\mathbb{C})$ denotes the compactly supported cohomology.

The residue theorem due to Heitsch is formulated as follows. For simplicity we state the theorem assuming that $Q(\mathcal{F})$ is trivial on a neighborhood of $\operatorname{Sing} X$. In such a case, secondary classes are obtained from $H^*(\mathrm{W}_q)$ and $H^*(\mathrm{W}_q^{\mathbb{C}})$ (see Definitions 1.1.19 and 1.1.14). Recall that there is a natural inclusion of $H^*(\mathrm{W}_q)$ into $H^*(\mathrm{W}_q^{\mathbb{C}})$.

Theorem 5.5 (Heitsch [43]). *Let X and U be as above and let s be a trivialization of $Q(\mathcal{F})$ on U. Then there is a well-defined element* $\operatorname{res}\omega(\mathcal{F}, X, s) \in H_c^{2q+1}(U;\mathbb{C})$ *for each $\omega \in H^{2q+1}(\mathrm{W}_q)$. This element depends on $\mathcal{F}_X$ and the homotopy type of s, and is called the residue of $\omega(\mathcal{F})$ with respect to X and s. Under the natural mapping from $H_c^*(U;\mathbb{C})$ to $H^*(M;\mathbb{C})$, the residue is mapped to $\omega(\mathcal{F})$.*

The original situation considered by Heitsch in [41], [43] is as follows. In what follows, coefficients of the cohomologies are chosen in $\mathbb{C}$. Let $M \cong W \times \mathbb{C}^{q+1}$ and assume that $W = W \times \{0\}$ is a compact leaf of $\mathcal{F}$. We also assume that $\operatorname{Sing} X = W$ and $\mathcal{F}_X$ is transversal to $M' = W \times S^{2q+1}$ so that $\mathcal{F}_X$ induces a foliation $\mathcal{F}$ of M', where S^{2q+1} is the unit sphere in $\mathbb{C}^{q+1}$. The Gysin exact sequence associated with M, M' and W is as follows:

$$\cdots \longrightarrow H^r(M) \longrightarrow H^r(M') \xrightarrow{\fint} H^{r-2q-1}(W) \longrightarrow H^{r+1}(M) \longrightarrow \cdots,$$

where $\displaystyle\fint$ denotes the integration along the fiber. Recall that $\displaystyle\fint$ is the composition $H^r(M') \xrightarrow{\partial} H^{r+1}(W \times D^{2(q+1)}, M') \cong H_c^{r+1}(M) \xrightarrow{\int_D} H^{r-2q-1}(W)$, where ∂ is the connecting homomorphism and $\displaystyle\int_D$ is the integration along the fiber of $W \times D^{q+1} \to W$. Let $\iota\colon M' \to M \setminus W$ be the inclusion. Then ι^* induces an isomorphism of cohomology and $\iota^*(\mathrm{Bott}_q(\mathcal{F}_X)) = \mathrm{Bott}_q(\mathcal{F})$ by the naturality. Hence $\mathrm{Bott}_q(\mathcal{F})$ is mapped to $\displaystyle\int_D v_1^{q+1}(\nabla^b)$ under $\displaystyle\fint$. Thus obtained class is the residue in the original sense.

EXAMPLE 5.6. Example 1.1.6 can be slightly modified. Let X_A be a holomorphic vector field determined by $A \in \mathrm{GL}(q+1;\mathbb{C})$, namely, let

$$X_A = \sum_{\substack{0 \le i \le q \\ 0 \le j \le q}} a_{ij} z_j \frac{\partial}{\partial z_i},$$

where a_{ij} denotes the (i,j)-entry of A. The vector field X_A has the origin as an isolated singularity. If X_A is transversal to S^{2q+1}, then the same construction as in Example 1.1.6 can be done. It is not always the case, however, X_A induces a foliation of a Hopf manifold as follows. Let $\widetilde{\mathcal{F}}_A$ be the foliation of $\mathbb{C}^{q+1} \setminus \{0\}$ by the orbits of X_A. Then $\widetilde{\mathcal{F}}_A$ is invariant under the $\mathbb{Z}$-action on $\mathbb{C}^{q+1} \setminus \{0\}$ defined by $v \cdot n = \lambda^n v$, where λ is a complex number such that $|\lambda| > 1$. If we set $H_\lambda = (\mathbb{C}^{q+1} \setminus \{0\})/\mathbb{Z}$, then $H_\lambda \cong S^{2q+1} \times S^1$ and $\widetilde{\mathcal{F}}_A$ induces a foliation $\mathcal{F}_A$ of H_λ.

The canonical bundle of $\mathcal{F}_A$ is trivial so that the classes $u_1 v_J(\mathcal{F}_A)$, $|J| = q$, are well-defined. To see the triviality, we fix the logarithm $\log \lambda$ of λ and define a function $\Psi\colon (0,+\infty) \to \mathbb{C}$ by $\Psi(r) = \exp\left(\dfrac{\log r}{\log|\lambda|}\log\lambda\right)$. Let $\|\cdot\|$ be the standard norm on $\mathbb{C}^{q+1}$ and set

$$\widetilde{\sigma}_A = \frac{1}{\Psi(\|z\|)^{q+1}}\sum_{i=0}^{q}\zeta_i dz_0\wedge\cdots\wedge\widehat{dz_i}\wedge\cdots\wedge dz_q,$$

where $\zeta = Az$ and $\zeta = {}^t(\zeta_0,\ldots,\zeta_q)$, then $\widetilde{\sigma}_A$ induces a trivialization of $K_{\mathcal{F}_A}$.

Let S^{2q+1} be the unit sphere in $\mathbb{C}^{q+1}$ and let S be the natural image of S^{2q+1} in H_λ. Then, we have

$$u_1v_J(\mathcal{F}_A) = \frac{v_1v_J(A)}{\det A}[S].$$

The formula is shown as follows. If we set

$$\omega = \frac{1}{\|z\|^2}z^*A^{-1}dz,$$

then $\omega(X_A) = 1$. We set $e_i = \Psi(\|z\|)\dfrac{\partial}{\partial z_i}$ for $i = 0,\ldots,q$. Then $e_0,\ldots,e_q$ are invariant under the multiplication by λ so that they induce vector fields on H_λ. Let ∇ be the unique connection on $\mathbb{C}^{q+1}\setminus\{0\}$ which satisfies $\nabla_Y e_i = \omega(Y)[X_A,e_i]$. The connection ∇ is a Bott connection for $\widetilde{\mathcal{F}}_A$ and induces a Bott connection for $\mathcal{F}_A$ on H_λ. In order to evaluate $u_1v_J(\mathcal{F}_A)$, it suffices to compute $\displaystyle\int_S u_1v_J(\mathcal{F}_A)$. Let $\rho\colon\mathbb{R}\to\mathbb{R}$ be a non-decreasing smooth function such that $\rho(r) = 0$ if $r\leq 0$ and that $\rho(r) = 1$ if $r\geq\dfrac{1}{2}$. If we set $\nabla' = \rho(\|z\|)\nabla$, then ∇' is a basic X_A-connection supported off U, where U is the open round ball of radius $\dfrac{2}{3}$ centered at the origin.

We denote by $u_1v_J(\nabla)$ and $u_1v_J(\nabla')$ the representatives of $u_1v_J(\widetilde{\mathcal{F}}_A)$ calculated by ∇ and ∇', respectively. Then,

$$\int_S u_1v_J(\mathcal{F}_A) = \int_{S^{2q+1}} u_1v_J(\nabla) = \int_{S^{2q+1}} u_1v_J(\nabla') = \int_{D^{2q+2}} v_1v_J(\Omega'_A),$$

where Ω'_A is the curvature of ∇'. The right-most term is by definition the residue of the trivial foliation of $\mathbb{C}^{q+1}$ at $\{0\}$ with respect to X_A. Therefore the formula follows from a more general formula of Baum–Bott [13, Proposition 8.67]. Under our setting, the proof is as follows. Let $\widetilde{\nabla}$ be the connection on $\mathbb{C}^{q+1}\setminus\{0\}$ such that $\widetilde{\nabla}\dfrac{\partial}{\partial z_i} = \omega\left[X_A,\dfrac{\partial}{\partial z_i}\right]$. Then $\widetilde{\nabla}$ is also a basic X_A-connection. If we denote by $\widetilde{\eta}$ the connection form of $\widetilde{\nabla}$ with respect to $\left\{\dfrac{\partial}{\partial z_0}, \dots, \dfrac{\partial}{\partial z_q}\right\}$, then $\widetilde{\eta} = -A\omega$. Note that $(d\omega)^{q+1} = 0$ by the Bott vanishing for $\widetilde{\mathcal{F}}_A$. We set $\widetilde{\nabla}' = \rho(\|z\|)\widetilde{\nabla}$. Then $\widetilde{\nabla}'$ is a basic X_A-connection supported off U. Hence $\displaystyle\int_{D^{2q+2}} v_1 v_J(\Omega'_A) = \int_{D^{2q+2}} v_1 v_J(\widetilde{\Omega}'_A)$, where $\widetilde{\Omega}'_A$ is the connection form of $\widetilde{\nabla}'$. We have

$$\begin{aligned}\widetilde{\Omega}'_A &= -\frac{d\rho}{dr}(\|z\|)\frac{z^*\,dz + z\,dz^*}{2\,\|z\|}A \wedge \omega - \rho(\|z\|)A\,d\omega \\ &= -A\left(\frac{d\rho}{dr}(\|z\|)\frac{z^*\,dz + z\,dz^*}{2\,\|z\|} \wedge \omega + \rho(\|z\|)\,d\omega\right).\end{aligned}$$

It follows that

$$\begin{aligned}v_1 v_J(\widetilde{\Omega}'_A) &= \left(\frac{-1}{2\pi\sqrt{-1}}\right)^{q+1} v_1 v_J(-A)\left(\frac{d\rho}{dr}(\|z\|)\frac{z^*\,dz + z\,dz^*}{2\,\|z\|} \wedge \omega + \rho(\|z\|)\,d\omega\right)^{q+1} \\ &= \left(\frac{-1}{2\pi\sqrt{-1}}\right)^{q+1} v_1 v_J(-A)(q+1)\frac{d\rho}{dr}\rho^q(\|z\|)\frac{z^*\,dz + z\,dz^*}{2\,\|z\|} \wedge \omega \wedge (d\omega)^q.\end{aligned}$$

We set $\omega' = z^* A^{-1} dz$. Then

$$d\omega = d\left(\frac{1}{\|z\|^2}\omega'\right) = -\frac{z^*\,dz + z\,dz^*}{\|z\|^4}\omega' + \frac{1}{\|z\|^2}\,d\omega'.$$

Hence, if we set $C = \left(\dfrac{-1}{2\pi\sqrt{-1}}\right)^{q+1}$, then we have

$$\begin{aligned}&v_1 v_J(\widetilde{\Omega}'_A) \\ &= C\,v_1 v_J(-A)(q+1)\frac{d\rho}{dr}(\|z\|)\rho(\|z\|)^q\frac{z^*\,dz + z\,dz^*}{2\,\|z\|^{2q+3}} \wedge \omega' \wedge (d\omega')^q \\ &= C\,v_1 v_J(-A)(q+1)\frac{d\rho}{dr}(\|z\|)\rho(\|z\|)^q\frac{z\,dz^*}{2\,\|z\|^{2q+3}} \wedge \omega' \wedge (d\omega')^q \\ &= C\,v_1 v_J(-A)(q+1)!\frac{d\rho}{dr}(\|z\|)\rho(\|z\|)^q\frac{1}{2\,\|z\|^{2q+1}} \wedge \det(A^{-1})\left(\bigwedge_{i=0}^{q} d\overline{z_i} \wedge dz_i\right) \\ &= \frac{1}{\pi^{q+1}}v_1 v_J(A)(q+1)\rho(r)^q\frac{d\rho}{dr}(\|z\|)\frac{1}{2\,\|z\|^{2q+1}} \wedge \det(A^{-1})\mathrm{vol}_{\mathbb{R}^{2q+2}} \\ &= v_1 v_J(A)(q+1)\rho(r)^q\frac{d\rho}{dr}(r)\,dr \wedge \det(A^{-1})\mathrm{vol}_{S^{2q+1}},\end{aligned}$$

where $\mathrm{vol}_{\mathbb{R}^{2q+2}}$ denotes the standard volume form of $\mathbb{R}^{2q+2}$, and $\mathrm{vol}_{S^{2q+1}}$ is the volume form of S^{2q+1} normalized so that $\mathrm{vol}(S^{2q+1}) = 1$. Therefore

$$\int_{D^{2q+2}} v_1 v_J(\widetilde{\Omega}'_A) = \frac{v_1 v_J(A)}{\det A}.$$

Note that if A is diagonal and X_A is transversal to S^{2q+1}, then $\mathcal{F}_A$ induces the foliation $\mathcal{F}_\lambda$ of S^{2q+1} in Example 1.1.6.

Example 3.3.6 is also related to residues as follows. Let $\widetilde{M} = \mathrm{SL}(q+1;\mathbb{C})\times\mathbb{C}^{q+1}$. Then $\mathrm{SL}(q+1;\mathbb{C})$ acts on $\widetilde{M}$ by $(g,v)h = (gh, h^{-1}v)$. In particular, $\mathrm{SU}(q+1)$ also acts on $\widetilde{M}$. We denote by π the quotient map and denote by $\widetilde{N}$ the image of π. Let $\widetilde{M}^* = \mathrm{SL}(q+1;\mathbb{C}) \times (\mathbb{C}^{q+1} \setminus \{0\})$ and $\widetilde{M}' = \mathrm{SL}(q+1;\mathbb{C}) \times S^{2q+1}$. Then $\widetilde{M}^*$ is invariant under the $\mathrm{SL}(q+1;\mathbb{C})$-action and $\widetilde{M}'$ is invariant under the $\mathrm{SU}(q+1)$-action. We set $\widetilde{N}^* = \pi(\widetilde{M}^*)$ and $\widetilde{N}' = \pi(\widetilde{M}')$. Let $\mathit{\Gamma}$ be a cocompact lattice of $\mathrm{SL}(q+1;\mathbb{C})$ such that $B = \mathit{\Gamma}\backslash\mathrm{SL}(q+1;\mathbb{C})/\mathrm{SU}(q+1)$ is a closed manifold. Set then $M = \mathit{\Gamma}\backslash\widetilde{M}$, $M^* = \mathit{\Gamma}\backslash\widetilde{M}^*$, $M' = \mathit{\Gamma}\backslash\widetilde{M}'$, $N = \mathit{\Gamma}\backslash\widetilde{N}$, $N^* = \mathit{\Gamma}\backslash\widetilde{N}^*$ and $N' = \mathit{\Gamma}\backslash\widetilde{N}'$, where $\mathrm{SL}(q+1;\mathbb{C})$ acts on these manifolds on the left by $h(g,v) = (hg,v)$.

Let $\widetilde{\mathcal{F}}$ be the foliation of $\widetilde{M}$ induced by the right $\mathrm{SL}(q+1;\mathbb{C})$-action. If T is the holomorphic vector field on $\mathbb{C}^{q+1}$ defined by $T = \displaystyle\sum_{i=0}^{q} z_i \frac{\partial}{\partial z_i}$, where $(z_0,\ldots,z_q)$ is the standard coordinates of $\mathbb{C}^{q+1}$, then T is a $\mathit{\Gamma}$-vector field for $\widetilde{\mathcal{F}}$. Note that T induces the Hopf fibration on S^{2q+1}. We also call the foliation of $\mathbb{C}^{2q+1} \setminus \{0\}$ by the orbits of T the Hopf fibration.

Let $\widetilde{\mathcal{F}}_T$ be the singular foliation $\widetilde{\mathcal{F}}_T$ of $\widetilde{M}$. Then $\widetilde{\mathcal{F}}_T$ is regular on $\widetilde{M}^*$. Indeed, if we define $f\colon \mathrm{SL}(q+1;\mathbb{C}) \times (\mathbb{C}^{q+1}\setminus\{0\}) \to \mathbb{C}^{q+1}\setminus\{0\}$ by $f(g,z) = gz$, then $\widetilde{\mathcal{F}}_T$ is the pull-back of the Hopf fibration of $\mathbb{C}^{q+1}\setminus\{0\}$ by f. Moreover, $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}_T$ naturally induce foliations $\mathcal{F}$ of M and $\mathcal{F}_T$ of M^* because they are invariant under the natural left action of $\mathrm{SL}(q+1;\mathbb{C})$ on $\widetilde{M}$ and $\widetilde{M}^*$. Finally, $\widetilde{\mathcal{F}}_T$ is transversal to $\widetilde{M}'$ so that it induces a foliation $\widetilde{\mathcal{F}}'_T$ of $\widetilde{M}'$. The foliation $\widetilde{\mathcal{F}}'_T$ is also invariant under the left $\mathrm{SL}(q+1;\mathbb{C})$-action so that it induces a foliation $\mathcal{F}'_T$ of M'. The foliations $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}_T$ and $\widetilde{\mathcal{F}}'_T$ are invariant under the right $\mathrm{SU}(q+1)$-action. Hence they naturally induce foliations of $\widetilde{N}$, $\widetilde{N}^*$ and $\widetilde{N}'$. We denote them by $\widetilde{\mathcal{G}}$, $\widetilde{\mathcal{G}}_T$ and $\widetilde{\mathcal{G}}'_T$, respectively. Foliations of N, N^* and N' can be constructed in a similar way. We denote them by $\mathcal{G}$, $\mathcal{G}_T$ and $\mathcal{G}'_T$, respectively.

The canonical bundles of $\widetilde{\mathcal{F}}_T$, $\widetilde{\mathcal{F}}'_T$, $\widetilde{\mathcal{G}}_T$ and $\widetilde{\mathcal{G}}'_T$ are trivial, indeed, if we define a q-form ω on $\mathbb{C}^{q+1}$ by

$$\omega = z_0\,dz_1 \wedge \cdots \wedge dz_q - z_1 dz_0 \wedge dz_2 \wedge \cdots \wedge dz_q + \cdots + (-1)^q z_q\,dz_0 \wedge \cdots \wedge dz_{q-1},$$

then $\omega(T) = 0$. The differential form $f^*\omega$ on $\widetilde{M}^*$ induces trivializations of canonical bundles. Since $f^*\omega$ is invariant under the left $\mathrm{SL}(q+1;\mathbb{C})$-action, the canonical bundles of $\mathcal{F}_T$, $\mathcal{F}'_T$, $\mathcal{G}_T$ and $\mathcal{G}'_T$ are also trivial. Therefore, the Bott class is well-defined for these foliations and the Godbillon–Vey class of these foliations are trivial. The vector field T gives rise to an S^1-action which preserves $\mathcal{F}_T$, $\mathcal{F}'_T$, $\mathcal{G}_T$ and $\mathcal{G}'_T$. Indeed, the action is essentially the Hopf fibration on $\mathbb{C}^{q+1} \setminus \{0\}$.

Let $\widetilde{M}'' = \mathrm{SL}(q+1;\mathbb{C}) \times \mathbb{C}P^q$. Then $\mathrm{SL}(q+1;\mathbb{C})$ acts on $\widetilde{M}''$ by $(g,[v])h = (gh, h^{-1}[v])$. Let $\widetilde{\mathcal{F}}''$ be the foliation of $\widetilde{M}''$ induced by the right $\mathrm{SL}(q+1;\mathbb{C})$-action. Let $\widetilde{N}''$ and $\widetilde{\mathcal{G}}''$ be the quotient by the induced action of $\mathrm{SU}(q+1)$, and we denote by π the quotient map by abuse of notation. We set $M'' = \varGamma\backslash\widetilde{M}''$ and $N'' = \varGamma\backslash\widetilde{N}''$. They are naturally equipped with foliations induced by $\widetilde{\mathcal{F}}''$ and $\widetilde{\mathcal{G}}''$. We denote them by $\mathcal{F}''$ and $\mathcal{G}''$. It is well-known that $(N'', \mathcal{G}'')$ is isomorphic to the foliation of $\varGamma\backslash\mathrm{SL}(q+1;\mathbb{C})/(T^1 \times SU_q)$ given in Example 3.3.6. We have the following commutative diagram:

$$\begin{array}{ccccccc}
M'' & \longleftarrow & M' & \hookrightarrow & M^* & \hookrightarrow & M \\
\Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\pi} \\
N'' & \underset{p}{\longleftarrow} & N' & \hookrightarrow & N^* & \hookrightarrow & N,
\end{array}$$

where $p\colon N' \to N''$ is the natural projection which is a fiberwise Hopf fibration. Note that $p^*(\mathrm{GV}_{2q}(\mathcal{G}'')) = \mathrm{GV}_{2q}(\mathcal{G}'_T) = 0$. The Gysin sequence associated with p is as follows:

$$\cdots \longrightarrow H^*(N') \xrightarrow{p_!} H^{*-1}(N'') \xrightarrow{\cup e} H^{*+1}(N'') \xrightarrow{p^*} H^{*+1}(N') \longrightarrow \cdots,$$

where $e = \dfrac{1}{(q+1)}\,\mathrm{ch}_1(\mathcal{G}'') = \dfrac{v_1(\mathcal{G}'') + \overline{v}_1(\mathcal{G}'')}{2(q+1)}$, and $p_!$ denotes the integration along the fiber. Therefore $\xi_q(\mathcal{G}'')\,\mathrm{ch}_1(\mathcal{G}'')^k$ is mapped to a non-zero multiple of

$\xi_q(\mathcal{G}'')\operatorname{ch}_1(\mathcal{G}'')^{k+1}$ under $\cup e$. If $k = q$, then $\xi_q(\mathcal{G}'')\operatorname{ch}_1(\mathcal{G}'')^q$ is a non-zero multiple of $\mathrm{GV}_{2q}(\mathcal{G}'')$, and the image under $\cup e$ is trivial because $\operatorname{ch}_1(\mathcal{G}'')^{q+1} = 0$. Hence there is an element of $H^{4q+2}(N')$ which is mapped to $\mathrm{GV}_{2q}(\mathcal{G}'')$. Indeed, $p_!(u_1v_1(\mathcal{F}_T)^q\overline{u}_1\overline{v}_1(\mathcal{F}_T)^q)$ is equal to a non-zero multiple of $\mathrm{GV}_{2q}(\mathcal{G}'')$ by Theorem 3.3.10.

The Gysin sequence associated with $S^{2q+1} \to N' \to B = \varGamma\backslash\mathrm{SL}(q+1;\mathbb{C})/\mathrm{SU}(q+1)$ is now decomposed as follows:

$$\begin{array}{ccccccc}
H^{4q+2}(B) \longrightarrow & H^{4q+2}(N') & \xrightarrow{p_!} & H^{4q+1}(N'') & \xrightarrow{\fint} & H^{2q+1}(B) \\
& \| & & & & \| \\
& H^{4q+2}(N') & \xrightarrow{\partial} & H^{4q+3}(N^*, N') & \xrightarrow{t^{-1}} & H^{2q+1}(B),
\end{array}$$

where $t\colon H^*(B) \to H^{*+2q+2}(N^*, N')$ is the Thom isomorphism, and $\fint$ is the integration along the fiber. We have

$$\fint p_!(u_1v_1^q(\mathcal{F}_T)\overline{u}_1\overline{v}_1^q(\mathcal{F}_T)) = \fint \mathrm{GV}_{2q}(\mathcal{G}'')$$

up to multiplications of non-zero constants, and

$$\begin{aligned}
t^{-1}\circ\partial(u_1v_1^q(\mathcal{F}'_T)\overline{u}_1\overline{v}_1^q(\mathcal{F}'_T)) &= t^{-1}(v_1(\mathcal{F}'_T)^{q+1}\overline{u}_1\overline{v}_1^q(\mathcal{F}'_T) - u_1v_1^q(\mathcal{F}'_T)\overline{v}_1^{q+1}(\mathcal{F}'_T)) \\
&= \operatorname{res}((v_1^{q+1}\overline{u}_1\overline{v}_1^q - u_1v_1^q\overline{v}_1^{q+1}), \mathcal{F}', T).
\end{aligned}$$

Thus Example 3.3.6 is related to the residue. It is possible to apply this construction to other examples involving $\mathrm{SO}(2n+1;\mathbb{C})$, $\mathrm{Sp}(n;\mathbb{C})$ and G_2 by using the Iwasawa decomposition and naturally associated S^1-bundles.

The fibration $N' \to N''$ is also relevant for studying derivatives of the Bott class with respect to deformations of foliations. By Theorem B2, $\xi_q(\mathcal{G}'')\operatorname{ch}_1^k(\mathcal{G}'')$ is rigid under deformations if $k > 0$. Indeed, $D_\mu B_q(\mathcal{G}'')\operatorname{ch}_1^k(\mathcal{G}'')$ is trivial if $k > 0$ by Corollary 4.3.30. On the other hand, if $k = 0$, then $D_\mu B_q(\mathcal{G}'')$ belongs to the kernel of $\cup e$. Hence there is an element of $H^{2q+2}(N')$ which is mapped to $D_\mu B_q(\mathcal{G}'')$. Such an element is obtained as follows.

Assume that $K_{\mathcal{F}}$ is trivial and let ω be a trivialization. We may assume that there is a family of local trivializations $\{\rho_i = {}^t(\rho_i^1, \dots, \rho_i^q)\}$ of $Q^*(\mathcal{F})$ such that

$\omega = \rho_i^1 \wedge \cdots \wedge \rho_i^q$ locally holds. Let $\mu \in H^1(M;\Theta_{\mathcal{F}})$ be an infinitesimal derivative and let σ be a representative. Let $\{\theta_i\}$ be the connection form of a Bott connection on $Q(\mathcal{F})$ with respect to the dual of $\{\rho_i\}$. If we set $\tau_i = \operatorname{tr}\theta_i$, then τ_i determines a globally well-defined 1-form τ thanks to the choice of $\{\rho_i\}$, and $d\omega = -\tau \wedge \omega$. Let θ' be the infinitesimal derivative of θ with respect to σ and set $\tau' = \operatorname{tr}\theta'$. If we set

$$\widetilde{\sigma} = \sum_{k=1}^{q} \rho_i^1 \wedge \cdots \wedge \rho_i^{k-1} \wedge \rho_i^k(\sigma) \wedge \rho_i^{k+1} \wedge \cdots \wedge \rho_i^q, \tag{5.7}$$

then $\widetilde{\sigma}$ is an infinitesimal deformation of $K_{\mathcal{F}}$ induced from σ ([10, Lemma 2.12]) and we have

$$d\widetilde{\sigma} + \tau \wedge \widetilde{\sigma} = \tau' \wedge \omega. \tag{5.8}$$

Lemma 5.9. *The $(2q+2)$-form $\tau' \wedge \tau \wedge (d\tau)^q$ is closed.*

Proof. We have $d(\tau' \wedge (d\tau)^q) = 0$ by Lemma 4.3.17. It follows that $d(\tau' \wedge \tau \wedge (d\tau)^q) = \tau \wedge d(\tau' \wedge (d\tau)^q) - \tau' \wedge d(\tau \wedge (d\tau)^q) = 0$. $\square$

Once the trivialization ω is fixed, by applying the results in Section 4.3, one can verify that the differential form $\tau' \wedge \tau \wedge (d\tau)^q$ determines a cohomology class which is independent of the choice of τ and τ' (see also [55]).

Definition 5.10 ([28, p. 248], [56], [55]). We denote by $T_\mu B_q(\mathcal{F}, \omega)$ the cohomology class in $H^{2q+2}(M;\mathbb{C})$ represented by $\left(\dfrac{-1}{2\pi\sqrt{-1}}\right)^{q+2} (q+1)\,\tau' \wedge \tau \wedge (d\tau)^q$. We call $T_\mu B_q(\mathcal{F}, \omega)$ the *Fuks–Lodder–Kotschick class of $\mathcal{F}$ with respect to ω.*

The class $T_\mu B_q(\mathcal{F}, \omega)$ is mentioned by Fuks [28] for families of codimension-one foliations. Lodder proved the well-definedness of the class in [56]. Then, Kotschick [55] pointed out and filled a gap in the proof, and extended the definition for arbitrary codimension case. To be precise, they defined a classes, which are denoted by $c(\zeta)$ in [56] and $TGV(\mathcal{F}_t)$ in [55]. We have $c(\zeta) = TGV(\mathcal{F}_t)(0)$, and if μ is the infinitesimal deformation associated with $\{\mathcal{F}_t\}$, then this is the class $T_\mu B_q(\mathcal{F}, \omega)$. It is easy to see that the construction is also valid for infinitesimal deformations, and also for transversely holomorphic foliations. The original construction was for real foliations so that we may assume $K_{\mathcal{F}}$ is trivial, and $T_\mu B_q(\mathcal{F},\omega)$ is independent of the choice of ω. This is the most difference, namely, if the construction is applied for transversely holomorphic foliations, then $T_\mu B_q(\mathcal{F},\omega)$ depends on the homotopy class of ω. Before giving a proof, we present an example.

EXAMPLE 5.11. Let $\mathcal{F}_\lambda$ be the foliation of S^3 given in Example 1.1.6 (with $n=1$). Let $M = S^3 \times S^1$ and $\pi\colon M \to S^3$ be the natural projection. We set $\mathcal{G}_\lambda = \pi^*\mathcal{F}_\lambda$. Let

$$\omega_m = t^m(\lambda_2 z_2\, dz_1 - \lambda_1 z_1\, dz_2)$$

be a trivialization of $K_{\mathcal{G}_\lambda} = Q^*(\mathcal{G}_\lambda)$, where t denotes the standard coordinates of S^1 considered as the unit circle in $\mathbb{C}$. Then, $d\omega_m = -\tau_m \wedge \omega_m$, where

$$\tau_m = -\frac{\lambda_1+\lambda_2}{|z_1|^2+|z_2|^2}\left(\frac{\bar z_1}{\lambda_1}\,dz_1 + \frac{\bar z_2}{\lambda_2}\,dz_2\right) - m\frac{dt}{t}.$$

Since we are working on S^3, we have

$$\begin{aligned}\tau_m &= -(\lambda_1+\lambda_2)\left(\frac{\bar z_1}{\lambda_1}\,dz_1 + \frac{\bar z_2}{\lambda_2}\,dz_2\right) - m\frac{dt}{t}\\ &= -\left(1+\frac{1}{\lambda}\right)\bar z_1\,dz_1 - (\lambda+1)\bar z_2\,dz_2 - m\frac{dt}{t}\end{aligned}$$

and

$$d\tau_m = -\left(1+\frac{1}{\lambda}\right)d\bar z_1\wedge dz_1 - (\lambda+1)\,d\bar z_2\wedge dz_2,$$

where $\lambda = \dfrac{\lambda_1}{\lambda_2}$. It follows that

$$\tau'_m = \lambda^{-2}\bar z_1\,dz_1 - \bar z_2\,dz_2$$

and that

$$\begin{aligned}&\tau'_m\wedge\tau_m\wedge d\tau_m\\ &= \left(\lambda^{-2}(\lambda+1)\bar z_1\,dz_1\wedge d\bar z_2\wedge dz_2 - \left(1+\frac{1}{\lambda}\right)\bar z_2\,dz_2\wedge d\bar z_1\wedge dz_1\right)\wedge m\frac{dt}{t}.\end{aligned}$$

Consequently,

$$T_\mu B_1(\mathcal{G}_\lambda,\omega_m) = -m\left(1-\frac{1}{\lambda^2}\right)[S^3\times S^1],$$

where $[S^3\times S^1]$ denotes the fundamental class and μ is an element of $H^1(M;\Theta_{\mathcal{G}})$ induced from $\{\mathcal{G}_\lambda\}$.

LEMMA 5.12. *$T_\mu B_q(\mathcal{F}, \omega)$ depends on the homotopy class of ω.*

PROOF. Let ω_0 and ω_1 be trivializations of $K_\mathcal{F}$ and let ω_s, $s \in [0,1]$, be a homotopy between ω_0 and ω_1. Then, we can choose a continuous family $\{f_s\}$ of smooth functions such that $\omega_s = e^{f_s}\omega_0$. It follows that we may assume that $\tau_s = \tau_0 - df_s$. If σ is an infinitesimal deformation of ω_0, then $d\sigma + \tau_0 \wedge \sigma = \tau_0' \wedge \omega_0$. We have

$$\begin{aligned} d(e^{f_s}\sigma) &= df_s \wedge (e^{f_s}\sigma) + e^{f_s} d\sigma \\ &= df_s \wedge (e^{f_s}\sigma) + e^{f_s}(-\tau_0 \wedge \sigma + \tau_0' \wedge \omega_0) \\ &= -\tau_s \wedge (e^{f_s}\sigma) + \tau_0' \wedge \omega_s. \end{aligned}$$

Hence we may assume that $\tau_s' = \tau_0'$ for any s. It follows that

$$\tau_s' \wedge \tau_s \wedge (d\tau_s)^q = \tau_0' \wedge (\tau_0 - df_s) \wedge (d\tau_0)^q = \tau_0' \wedge \tau_0 \wedge (d\tau_0)^q + d(f_s \tau_0' \wedge (d\tau_0)^q)$$

because $d(\tau_0' \wedge (d\tau_0)^q) = 0$. $\square$

REMARK 5.13. Actually, it suffices to begin with a Bott connection on $K_\mathcal{F}$ in order to define the class $T_\mu B_q(\mathcal{F})$. Indeed, the notions of infinitesimal deformation of $K_\mathcal{F}$ and infinitesimal derivative of Bott connections can be introduced by the formulae (5.7) and (5.8) [10]. Note that (5.8) follows from (4.3.8) if connections on $Q(\mathcal{F})$ (or $Q^*(\mathcal{F})$) are considered.

We will retain the notations in Theorem 3.3.10. We have then

THEOREM 5.14 (cf. [8, Theorem 2.3]). *If $\mu \in H^1(M; \Theta_\mathcal{F})$, then*

$$\pi_{m!} T_{\pi_m^* \mu} B_q(\mathcal{G}_m, \omega_m) = -m D_\mu B_q(\mathcal{F}).$$

PROOF. We denote π_m and ω_m by π and ω for simplicity. Let $\underline{\sigma}$ be a representative of μ. If we set

$$\underline{\sigma}_i = \sum_{k=1}^{q} dz_i^1 \wedge \cdots \wedge dz_i^{k-1} \wedge dz_i^k(\underline{\sigma}) \wedge dz_i^{k+1} \wedge \cdots \wedge dz_i^q,$$

then we have $d\underline{\sigma}_i + \alpha_i \wedge \underline{\sigma}_i = \alpha'_i \wedge \nu_i$, where $\{\alpha'_i\}$ is the infinitesimal derivative of $\{\alpha_i\}$ with respect to $\underline{\sigma}$. As $K_{\mathcal{G}_m}$ is trivialized by $\{\omega_i\}$, $\pi^*\mu$ is represented by $\{\sigma_i\}$, where $\sigma_i = t^{-m}\pi^*\underline{\sigma}_i$. It follows that

$$d\pi^*\underline{\sigma}_i + \tau \wedge \pi^*\underline{\sigma}_i - m\frac{dt}{t} \wedge \pi^*\underline{\sigma}_i = t^m \pi^*\alpha'_i \wedge \omega_i.$$

Hence we have

$$d\sigma_i + \tau \wedge \sigma_i = \pi^*\alpha'_i \wedge \omega_i.$$

Therefore, $T_{\pi_m^*\mu} B_q(\mathcal{G}_m, \omega)$ is locally represented by

$$\left(\frac{-1}{2\pi\sqrt{-1}}\right)^{2q+2} (q+1)(\pi^*\alpha'_i) \wedge \tau \wedge (d\tau)^q.$$

On the other hand, $D_\mu B_q(\mathcal{F})$ is locally represented by

$$\left(\frac{-1}{2\pi\sqrt{-1}}\right)^{2q+1} (q+1)\, \alpha'_i \wedge (d\alpha_i)^q.$$

It is easy to see that

$$\begin{aligned} \int_{S^1} (\pi^*\alpha'_i) \wedge \tau \wedge (d\tau)^q &= \int_{S^1} \pi^*(\alpha'_i \wedge (d\alpha_i)^q) \wedge m\frac{dt}{t} \\ &= 2m\pi\sqrt{-1}\alpha'_i \wedge (d\alpha_i)^q. \end{aligned}$$

□

By applying Theorem 5.14 to $(N', \mathcal{F}_T)$ and $(N'', \mathcal{G}'')$, one sees that $D_\mu B_q(\mathcal{G}'')$ belongs to the image of the integration along the fiber.