Part III

SPACES OF COMPLEX STRUCTURES

Chapter 10

Virasoro group and its coadjoint orbits

In this Chapter we introduce the Virasoro group Vir, which is a central extension of the diffeomorphism group of the circle $\mathrm{Diff}_+(S^1)$, and study its coadjoint representation. We are especially interested in the coadjoint orbits, which have, along with the natural symplectic form, also a compatible complex structure. These Kähler coadjoint orbits of Vir are studied in Sec. 10.3 of this Chapter.

10.1 Virasoro group and Virasoro algebra

The Virasoro group is a central extension of the diffeomorphism group of the circle $\mathrm{Diff}_+(S^1)$. To describe it explicitly, we find first central extensions of the Lie algebra $\mathrm{Vect}(S^1)$ of $\mathrm{Diff}_+(S^1)$, being the algebra of tangent vector fields on S^1.

As we have pointed out in Sec. 4.1, any central extension of $\mathrm{Vect}(S^1)$ is determined by some 2-cocycle w on the algebra $\mathrm{Vect}(S^1)$. We extend this cocycle complex-linearly to the complexification $\mathrm{Vect}^{\mathbb{C}}(S^1)$ of the algebra $\mathrm{Vect}(S^1)$. The extended cocycle, denoted by the same letter w, is uniquely determined by its values $w_{m,n} := w(e_m, e_n)$ on the basis vector fields

$$e_m = ie^{im\theta}\frac{d}{d\theta} , \quad m = 0, \pm1, \pm2, \dots ,$$

of $\mathrm{Vect}^{\mathbb{C}}(S^1)$ (cf. Sec. 2.2). The cocycle condition for w, written for three vector fields (e_0, e_m, e_n):

$$w([e_0, e_m], e_n) + w(e_m, [e_0, e_n]) = w(e_0, [e_m, e_n]) ,$$

implies that the cohomology class $[w]$ does not change under the action of rotations (generated by the vector field e_0). So the cocycle, obtained from w by averaging over S^1, belongs to the same cohomology class, as w. Therefore we can suppose from the beginning that the cocycle w is invariant under rotations, i.e.

$$w([e_0, e_m], e_n) + w(e_m, [e_0, e_n]) = 0$$

on the basis vector fields e_m, e_n. Due to the commutation relations for basis vector fields

$$[e_m, e_n] = (m - n)e_{m+n} ,$$

it means that

$$mw_{m,n} + nw_{m,n} = 0 \ . \tag{10.1}$$

The latter relation implies that $w_{m,n} = 0$ for $m + n \neq 0$. So we set $w_m := w_{m,-m}$ and note that $w_{-m} = -w_m$ due to the skew-symmetricity of w. It remains to find out the values of w_m for natural m.

The cocycle condition for w on three basis vector fields (e_m, e_n, e_{m+n}) means that

$$(m-n)w_{m+n} = (m+2n)w_m - (2m+n)w_n \ , \tag{10.2}$$

so we get a finite-difference equation of the 2nd order for the computation of values w_m. In order to find a general solution of (10.2), it's sufficient to find its two particular solutions. But it's easy to see that $w_m = m$ and $w_m = m^3$ are two independent solutions of (10.2). Hence a general solution of (10.2) has the form

$$w_m = \alpha m^3 + \beta m \tag{10.3}$$

with arbitrary complex coefficients α, β.

Note that the cocycle w with $w_m = m$ is a coboundary, since in this case

$$w(e_m, e_n) = d\theta(e_m, e_n) = \theta([e_n, e_m]) \ ,$$

where θ is a 1-cochain on $\mathrm{Vect}^{\mathbb{C}}(S^1)$, defined by: $\theta(e_0) = -\frac{1}{2}$ and $\theta(e_m) = 0$ for $m \neq 0$. So the value of β in the formula (10.3) is not essential. Hence all cocycles w, defining non-trivial central extensions of the algebra $\mathrm{Vect}(S^1)$, up to coboundaries, are proportional to each other. In other words, we have proved the following

Proposition 19. *The cohomology group $H^2(\mathrm{Vect}(S^1), \mathbb{R})$ has dimension 1. A general central extension of the algebra $\mathrm{Vect}(S^1)$ is determined by a cocycle w of the form*

$$w(e_m, e_n) = \begin{cases} \alpha m(m^2-1) & \text{for } m+n=0, \alpha \in \mathbb{R}, \\ 0 & \text{for } m+n \neq 0 \ . \end{cases}$$

We have chosen the parameter $\beta = -\alpha$ in order to annihilate the restriction of the cocycle w to the subalgebra $\mathrm{sl}(2,\mathbb{R})$ in $\mathrm{Vect}(S^1)$, generated by the vectors e_0, e_1, e_{-1} (this subalgebra coincides with the Lie algebra of the Möbius group $\mathrm{PSL}(2,\mathbb{R})$ of diffeomorphisms of the circle S^1, extending to the fractional-linear automorphisms of the unit disc Δ).

We note that the *Gelfand–Fuks cocycle*

$$w(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \xi'(\theta) d\eta'(\theta) \ , \quad \xi = \xi(\theta)\frac{d}{d\theta}, \ \eta = \eta(\theta)\frac{d}{d\theta} \in \mathrm{Vect}(S^1) \ ,$$

found in [25], has the basis values, equal to $w_m = im^3$, $m \in \mathbb{Z}$.

One can also change the value of α, multiplying the central element by a number. The usual choice for α (based on physical analogies) is $\alpha = \frac{1}{12}$. The corresponding central extension of the algebra $\mathrm{Vect}(S^1)$ is called the *Virasoro algebra* and denoted by vir. The Virasoro algebra is generated (as a vector space) by the basis vector fields $\{e_m\}$ of the algebra $\mathrm{Vect}(S^1)$ and a central element κ, satisfying the commutation relations of the form

$$[e_m, \kappa] = 0 \ , \quad [e_m, e_n] = (m-n)e_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}\kappa \ .$$

This central extension of the Lie algebra $\mathrm{Vect}(S^1)$ corresponds to a central extension of the Lie group $\mathrm{Diff}_+(S^1)$, which we describe next.

Since the Frechet manifold $\mathrm{Diff}_+(S^1)$ is homotopy equivalent to the circle S^1 (cf. Sec. 1.2.1), all S^1-bundles over $\mathrm{Diff}_+(S^1)$ are topologically trivial and any central extension of the group $\mathrm{Diff}_+(S^1)$ is determined by some 2-cocycle c on $\mathrm{Diff}_+(S^1)$ (cf. Sec. 4.1). In other words, such a central extension consists of elements of the form

$$(f, \lambda) , \quad f \in \mathrm{Diff}_+(S^1),\ \lambda \in S^1 ,$$

and the product is given by the formula

$$(f, \lambda) \cdot (g, \mu) = (f \circ g, \lambda\mu e^{ib(f,g)}) ,$$

where $c(f, g) = e^{ib(f,g)}$ is the 2-cocycle on $\mathrm{Diff}_+(S^1)$, defining the central extension. The cocycle condition in terms of b takes the form

$$b(f, g) + b(f \circ g, h) = b(f, g \circ h) + b(g, h) . \tag{10.4}$$

An explicit solution of this functional equation, found by Bott [11], has the form

$$b_0(f, g) = \frac{1}{2\pi}\int_0^{2\pi} \ln(f \circ g)' \, d \ln g' .$$

Note that the Bott group cocycle corresponds on the Lie algebra level to the Gelfand–Fuks cocycle of the Lie algebra $\mathrm{Vect}(S^1)$.

A general solution of (10.4) coincides with b_0 up to a coboundary, more precisely, it has the form

$$b(f, g) = \alpha b_0(f, g) + a(f \circ g) - a(f) - a(g) ,$$

where $\alpha = \mathrm{const} \in \mathbb{R}$, and a is an arbitrary smooth real functional on $\mathrm{Diff}_+(S^1)$.

The central extension of the group $\mathrm{Diff}_+(S^1)$, determined by the Bott cocycle, is called the *Virasoro group* (or *Virasoro–Bott group*) and is denoted by Vir.

10.2 Coadjoint action of the Virasoro group

Consider the coadjoint action of the diffeomorphism group of the circle $\mathrm{Diff}_+(S^1)$ and its central extension, the Virasoro group Vir, on the dual spaces of their Lie algebras.

We study first the coadjoint action of the diffeomorphism group $\mathrm{Diff}_+(S^1)$ on the space $\mathrm{Vect}^*(S^1)$, dual to the Lie algebra $\mathrm{Vect}(S^1)$ of $\mathrm{Diff}_+(S^1)$. The space $\mathrm{Vect}^*(S^1)$, dual to the Frechet space $\mathrm{Vect}(S^1)$, can be identified with the tensor product

$$\Omega^1(S^1) \otimes_{\mathcal{D}(S^1)} \mathcal{D}'(S^1)$$

over the ring $\mathcal{D}(S^1)$, consisting of all C^∞-smooth (real-valued) functions on S^1. Here, $\Omega^1(S^1)$ is the Frechet space of C^∞-smooth 1-forms on S^1, and $\mathcal{D}'(S^1)$ is the space of distributions on S^1, i.e. of linear continuous functionals on $\mathcal{D}(S^1)$ (note that $\mathcal{D}'(S^1)$ is not a Frechet space!). The above tensor product should be taken in the category of topological vector spaces, we recall its definition for convenience.

Digression 3 (Tensor product of topological vector spaces). The tensor product $E\otimes F$ of topological vector spaces E and F is provided with the projective topology, generated by the seminorms $p\otimes q$, where $\{p\}$ and $\{q\}$ are families of seminorms on E and F respectively. The seminorm $p\otimes q$ is defined as

$$(p\otimes q)(z)=\inf\left\{\sum_i p(x_i)q(y_i)\ :\ z=\sum x_i\otimes y_i\right\}\ ,$$

where the infimum is taken over all possible representations of $z\in E\otimes F$ as finite sums of the form $\sum x_i\otimes y_i$ with $x_i\in E$, $y_i\in F$.

The elements of the completion $\widetilde{E\otimes F}$ of the space $E\otimes F$ with respect to this topology in the case of metrizable spaces E and F can be given by series of the form

$$\widetilde{E\otimes F}\ni z=\sum_{i=1}^{\infty}\lambda_i x_i\otimes y_i\ ,$$

where $\sum_{i=1}^{\infty}|\lambda_i|<\infty$ and the sequences $\{x_i\}$, $\{y_i\}$ tend to zero in E and F respectively.

For the nuclear spaces E and F the topology, introduced on $\widetilde{E\otimes F}$, coincides with the topology of the uniform equicontinuous convergence (i.e. topology of uniform convergence on the sets of the form $S\otimes T$, where S and T are uniformly equicontinuous subsets in E' and F' respectively).

We return to the dual space $\mathrm{Vect}^*(S^1)$, which is identified with the tensor product $\Omega^1(S^1)\otimes_{\mathcal{D}(S^1)}\mathcal{D}'(S^1)$ by the map, associating with an element $(\alpha,\varphi)\in\Omega^1(S^1)\otimes_{\mathcal{D}(S^1)}\mathcal{D}'(S^1)$ a linear continuous functional on $\mathrm{Vect}(S^1)$ of the form

$$T_{(\alpha,\varphi)}(\xi)=\varphi[\alpha(\xi)]\ ,\quad \xi\in\mathrm{Vect}(S^1)\ .$$

As in Sec. 8.3, we restrict ourselves to the study of the coadjoint action of the group $\mathrm{Diff}_+(S^1)$ on the "smooth" part of the space $\mathrm{Vect}^*(S^1)$, identified with the tensor product of Frechet spaces

$$\Omega^1(S^1)\otimes_{\mathcal{D}(S^1)}\Omega^1(S^1)\ .$$

An element (α,β) of this space determines a linear continuous functional on $\mathrm{Vect}(S^1)$ by the formula

$$\mathrm{Vect}(S^1)\ni\xi\longmapsto T_{(\alpha,\beta)}(\xi)=\frac{1}{2\pi}\int_0^{2\pi}\beta(\xi(\theta))\alpha(\theta)\ .$$

In other words, the smooth part of the space $\mathrm{Vect}^*(S^1)$ may be identified with the *space* $Q(S^1)$ *of quadratic differentials* on S^1 of the form

$$q=q(\theta)(d\theta)^2\ ,$$

where q is a smooth 2π-periodic function of θ.

From another point of view, one can consider $Q(S^1)$ as a set of pseudometrics on S^1 (the term "pseudo" indicates that the function $q(\theta)$ may have zeros on S^1).

The coadjoint action of the group $\mathrm{Diff}_+(S^1)$ on $Q(S^1)$ coincides with the natural action of the group $\mathrm{Diff}_+(S^1)$ on quadratic differentials

$$\mathrm{Diff}_+(S^1) \ni f \longmapsto K(f)q = q \circ f^{-1} := q(g(\theta))g'(\theta)^2 d\theta^2 \ ,$$

where $g(\theta) = f^{-1}(\theta)$.

We consider next the coadjoint action of the group $\mathrm{Diff}_+(S^1)$ on the dual space vir^* of the Virasoro algebra vir. Since the Virasoro algebra coincides with $\mathrm{vir} = \mathrm{Vect}(S^1) \oplus \mathbb{R}$ (as a vector space), we have $\mathrm{vir}^* = \mathrm{Vect}^*(S^1) \oplus \mathbb{R}$. So the smooth part of vir^* may be identified with the space

$$Q(S^1) \oplus \mathbb{R} = \{(q,s) : \ q \text{ is a quadratic differential, } s \in \mathbb{R}\} \ .$$

The coadjoint action of the group $\mathrm{Diff}_+(S^1)$ on $Q(S^1) \oplus \mathbb{R}$ associates with an element $f \in \mathrm{Diff}_+(S^1)$ a linear transformation $\widetilde{K}(f)$ of the space $Q(S^1) \oplus \mathbb{R}$, acting by the formula

$$\widetilde{K}(f)(q,s) = (K(f)q + sS(f) \circ f^{-1}, s) = ((q + sS(f)) \circ f^{-1}, s) \ , \tag{10.5}$$

where S is a 1-cocycle on the group $\mathrm{Diff}_+(S^1)$, satisfying the relation

$$S(f \circ h) = (S(f) \circ h) + S(h) \ . \tag{10.6}$$

A non-trivial particular solution of this equation is given by the *Schwarzian*

$$S[f] = \left(\frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 \right) d\theta^2 = d^2 \ln f' - \frac{1}{2}(d \ln f')^2 \ , \tag{10.7}$$

while a general solution has the form

$$S[f] + q \circ f - q \ ,$$

where $q \in Q(S^1)$ is a quadratic differential.

Digression 4 (Schwarzian). A characteristic property of the Schwarzian is its conformal invariance:

$$S \left[\frac{af+b}{cf+d} \right] = S[f]$$

for any fractional-linear transformation $z \mapsto \frac{az+b}{cz+d}$ from the Möbius group $\mathrm{M\ddot{o}b}(S^1) := \mathrm{PSL}(2,\mathbb{R})$. This property follows immediately from the transformation rule for the Schwarzian

$$S[f \circ h] = (S[f] \circ h)\,(h')^2 + S[h] \ , \tag{10.8}$$

which is just a decoded version of (10.6).

The Schwarzian $S[f]$ of a diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ measures its deviation from conformal automorphisms of the unit disc in the sense that

$$S[f] = 0 \iff f \text{ is fractional-linear} \ .$$

Moreover, one can define the Schwarz derivative $S[f]$ of any conformal map $f : \Delta \to \mathbb{C}$ by the same formula (10.7). Then $S[f]$ measures again the deviation of a conformal map f in Δ from fractional-linear automorphisms of Δ, and the maximal deviation may be explicitly computed. Introduce a natural norm on Schwarz derivatives $S[f]$, coinciding with the hyperbolic norm on quadratic differentials in Δ:

$$\|S[f]\|_2 := \sup_{z\in\Delta} |S[f](z)|(1-|z|^2)^2 \ .$$

There is a following remarkable theorem, known as *Nehari theorem.*

Theorem 11 ((cf. [49], Theor. II.1.3)). *For any conformal map f of the unit disc Δ the following sharp estimate holds*

$$\|S[f]\|_2 \leq 6 \ .$$

The upper bound is attained on the Koebe function $z \mapsto z/(1+z)$.

The infinitesimal variant of the coadjoint representation (10.5) is given by the representation of the Lie algebra $\mathrm{Vect}(S^1)$ on the space $Q(S^1)\oplus\mathbb{R}$, defined by the formula

$$\widetilde{k}(\xi)(q,s) = (-D_{q,s}\xi, s) \ , \tag{10.9}$$

where $\xi = \xi(\theta)\frac{d}{d\theta} \in \mathrm{Vect}(S^1)$, $q = q(\theta)(d\theta)^2 \in Q(S^1)$, and the operator $D_{q,s}$ has the form

$$D_{q,s} = s\frac{d^3}{d\theta^3} + q\frac{d}{d\theta} + \frac{d}{d\theta}q \ .$$

What can be said about the orbits of the coadjoint representation of $\mathrm{Diff}_+(S^1)$? The orbit of a regular element $(q,s) \in Q(S^1)\oplus\mathbb{R}$ under the action of the group $\mathrm{Diff}_+(S^1)$ is completely determined by the isotropy subgroup $G_{q,s}$ with respect to the coadjoint action. The Lie algebra $\mathfrak{g}_{q,s}$ of this subgroup consists of vector fields $\xi = \xi(\theta)\frac{d}{d\theta} \in \mathrm{Vect}(S^1)$, satisfying the condition: $D_{q,s}\xi = 0$. In other words, to describe the subalgebra $\mathfrak{g}_{q,s}$, one should find periodic solutions $\xi(\theta)$ of the linear differential equation

$$s\xi^{'''} + 2q\xi' + q'u = 0 \ . \tag{10.10}$$

Referring for the general solution of this problem to the papers [40, 30], we consider here only its particular case, when a regular element (q,s) has the form $(q(d\theta)^2, s)$ with $q \equiv \mathrm{const} =: c$, $s \neq 0$. In this case the equation (10.10) takes on the form

$$s\xi^{'''} + 2c\xi' = 0 \ , \tag{10.11}$$

which, after the change of variable $\eta := \xi'$, reduces to the equation

$$s\eta^{''} + 2c\eta = 0 \ .$$

The latter equation has non-trivial periodic solutions only for $2c = n^2$, where n is a natural number, and all these solutions are linear combinations of the functions $\cos n\theta$ and $\sin n\theta$. In other words, the only periodic solutions of the equation (10.11) for $\frac{2c}{s} \neq n^2$ are given by constants, while for $\frac{2c}{s} = n^2$ they are linear combinations of the functions 1, $\frac{1}{n}\cos n\theta$ and $\frac{1}{n}\sin n\theta$.

The isotropy subalgebra $\mathfrak{g}_{q,s}$ in the first case coincides with $\mathbb{R}$, and in the second case with the algebra $\mathrm{sl}(2,\mathbb{R})$. Respectively, the isotropy subgroup $G_{q,s}$ in the first case coincides with the rotation group $S^1 \subset \mathrm{Diff}_+(S^1)$, and in the second case with the group $\mathrm{PSL}^{(n)}(2,\mathbb{R})$, which is the n-fold covering of the Möbius group $\text{Möb}(S^1) = \mathrm{PSL}(2,\mathbb{R})$. We have already encountered this group in Sec. 2.2. Recall that a diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ belongs to the group $\mathrm{PSL}^{(n)}(2,\mathbb{R})$ if and only if there exists a transformation $\varphi \in \mathrm{PSL}(2,\mathbb{R})$ such that

$$\lambda_n \circ f = \varphi \circ \lambda_n$$

where $\lambda_n : z \mapsto z^n$ is the map, defining the n-fold covering of the circle S^1.

It follows from the description of isotropy subgroups that the coadjoint orbit of a constant element $(q, s) = (c(d\theta)^2, s)$ coincides with the homogeneous space $\mathrm{Diff}_+(S^1)/S^1$, when $2c/s$ is not a square of a natural number, and with the homogeneous space $\mathrm{Diff}_+(S^1)/\mathrm{PSL}^{(n)}(2,\mathbb{R})$, when $2c/s = n^2$.

As we have explained earlier in Subsec. 3.2.3, all coadjoint orbits have a natural symplectic structure, given by the Kirillov form. In the case, we are considering, the value of this form at a point $(q, s) \in Q(S^1) \oplus \mathbb{R}$ of an orbit O of the group $\mathrm{Diff}_+(S^1)$ may be computed in the following way. Let $\delta\xi$ and $\delta\eta$ be tangent vectors from $T_{q,s}O$, which are the images of tangent vectors $\xi, \eta \in \mathrm{Vect}(S^1)$ under the map $\widetilde{k}$ from (10.9):

$$\delta\xi = \widetilde{k}(\xi)(q, s) \ , \quad \delta\eta = \widetilde{k}(\eta)(q, s) \ .$$

Then the value of the form ω_O on these vectors is equal to

$$\omega_O(\delta\xi, \delta\eta) = -\int_{S^1} (D_{q,s}\xi)(\theta)\eta(\theta)d\theta \ .$$

Thus every coadjoint orbit of Vir has a symplectic structure. But not all of them can be provided with a compatible complex structure. In fact, among the coadjoint orbits of the group Vir, described above, only the orbits

$$\mathrm{Diff}_+(S^1)/S^1 \ , \quad \mathrm{Diff}_+(S^1)/\,\text{Möb}(S^1) = \mathrm{Diff}_+(S^1)/\mathrm{PSL}(2,\mathbb{R})$$

are Kähler (cf. [79]). In other words, only these orbits admit $\mathrm{Diff}_+(S^1)$-invariant complex structures, compatible with the symplectic structure ω_O. We shall concentrate our attention on these Kähler orbits.

Example 29. We give now an interesting interpretation of the coadjoint action of the Virasoro group in terms of Hill operators, due to Lazutkin and Pankratova [48].

Recall that a *Hill operator* is a differential operator of the 2nd order, having the form

$$L = \frac{d^2}{d\theta^2} + u(\theta) \ ,$$

where $u = u(\theta)$ is a potential, given by a C^∞-smooth 2π-periodic function on $\mathbb{R}$. The corresponding ordinary differential equation

$$y'' + uy = 0$$

is called the *Hill equation*. Its solutions form a two-dimensional vector space V, provided with a natural symplectic 2-form, given by the Wronskian of two solutions. The shift of a solution y of the Hill equation $Ly = 0$ to the period 2π transforms it into another solution, obtained from y by the action of an operator $M \in \mathrm{SL}(V)$, called the *monodromy matrix* of the operator L.

If $\{y_1, y_2\}$ is a fundamental system of solutions, i.e. a basis in the space V of solutions of the Hill equation, then one can reconstruct the potential u from this system by the Schwarz formula:

$$u(\theta) = \begin{cases} \frac{1}{2}S[y_1/y_2](\theta) \ , & \text{if } y_2(\theta) \neq 0 \ , \\ \frac{1}{2}S[y_2/y_1](\theta) \ , & \text{if } y_1(\theta) \neq 0 \ , \end{cases}$$

where $S[y]$ is the Schwarzian of y.

The diffeomorphism group $\mathrm{Diff}_+(S^1)$ acts in a natural way on the space of Hill operators. Namely, we can associate with any diffeomorphism $f \in \mathrm{Diff}_+(S^1)$, which lifts to a diffeomorphism $\tilde{f}$ of the real line $\mathbb{R}$, a transformation, which sends a given Hill operator $L = \frac{d^2}{d\theta^2} + u(\theta)$ to another Hill operator $f^*L = \frac{d^2}{d\theta^2} + f^*u(\theta)$ with

$$f^*u(\theta) := u(\tilde{f}(\theta)) \cdot (\tilde{f}'(\theta))^2 + \frac{1}{2}S[\tilde{f}](\theta) \ .$$

Under this transformation a solution y of the Hill equation $Ly = 0$ is transferred to a solution z of the Hill equation $(f^*L)z = 0$ with

$$z(\theta) := y(\tilde{f}(\theta)) \cdot (\tilde{f}'(\theta))^{-\frac{1}{2}} \ .$$

Note that, due to the periodicity of the potential u, the action of f on potentials does not depend on the choice of the lift $\tilde{f}$ of the diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ and so defines an action of the group $\mathrm{Diff}_+(S^1)$ on Hill operators. This action coincides with the coadjoint action of the group $\mathrm{Diff}_+(S^1)$ on elements $(u, \frac{1}{2})$ of the space $Q(S^1) \oplus \mathbb{R}$, given by (10.5).

But the action of f on solutions of the Hill equation depends on the choice of the lift $\tilde{f}$, because of the monodromy. In accordance with the above formula, solutions of the Hill equation transform under the action of diffeomorphisms $\tilde{f}$, as densities of order $-1/2$ on the line $\mathbb{R}$.

The constructed action of the group $\mathrm{Diff}_+(S^1)$ on Hill operators was studied in the Lazutkin–Pankratova's paper [48]. The authors formulate, in particular, a conjecture that any Hill operator with the help of the above action can be brought to the *Matieu normal form* of the type:

$$L = \frac{d^2}{d\theta^2} + a\cos(2\pi n\theta) + b \ .$$

10.3 Kähler structure of the spaces $\mathrm{Diff}_+(S^1)\big/\,\mathrm{M\ddot{o}b}(S^1)$ and $\mathrm{Diff}_+(S^1)/S^1$

As we have pointed out in the previous Section, among the coadjoint orbits of the Virasoro group Vir only two are Kähler, namely:

$$\mathcal{R} := \mathrm{Diff}_+(S^1)/S^1 \quad \text{and} \quad \mathcal{S} := \mathrm{Diff}_+(S^1)\big/\,\mathrm{M\ddot{o}b}(S^1) \ .$$

In this Section we study their Kähler structure in detail.

As coadjoint orbits of the group Vir, these spaces have a natural symplectic structure ω, given by the Kirillov form.

We introduce now a complex structure J on the space $\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$, invariant under the action of the diffeomorphism group $\mathrm{Diff}_+(S^1)$ by left translations. Due to its invariance, it's sufficient to define this complex structure only at the origin $o \in \mathcal{S}$.

The tangent space $T_o\mathcal{S}$ may be identified with the quotient of the Lie algebra $\mathrm{Vect}(S^1)$ of tangent vector field on S^1 modulo its subalgebra $\mathrm{sl}(2,\mathbb{R})$. In terms of Fourier decompositions vector fields $v = v(\theta)\frac{d}{d\theta} \in T_o\mathcal{S}$ are given by series of the form

$$v(\theta) = \sum_{n\neq -1,0,1} v_n e^{in\theta} , \quad v_n \in \mathbb{C} ,$$

subject to the condition: $v_{-n} = \bar{v}_n$. In these terms the restriction of the $\mathrm{Diff}_+(S^1)$-invariant complex structure J to $T_o\mathcal{S}$ is given by the formula

$$Jv(\theta) = -i\sum_{n>1} v_n e^{in\theta} + i\sum_{n<-1} v_n e^{in\theta}$$

for $v = v(\theta)\frac{d}{d\theta} \in T_o\mathcal{S}$. It's easy to see that the constructed complex structure on $\mathcal{S}$ is formally integrable (i.e. the bracket of two tangent vector fields of type $(1,0)$ with respect to this complex structure is again a vector field of type $(1,0)$). Moreover, this complex structure is compatible with the symplectic structure ω on $\mathcal{S}$, mentioned above.

The symplectic form ω on $\mathcal{S}$ together with the complex structure J define a Kähler metric g on $\mathcal{S}$. In terms of Fourier decompositions this metric can be defined in the following way. Suppose that tangent vectors $u, v \in T_o\mathcal{S}$ are given by the Fourier series

$$u = \sum_{n\neq -1,0,1} u_n e_n \quad \text{and} \quad v = \sum_{n\neq -1,0,1} v_n e_n . \tag{10.12}$$

Then the value of the metric g on these vectors is equal to

$$g(u,v) = 2\,\mathrm{Re}\left(\sum_{n=2}^{\infty} u_n\bar{v}_n(n^3 - n)\right) . \tag{10.13}$$

The infinite series in the right hand side of (10.13) is absolutely converging, if the Fourier series (10.12) correspond to the vector fields u, v of the class $C^{3/2+\epsilon}$ on S^1.

We turn now to the orbit $\mathcal{R} := \mathrm{Diff}_+(S^1)/S^1$. It can be identified (as a homogeneous space) with a subgroup of $\mathrm{Diff}_+(S^1)$, consisting of diffeomorphisms $f \in \mathrm{Diff}_+(S^1)$, fixing the point $1 \in S^1$: $f(1) = 1$.

The embedding of the rotation group of the circle S^1 into the Möbius group $\mathrm{Möb}(S^1)$ generates a homogeneous bundle

$$\mathcal{R} = \mathrm{Diff}_+(S^1)/S^1 \longrightarrow \mathcal{S} ,$$

having the unit disc Δ as a fibre.

We describe explicitly the symplectic structure on $\mathcal{R}$, given by the Kirillov form. This form, being invariant under the left translations of the group $\mathrm{Diff}_+(S^1)$, is completely determined by its restriction to the tangent space at the origin $T_o\mathcal{R}$.

The tangent space $T_o\mathcal{R}$ is identified with the space $\mathrm{Vect}_0(S^1)$, consisting of vector fields $v = v(\theta)\frac{d}{d\theta}$, whose coefficients $v(\theta)$ are 2π-periodic functions with zero average:

$$\frac{1}{2\pi}\int_0^{2\pi} v(\theta)d\theta = 0 \ .$$

In terms of Fourier decompositions tangent vectors $v \in T_o\mathcal{R}$ are given by the series of the form $v = \sum_{n\neq 0} v_n e_n$, subject to the condition: $v_{-n} = \bar{v}_n$.

An invariant symplectic structure on $\mathcal{R}$ is defined by a 2-cocycle w on the Lie algebra $\mathrm{Vect}^{\mathbb{C}}(S^1)$, invariant under rotations. Such a cocycle is determined by its values $w(e_m, e_n)$ on the basis elements $\{e_m\}$. These basis values necessarily have the form (cf. Prop. 19 in Sec. 10.1):

$$w(e_m, e_n) = (\alpha m^3 + \beta m)\delta_{m,-n}$$

for some real α, β. Denote the form, corresponding to the parameters α, β, by $w_{\alpha,\beta}$. It's easy to see that it is non-degenerate on $\mathrm{Vect}_0(S^1)$ if and only if

$$\alpha m^3 + \beta m \neq 0 \quad \text{for all natural } m \ .$$

The latter condition is satisfied, if either $\alpha = 0$, $\beta \neq 0$, or $-\beta/\alpha$ is not a square of a natural number. In the first case the form $w_{\alpha,\beta}$ is exact (cf. Sec. 10.1), so we choose the second possibility.

The form $w_{\alpha,\beta}$ defines a symplectic structure on $\mathrm{Vect}_0(S^1)$, which can be written in a more invariant way as

$$w_{\alpha,\beta}(u,v) = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\left(\beta v'(\theta) - \alpha v'''(\theta)\right) d\theta \ ,$$

where $u, v \in \mathrm{Vect}_0(S^1)$. In terms of Fourier decompositions

$$u = \sum_{n\neq 0} u_n e^{in\theta} \ , \quad v = \sum_{n\neq 0} v_n e^{in\theta} \ ,$$

we get

$$w_{\alpha,\beta}(u,v) = 2\mathrm{Im}\left(\sum_{n\geq 1}(\alpha n^3 + \beta n)\xi_n\bar{\eta}_n\right) \ .$$

The constructed 2-parameter family of symplectic structures on $\mathcal{R}$ has a natural interpretation in terms of the coadjoint action of the group $\mathrm{Diff}_+(S^1)$. Recall that the orbit of an element $(c(d\theta)^2, s)$ coincides with $\mathcal{R}$, if $2c/s$ is not a square of a natural number. By identifying the homogeneous space $\mathcal{R}$ with the orbit of an element $(c(d\theta)^2, s)$ and providing it with the canonical symplectic structure, given by the Kirillov form, we shall obtain, for different choices of (c, s) with $2c/s \neq n^2$, the two-parameter family of symplectic structures on $\mathcal{R}$, constructed above.

Introduce a $\mathrm{Diff}_+(S^1)$-invariant complex structure J on the space $\mathcal{R}$. Its restriction to $T_o\mathcal{R} = \mathrm{Vect}_0(S^1)$ is given by the Hilbert transform, which assigns to a tangent vector $v \in \mathrm{Vect}_0(S^1)$ the vector

$$(Jv)(\theta) = \frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\theta - \psi}{2}\, v(\psi)d\psi \ , \quad 0 \leq \theta \leq 2\pi \ .$$

In terms of the Fourier decomposition $v = \sum_{n\neq 0} v_n e_n \in \mathrm{Vect}_0(S^1)$ we get

$$Jv = -i\sum_{n>0} v_n e_n + i\sum_{n<0} v_n e_n \ .$$

The complex structure J is formally integrable, i.e. the bracket of two tangent vector fields of type $(1,0)$ with respect to this complex structure is again a vector field of type $(1,0)$. Moreover, it can be shown that this complex structure is a unique formally integrable $\mathrm{Diff}_+(S^1)$-invariant complex structure on $\mathcal{R}$.

The constructed complex structure J is compatible with all symplectic structures $w_{\alpha,\beta}$, so it generates a 2-parameter family of Kähler metrics $g_{\alpha,\beta}(u,v) := w_{\alpha,\beta}(u,Jv)$ on $\mathcal{R}$, given at the origin by the formula:

$$g_{\alpha,\beta}(u,v) = 2\,\mathrm{Re}\left(\sum_{n\geq 1}(\alpha n^3 + \beta n)u_n\bar{v}_n\right) \ ,$$

where $u = \sum_{n\neq 0} u_n e_n$, $v = \sum_{n\neq 0} v_n e_n \in T_o\mathcal{R}$. Hence, $\mathcal{R}$ is a Kähler Frechet manifold with a 2-parameter family of Kähler metrics $g_{\alpha,\beta}$.

As we know, the existence of a formally integrable complex structure on an infinite-dimensional manifold does not guarantee the existence of an atlas of local complex coordinates on it. We shall introduce local complex coordinates on $\mathcal{R}$, following an idea, proposed by Kirillov and Yuriev [44]. Namely, we shall realize $\mathcal{R}$ as the space of holomorphic univalent functions in the unit disc Δ.

Denote by $\mathcal{A}$ the complex Frechet space of all C^∞-smooth complex-valued functions in the closure $\overline{\Delta}$ of the unit disc Δ, which are holomorphic inside Δ and vanish at the origin. Let $\mathcal{A}_0$ be a subset of $\mathcal{A}$, consisting of all $f \in \mathcal{A}$, which define a C^∞-smooth embedding of the closed disc $\overline{\Delta}$ into $\mathbb{C}$. It is an open subset in $\mathcal{A}$, which inherits a complex Frechet manifold structure. Denote by $\mathfrak{S}$ the set of functions $f \in \mathcal{A}_0$, such that $f'(0) = 1$, which is a smooth hypersurface in $\mathcal{A}_0$. The functions $f \in \mathfrak{S}$ are holomorphic and univalent in Δ, they define C^∞-smooth embeddings $\overline{\Delta} \to \overline{f(\Delta)}$ and satisfy the normalizing conditions: $f(0) = 0$, $f'(0) = 1$. They can be given by power series of the form

$$f(z) = z + c_2 z^2 + c_3 z^3 + \dots \ ,$$

whose coefficients satisfy, according to de Branges theorem, the relations: $|c_k| < k$. The coefficients $\{c_k\}$ may be chosen for local complex coordinates in a neighborhood of $f(z) \equiv z$ in $\mathfrak{S}$.

We construct now a map from $\mathfrak{S}$ to $\mathcal{R}$. For that we associate with a function $f \in \mathfrak{S}$ the contour $K := f(S^1)$. The function $f := f_K$ maps conformally the unit disc $\Delta := \Delta_+$ onto the domain D_K, bounded by the contour K. Denote by

$$g_K : \Delta_- \longrightarrow \overline{\mathbb{C}} \setminus \overline{D}_K$$

the conformal map of the complement $\Delta_- := \overline{\mathbb{C}} \setminus \overline{\Delta}_+$ of the closed unit disc $\overline{\Delta}_+$ on the Riemann sphere $\overline{\mathbb{C}}$ onto the domain $\overline{\mathbb{C}} \setminus \overline{D}_K$, normalized by the conditions:

$$g_K(\infty) = \infty \ , \quad g'_K(\infty) > 0 \ .$$

The map g_K extends to a diffeomorphism of $\partial\Delta_- = S^1$ onto ∂D_K. We associate with $f \in \mathfrak{S}$ the diffeomorphism

$$\gamma_K := f_K^{-1} \circ g_K|_{S^1} \ .$$

In order to construct an inverse map from $\mathcal{R}$ to $\mathfrak{S}$, note that, using an arbitrary diffeomorphism $\gamma \in \mathcal{R}$, we can construct a new complex structure on the Riemann sphere $\overline{\mathbb{C}}$. Indeed, denote by $\overline{\mathbb{C}}_\gamma$ the smooth manifold, obtained by gluing Δ_+ with Δ_- with the help of γ. In other words, $\overline{\mathbb{C}}_\gamma$ is obtained from the disconnected union $\overline{\Delta}_+ \sqcup \overline{\Delta}_-$ by the identification of points from $S^1 = \partial\Delta_+ = \partial\Delta_-$ via the rule:

$$z \in S^1 = \partial\Delta_+ \longleftrightarrow \gamma^{-1}(z) \in S^1 = \partial\Delta_- \ .$$

The complex manifold $\overline{\mathbb{C}}_\gamma$ is diffeomorphic to the Riemann sphere $\overline{\mathbb{C}}$. But, according to the theorem of Ahlfors, there exists a unique complex structure on the Riemann sphere $\overline{\mathbb{C}}$. So the two manifolds are biholomorphic to each other, i.e. there exists a biholomorphic map

$$F : \overline{\mathbb{C}}_\gamma \longrightarrow \overline{\mathbb{C}} \ ,$$

which is uniquely defined, being normalized by the following conditions:

$$F(0) = 0 \ , \quad F(\infty) = \infty \ , \quad F'(0) = 1 \ .$$

The biholomorphism F is given by a pair of functions (f, g), where the function f is holomorphic in Δ_+ and C^∞-smooth up to $S^1 = \partial\Delta_+$, and the function g is holomorphic in Δ_- and C^∞-smooth up to $S^1 = \partial\Delta_-$, while

$$f = g \circ \gamma^{-1} \quad \text{on} \quad S^1 \ .$$

Setting $K := f(S^1)$, we get that $\gamma = \gamma_K \mod S^1$ (since the normalization of F does not fix $\arg g(\infty)$).

As it is pointed out by Lempert [50], one can construct the inverse map by using, instead of the Ahlfors theorem, the factorization theorem of Pflüger [62], which asserts that any diffeomorphism $\gamma \in \mathcal{R}$ may be represented in the form

$$\gamma = f^{-1} \circ g \ ,$$

where f and g have the same properties, as above.

The constructed one-to-one map from $\mathfrak{S}$ to $\mathcal{R}$ is smooth and defines a diffeomorphism

$$\kappa : \mathcal{R} \longrightarrow \mathfrak{S} \ .$$

It's easy to describe its tangent map

$$d_0\kappa : T_0\mathcal{R} \longrightarrow T_1\mathfrak{S} \ .$$

The tangent space $T_1\mathfrak{S}$ is identified with the space Φ, consisting of functions φ, which are holomorphic in Δ, C^∞-smooth up to $\partial\Delta$ and normalized by the conditions: $\varphi(0) = 0$, $\varphi'(0) = 0$. (Indeed, any such vector φ is tangent to the curve $f_t(z) = z + t\varphi(z)$, which is contained in $\mathfrak{S}$ for $0 \le t \le \epsilon$.) The map $d_0\kappa$ associates with a vector $v \in T_0\mathcal{R}$ a function $\varphi \in T_1\mathfrak{S}$ by the formula

$$2\,\mathrm{Re}\,\varphi(e^{i\theta}) = (Jv)(\theta) \ ,$$

where J is the Hilbert transform on $T_0\mathcal{R}$. The Hilbert transform J on $T_0\mathcal{R}$ corresponds to the multiplication by i in the space $T_1\mathfrak{S}$, hence the map, inverse to $d_0\kappa$, is given by the formula: $v(\theta) = -2\,\mathrm{Im}\,\varphi(e^{i\theta})$.

It follows from the definition of complex structures on $\mathcal{R}$ and $\mathcal{S}$ that the homogeneous disc bundle $\mathcal{R} \to \mathcal{S}$ is, in fact, holomorphic.

We note also that on the Virasoro group Vir itself there exists a complex structure, induced by the complex structure on $\mathcal{R}$, such that the natural projection

$$\pi : \mathrm{Vir} \longrightarrow \mathcal{R}$$

is a holomorphic $\mathbb{C}^*$-bundle with respect to this complex structure (cf. [50]).

Bibliographic comments

The Virasoro group and Virasoro algebra are considered in different books, dealing with infinite-dimensional groups and algebras. Apart from the Pressley-Segal book [65], see also [38, 22]. The coadjoint representation of the Virasoro group and its orbits are studied in [40, 30]. The study of the Kähler structure of the space $\mathcal{R}$ was initiated by Bowick–Rajeev [14] and Kirillov [41]. A relation between this space and the space of holomorphic univalent functions in the unit disc was established in the Kirillov–Yuriev paper [44].

Chapter 11

Universal Techmüller space

In this Chapter we study the Kähler geometry of the universal Teichmüller space $\mathcal{T}$, which can be defined as the space of normalized homeomorphisms of S^1, extending to quasiconformal maps of the unit disc Δ. It may be also realized as an open subset in the complex Banach space of holomorphic quadratic differentials in a disc. All classical Teichmüller spaces $T(G)$, where G is a Fuchsian group, are contained in $\mathcal{T}$ as complex Kähler submanifolds. The homogeneous space $\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$, introduced in the previous Chapter 10, may be considered as a "regular" part of $\mathcal{T}$.

11.1 Definition of the universal Techmüller space

Definition 37. A homeomorphism $f : S^1 \to S^1$ is called *quasisymmetric*, if it can be extended to a quasiconformal homeomorphism of the unit disc Δ.

This definition agrees with the definition of a quasisymmetric homeomorphism as an orientation-preserving homeomorphism of S^1, satisfying the Beurling–Ahlfors condition (6.5), given in Sec. 6.1. The equivalence of two definitions is established in the Beurling–Ahlfors theorem in Sec. 6.1.

We denote by $\mathrm{QS}(S^1)$ the set of all orientation-preserving quasisymmetric homeomorphisms of S^1. This is a group with respect to the composition of homeomorphisms.

Any diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ extends to a diffeomorphism of the closed unit disc $\overline{\Delta}$, and so to a quasiconformal homeomorphism $\tilde{f}$ (recall that the Jacobian of a diffeomorphism f is equal to $|f_z|^2 - |f_{\bar{z}}|^2$). Hence, $\mathrm{Diff}_+(S^1) \subset \mathrm{QS}(S^1)$. Since the Möbius group $\mathrm{Möb}(S^1)$ of fractional-linear automorphisms of the disc is contained in $\mathrm{Diff}_+(S^1)$, we obtain the following chain of embeddings

$$\mathrm{Möb}(S^1) \subset \mathrm{Diff}_+(S^1) \subset \mathrm{QS}(S^1) \subset \mathrm{Homeo}(S^1) .$$

Definition 38. The quotient space

$$\mathcal{T} := \mathrm{QS}(S^1)/\mathrm{Möb}(S^1)$$

is called the *universal Teichmüller space*. It can be identified with the space of *normalized* quasisymmetric homeomorphisms of S^1, fixing the points ± 1 and $-i$.

The reasons for choosing the name "universal Teichmüller space" for the introduced object will become clear later.

As we have just pointed out, we have an inclusion

$$\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{M\ddot{o}b}(S^1) \hookrightarrow \mathcal{T} = \mathrm{QS}(S^1)/\mathrm{M\ddot{o}b}(S^1) \ .$$

Using the existence theorem for quasiconformal maps (Theor. 5 from Sec. 6.2), we can describe the universal Teichmüller space in terms of Beltrami differentials. Denote by $B(\Delta)$ the set of Beltrami differentials in the unit disc Δ. It can be identified, as we have pointed out in Sec. 6.1, with the unit ball in the complex Banach space $L^\infty(\Delta)$.

Given a Beltrami differential $\mu \in B(\Delta)$, we can extend it by symmetry (cf. Sec. 6.2) to the Beltrami differential $\hat{\mu}$ on the whole plane. Theor. 5 from Sec. 6.2 implies the existence of a unique normalized quasiconformal homeomorphism w_μ on the extended complex plane $\overline{\mathbb{C}}$ with complex dilatation $\hat{\mu}$. Moreover, this homeomorphism preserves the unit disc Δ, so we can associate with μ the quasisymmetric homeomorphism $w_\mu|_{S^1}$ of the unit circle S^1. Introduce an equivalence relation between Beltrami differentials in Δ: $\mu \sim \nu$ *if and only if*

$$w_\mu = w_\nu \quad \text{on } S^1 \ .$$

Then *the universal Teichmüller space $\mathcal{T}$ will be identified with the quotient of the space $B(\Delta)$ of Beltrami differentials modulo this equivalence relation*:

$$\mathcal{T} = B(\Delta)/\sim \ .$$

Or, to put it in another words, *$\mathcal{T}$ coincides with the space of normalized quasiconformal self-homeomorphisms of the unit disc Δ.*

We can give still another definition of the universal Teichmüller space $\mathcal{T}$, using the extension of a given Beltrami differential μ by zero outside the unit disc Δ (cf. Sec. 6.2). In more detail, we denote by $\check{\mu}$ the Beltrami differential on the complex plane, obtained by the extension of μ by zero outside Δ. Then by Theor. 5 from Sec. 6.2 we obtain a normalized quasiconformal homeomorphism w^μ of the extended complex plane $\overline{\mathbb{C}}$, which is conformal on the exterior Δ_- of the closed unit disc $\overline{\Delta} \subset \overline{\mathbb{C}}$ and fixes the points $\pm 1, -i$. Recall that the image $\Delta^\mu := w^\mu(\Delta)$ of the unit disc Δ under the quasiconformal map w^μ is called the quasidisc. We associate with the Beltrami differential $\mu \in B(\Delta)$ the normalized quasidisc Δ^μ.

Introduce now another equivalence relation between Beltrami differentials in Δ by saying that two Beltrami differentials μ and ν are equivalent, if $w^\mu|_{\Delta_-} = w^\nu|_{\Delta_-}$. We claim that this new equivalence relation between Beltrami differentials coincides with the previous one. More precisely, we have the following

Lemma 4. *Two Beltrami differentials $\mu, \nu \in B(\Delta)$ are equivalent if and only if*

$$w_\mu|_{S^1} = w_\nu|_{S^1} \Longleftrightarrow w^\mu|_{\Delta_-} = w^\nu|_{\Delta_-} \ .$$

The proof of Lemma will be given below. Note that it implies that *the universal Teichmüller space $\mathcal{T}$ can be identified with the space of normalized quasidiscs in $\overline{\mathbb{C}}$.*

This last definition of $\mathcal{T}$ allows us to consider the elements of $\mathcal{T}$ as univalent holomorphic functions in Δ_- (which extend to quasiconformal homeomorphisms of the extended complex plane $\overline{\mathbb{C}}$ and fix the points ± 1 and $-i$). For such functions it is standard to use an alternative normalization by fixing their Laurent decompositions at ∞ in the form

$$f(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots \ .$$

The complex numbers $b_1, b_2, \dots$ play the role of complex coordinates on $\mathcal{T}$. According to the classical *area theorem*, they satisfy the inequality

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1 \ .$$

A relation between two different interpretations of Teichmüller space $\mathcal{T}$, namely, as the space of normalized quasisymmetric homeomorphisms of S^1 and the space of normalized quasidiscs in $\overline{\mathbb{C}}$, can be established in the following way.

If f is a given quasisymmetric homeomorphism of S^1, then it can be extended to a quasiconformal homeomorphism of the unit disc Δ, associated with some Beltrami differential μ. Then the corresponding quasidisc

$$\Delta^\mu = w^\mu(\Delta)$$

will not depend on the choice of the quasiconformal extension of f to Δ.

Conversely, let Δ^μ be the quasidisc, corresponding to a quasiconformal map with the complex dilatation μ. Since both maps $w^\mu : \Delta \to \Delta^\mu$ and $w_\mu : \Delta \to \Delta$ are quasiconformal and have the same Beltrami potential μ in Δ, the map $\rho := w^\mu \circ w_\mu^{-1}$ defines a conformal transform of the unit disc Δ onto the quasidisc Δ^μ. Denote this map by ρ_+, and by $\rho_- : \Delta_- \to \Delta_-^\mu$ a conformal map of Δ_- onto the exterior Δ_-^μ of the closed quasidisc $\overline{\Delta^\mu}$, provided by the Riemann mapping theorem. We associate with the quasidisc Δ^μ the quasisymmetric homeomorphism of S^1, given by the formula

$$f := \rho_+^{-1} \circ \rho_- \,|_{S^1} \ .$$

The constructed correspondences preserve the normalizations and so establish a relation between two different interpretations of the universal Teichmüller space $\mathcal{T}$.

We give now the proof of the Lemma, formulated above.

Proof of Lemma. Suppose first that $w^\mu|_{\Delta_-} = w^\nu|_{\Delta_-}$. Then the maps $w^\mu \circ w_\mu^{-1}$ and $w^\nu \circ w_\nu^{-1}$ are both conformal in Δ_+, which they map onto the same quasidisc. Being normalized, they should agree on S^1. But $w^\mu|_{S^1} = w^\nu|_{S^1}$, so we should also have $w_\mu|_{S^1} = w_\nu|_{S^1}$.

Conversely, suppose that $w_\mu|_{S^1} = w_\nu|_{S^1}$. Consider a map w of the extended complex plane $\overline{\mathbb{C}}$, given by

$$w = \begin{cases} w^\mu \circ (w^\nu)^{-1} & \text{on } w^\nu(\overline{\Delta}_-) \ , \\ [w^\mu \circ (w_\mu)^{-1}] \circ [w_\nu \circ (w^\nu)^{-1}] & \text{on } w^\nu(\Delta_+) \ . \end{cases}$$

It follows from the assumption $w_\mu|_{S^1} = w_\nu|_{S^1}$ that w is a homeomorphism of $\overline{\mathbb{C}}$. Moreover, w is conformal on $w^\nu(\Delta_-)$ by construction and w is conformal on $w^\nu(\Delta_+)$, since both maps $w^\mu \circ (w_\mu)^{-1}$ and $w_\nu \circ (w^\nu)^{-1}$ are conformal there. It follows from the quasiconformal extension property (cf. [49], Lemma I.6.1) that w extends to a conformal map of $\overline{\mathbb{C}}$, i.e. to a fractional-linear automorphism of $\overline{\mathbb{C}}$. Since it is normalized, it should be equal to identity, so $w^\mu|_{\Delta_-} = w^\nu|_{\Delta_-}$. □

The universal Teichmüller space $\mathcal{T}$ can be provided with a natural metric, called the *Teichmüller distance*, which can be defined as follows. Representing the points of $\mathcal{T}$ as normalized quasiconformal self-homeomorphisms of Δ, fixing the points ± 1 and $-i$, we can define the distance between two points $[w_1]$, $[w_2]$ of $\mathcal{T}$ as

$$\tau([w_1],[w_2]) := \frac{1}{2}\inf\{\log K_{w_2\circ w_1^{-1}} : w_1 \in [w_1], w_2 \in [w_2]\} ,$$

where K_w is the maximal dilatation of a quasiconformal map w (cf. Sec. 6.1). This metric converts $\mathcal{T}$ into a complete metric space (cf. [49], Sec. III.3.2). Moreover, it can be shown that $\mathcal{T}$ is contractible (cf. [49], Theor. III.3.2).

11.2 Kähler structure of the universal Techmüller space

We shall study the Kähler geometry of the universal Teichmüller space $\mathcal{T}$, using an embedding of $\mathcal{T}$ into the space of quadratic differentials, proposed by L.Bers. This embedding will allow us to introduce complex coordinates on $\mathcal{T}$. It is convenient to use for its definition the model of $\mathcal{T}$ as the space of normalized quasidiscs $\Delta^\mu = w^\mu(\Delta_+)$ or, which is the same, the space of normalized conformal maps w^μ of Δ_-. By using a suitable Möbius transform, we can substitute here the disc Δ_+ by the upper halfplane H_+ and represent $\mathcal{T}$ as the space of normalized quasidiscs $w^\mu(H_+)$, i.e. the images of the upper halfplane H_+ under quasiconformal homeomorphisms w^μ of the extended complex plane $\overline{\mathbb{C}}$, which are conformal on H_- and fix the points $0, 1, \infty$.

Suppose that $[\mu]$ is an arbitrary point of $\mathcal{T}$, represented by a normalized quasidisc $w^\mu(H_+)$, and define a map

$$\Psi : [\mu] \longmapsto \psi[\mu] := S[w^\mu|_{H_-}] , \tag{11.1}$$

where S denotes the Schwarzian (cf. Sec. 10.2). Due to the invariance of the Schwarzian under the Möbius transformations, the image of this map $\psi[\mu]$ depends only on the class $[\mu]$ of the Beltrami differential μ in $\mathcal{T}$ and is a holomorphic function in H_-. The converse is also true: *if* $\psi[\mu] = \psi[\nu]$, *then* $[\mu] = [\nu]$ *in* $\mathcal{T}$. Indeed, consider the conformal map $h := w^\mu \circ (w^\nu)^{-1}$ from $w^\nu(H_-)$ to $w^\mu(H_-)$. Then, applying the transformation rule (10.8) for the Schwarzian on H_-, we shall have

$$S[w^\mu] = S[h \circ w^\nu] = (S[h] \circ w^\nu)\,(w^\nu)'^2 + S[w^\nu] .$$

Since $S[w^\mu] = S[w^\nu]$ in H_-, it follows that $S[h] = 0$ in H_-. So h is a fractional-linear transformation (cf. Sec. 10.2), which is normalized (i.e. fixes the points $0, 1, \infty$). Hence, h is the identity, which implies that $[\mu] = [\nu]$ in $\mathcal{T}$.

The transformation rule for the Schwarzian (10.8) suggests that the image $\psi[\mu]$ of a Beltrami differential $\mu \in B(H_-)$ is a holomorphic quadratic differential in H_-. So the map (11.1) defines an embedding of the universal Teichmüller space $\mathcal{T}$ into the space of holomorphic quadratic differentials in H_-, called the *Bers embedding*.

We have already considered in Sec. 10.2 a natural hyperbolic norm on the space of quadratic differentials. In the case of H_- it is equal to

$$\|\psi\|_2 := \sup_{z \in H_-} 4y^2|\psi(z)|$$

for a quadratic differential ψ. It follows from Theor. 11 in Sec. 10.2 that

$$\|\psi[\mu]\|_2 \leq 6$$

for any Beltrami differential $\mu \in B(H_-)$. Denote by $B_2(H_-)$ the space of holomorphic quadratic differentials in H_- with a finite norm:

$$B_2(H_-) = \{\text{holomorphic quadratic differentials } \psi \text{ on } H_- : \ \|\psi\|_2 < \infty\} \ .$$

So we have an embedding

$$\Psi : \mathcal{T} \longrightarrow B_2(H_-)$$

of $\mathcal{T}$ into a bounded subset in $B_2(H_-)$. It can be shown that it is a homeomorphism (with respect to the topology on $\mathcal{T}$, determined by the Teichmüller distance) onto the image of Ψ (cf. [49], Theor. III.4.1). The image $\Psi(\mathcal{T})$ is an open subset in $B_2(H_-)$, which contains the ball of radius 1/2 centered at zero (cf. [1]). Moreover, it is known (cf. [20]) that it is a connected contractible set.

Using Bers embedding, we can introduce a complex structure and complex coordinates on the universal Teichmüller space $\mathcal{T}$ by pulling them back from the complex Banach space $B_2(H_-)$. It provides $\mathcal{T}$ with the structure of a complex Banach manifold. Consider now the natural projection of the space of Beltrami differentials to the universal Teichmüller space, defined in the beginning of Sec. 11.1. In our realization of $\mathcal{T}$ this map is given by

$$\Phi : B(H_+) \longrightarrow \mathcal{T} = B(H_+)/\sim \ .$$

It is holomorphic with respect to the introduced complex structure on $\mathcal{T}$ (cf. [56], Ch. 3.4). So the composition map

$$F := \Psi \circ \Phi : B(H_+) \longrightarrow B_2(H_-)$$

is also holomorphic.

We study next the tangent structure of this map, i.e. the differential of F. We describe the tangent bundle $T\mathcal{T}$, using the definition of $\mathcal{T}$ in terms of Beltrami differentials

$$\mathcal{T} = B(H_+)/\sim \ .$$

Due to the homogeneity of $\mathcal{T}$ with respect to the right action of quasisymmetric homeomorphisms, it's sufficient to determine the tangent space $T_0\mathcal{T}$ at the origin, corresponding to the identity homeomorphism, associated with $\mu = 0$.

Let $\mu \in L^\infty(H_+)$ represents an arbitrary tangent vector from $T_0B(H_+)$. Then for the corresponding quasiconformal map $w^{t\mu}$ we'll have an expansion

$$w^{t\mu}(z) = z + tw_1(z) + o(t)$$

for $t \to 0$, where $o(t) := t\delta(z,t)$ and $\delta(z,t) \to 0$ uniformly in z, when z belongs to a compact subset in $\mathbb{C}$. The term

$$w_1(z) \equiv \dot{w}[\mu](z)$$

represents the first variation of the quasiconformal map $w^{t\mu}$ with respect to μ. We substitute $w^{t\mu}$ into the Beltrami equation and differentiate it with respect to t at

$t = 0$. Since $\partial/\partial t$ commutes with $\partial/\partial z$ and $\partial/\partial\bar{z}$ for almost all z, being applied to $w^{t\mu}(z)$ (cf. [2]), we obtain that

$$\frac{\partial}{\partial\bar{z}}\left(\dot{w}[\mu](z)\right) = \mu(z)$$

for almost all z, i.e. $\dot{w}[\mu](z)$ satisfies the $\bar{\partial}$-equation. Hence it can be represented by the Cauchy-Green integral: if μ has a compact support in $\mathbb{C}$ it has the form

$$-\frac{1}{\pi}\int_{\mathbb{C}}\frac{\mu(\zeta)}{\zeta - z}\,d\xi d\eta \quad \text{for } \zeta = \xi + i\eta$$

plus an arbitrary entire function, which in our case can be only a linear function of the form (cf. [1])

$$A + Bz = (z-1)\int_{\mathbb{C}}\frac{\mu(\zeta)}{\zeta}d\xi d\eta - z\int_{\mathbb{C}}\frac{\mu(\zeta)}{\zeta - 1}\,d\xi d\eta\ .$$

Altogether it gives the following formula for $\dot{w}[\mu](z)$

$$w_1(z) = \dot{w}[\mu](z) = -\frac{z(z-1)}{\pi}\int_{\mathbb{C}}\frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-z)}\,d\xi d\eta\ , \qquad (11.2)$$

which holds for <u>all</u> $\mu \in L^\infty(H_+)$ (the restriction on the support of μ being compact is removed by a standard approximation argument, cf. [1]).

We are now able to prove the following

Proposition 20 ([1, 56]). *The differential of the map*

$$F = \Psi\circ\Phi : B(H_+) \longrightarrow B_2(H_-)$$

at zero is given by the formula

$$d_0\left(\Psi\circ\Phi\right)[\mu](z) = -\frac{6}{\pi}\int_{H_+}\frac{\mu(\zeta)}{(\zeta - z)^4}\,d\xi d\eta\ , \quad z\in H_-\ , \qquad (11.3)$$

for $\mu \in B(H_+)$.

Proof. Fix $z_0 \in H_-$. We want to find the derivative of the function

$$\varphi(t,z) := S[w^{t\mu}](z) = F[t\mu](z)$$

at $t = 0$. By denoting $w := w^{t\mu}$, the derivative with respect to t by "dot", and derivative with respect to z by "prime", we get

$$\dot{\varphi} = \left(\frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2\right)^{\cdot} = \frac{(w')^3\dot{w}''' - \dot{w}'(w')^2w''' - 3\dot{w}''(w')^2w'' + 3\dot{w}'w'(w'')^2}{(w')^4}\ .$$

For $t = 0$ we have $w(z) \equiv z$, so $w' \equiv 1$, $w'' = w''' \equiv 0$. Hence, the above formula reduces to

$$\left.\frac{\partial\varphi}{\partial t}\right|_{t=0} = \frac{(w')^3\dot{w}'''}{(w')^4} = \dot{w}'''\ .$$

But the formula (11.2) implies that

$$\dot{w}(z) = -\frac{z(z-1)}{\pi} \int_{H_+} \frac{\mu(\zeta)}{\zeta(\zeta-1)(\zeta-z)} \, d\xi d\eta$$

(note that $\mu \equiv 0$ on H_-). Differentiating this formula three times over z, we obtain the desired formula (11.3). □

In addition to formula (11.3), it may be proved (cf. [56], Theor. 3.4.5) that the operator d_0F is a bounded linear operator and estimate its norm by an absolute constant.

We describe the kernel of the differential d_0F. We note that there is a natural pairing between the space $A_2(H_+)$ of L^1-integrable holomorphic quadratic differentials in H_+ and the space $B(H_+)$ of Beltrami $(-1,1)$-differentials in H_+, denoted by

$$< \mu, \psi > := \int_{H_+} \mu\psi \ . \tag{11.4}$$

In terms of this pairing, the kernel of d_0F can be identified as follows.

Theorem 12 (Teichmüller lemma)**.** *The kernel of d_0F coincides with the subspace*

$$N \equiv A_2(H_+)^\perp = \{\mu \in L^\infty(H_+) : < \mu, \psi > = 0 \textit{ for all } \psi \in A_2(H_+)\} \ .$$

The proof of this Lemma may be found in ([1], Sec.IV(D); [56], Sec.3.7).

It will be useful to summarize the previous results also in the case of the unit disc $\Delta = \Delta_+$. The Bers embedding for this case coincides with the map

$$F : B(\Delta_+) \longrightarrow B_2(\Delta_-) \ ,$$

associating with a Beltrami differential $\mu \in B(\Delta_+)$ in the unit disc Δ_+ the restriction $S[w^\mu]|_{\Delta_-}$ of the Schwarzian $S[w^\mu]$ to the exterior $\Delta_- = \{|z| > 1\} \cup \infty$ of the closed unit disc $\overline{\Delta}_+$ on the Riemann sphere $\overline{\mathbb{C}}$. The image of this map is contained in the space of holomorphic quadratic differentials in Δ_- with a finite norm

$$\|\psi\|_2 := \sup_{z \in \Delta_-} (1 - |z|^2)^2 |\psi(z)| < \infty \ .$$

The formula for the differential d_0F is given by

$$d_0F[\mu](z) = -\frac{6}{\pi} \int_{\Delta_+} \frac{\mu(\zeta)}{(\zeta - z)^4} \, d\xi d\eta \ , \quad z \in \Delta_- \ , \tag{11.5}$$

for $\mu \in L^\infty(\Delta_+)$. The kernel of d_0F is equal to

$$N \equiv A_2(\Delta_+)^\perp = \{\mu \in L^\infty(\Delta_+) : < \mu, \psi > = 0 \text{ for all } \psi \in A_2(\Delta_+)\} \ .$$

This definition is equivalent to the following (cf. [56], Sec. 3.7.2)

$$N = \{\mu \in L^\infty(\Delta) : \int_\Delta \frac{\mu(\zeta)}{(\zeta - z)^4} \, d\xi d\eta = 0 \text{ for all } z \in \Delta_-\} \ .$$

The formulas (11.3),(11.5) suggest how a Kähler metric on $\mathcal{T}$ can be defined. Namely, we employ the *Ahlfors map* (cf. [3]): $L^\infty(\Delta) \longrightarrow B_2(\Delta)$, given by

$$L^\infty(\Delta) \ni \mu \longmapsto \varphi[\mu](z) = \int_\Delta \frac{\overline{\mu(\zeta)}}{(1-z\bar{\zeta})^4}\, d\xi d\eta \ .$$

It associates with any $\mu \in L^\infty(\Delta)$ a holomorphic quadratic differential $\varphi[\mu]$ with a finite norm $\|\varphi\|_2 = \sup_{z\in\Delta}(1-|z|^2)^2|\varphi(z)| < \infty$. The kernel of this map coincides with $N = A_2(\Delta_+)^\perp$. Now we can define formally a Hermitian metric on $\mathcal{T}$ by setting for two tangent vectors μ, ν in $T_0\mathcal{T} = L^\infty(\Delta)/N$:

$$(\mu,\nu) := <\mu, \varphi[\nu]> = \int_\Delta \int_\Delta \frac{\mu(z)\overline{\nu(\zeta)}}{(1-z\bar{\zeta})^4}\, d\xi d\eta\, dxdy \ . \tag{11.6}$$

However, this metric is only densely defined. More precisely (cf. [59]), for a general $\mu \in L^\infty(\Delta)$ its image $\varphi[\mu]$ in $B_2(\Delta)$ may be not integrable, i.e. it does not belong, in general, to $A_2(\Delta)$, in which case the integral in (11.6) will diverge. In fact, the formula (11.6) is correctly defined, if the tangent vectors μ, ν in $T_0\mathcal{T}$ are sufficiently smooth. To formulate this smoothness condition more precisely, we realize $\mathcal{T}$ as the space of normalized quasisymmetric homeomorphisms of S^1. Then a tangent vector $\mu \in L^\infty(\Delta) = T_0B(\Delta)$ will correspond under the differential $d_0\Phi$ to the vector field $v = v(\theta)\partial/\partial\theta$ on S^1 of the form

$$v(\theta)\frac{\partial}{\partial\theta} = \dot{w}[\mu](z)\frac{\partial}{\partial z} \ , \quad z = e^{i\theta} \ ,$$

where $\dot{w}[\mu]$ is the derivative with respect to t of the one-parameter flow $w_{t\mu}$ of quasisymmetric homeomorphisms:

$$w_{t\mu}(z) = z + t\dot{w}[\mu](z) + o(t) \quad \text{for } t \to 0 \ .$$

Then it may be proved (cf. [59]) that the integral in (11.6) converges, if the tangent vectors μ, ν in $T_0\mathcal{T}$ correspond to $C^{3/2+\epsilon}$-smooth vector fields on S^1. Whenever the metric (11.6) is well-defined, it determines a Kähler metric, in particular, it defines a Kähler metric on the "regular" part of $\mathcal{T}$.

11.3 Teichmüller spaces $T(G)$ and $\mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$

The universal Teichmüller space $\mathcal{T}$ contains, as its complex submanifolds, all classical Teichmüller spaces $T(G)$, where G is a *Fuchsian group* (cf. [49, 56]). In particular, it is true for all Teichmüller spaces of compact Riemann surfaces. This property of $\mathcal{T}$ motivates the use of the term "universal" in the name of $\mathcal{T}$.

With an arbitrary Fuchsian group G we associate the Riemann surface $X := \Delta/G$, uniformized by the unit disc Δ. By definition, $T(G)$ consists of quasisymmetric homeomorphisms $f \in \mathrm{QS}(S^1)$, which are *G-invariant* in the following sense:

$$f \circ g \circ f^{-1} \text{ belongs to } \mathrm{Möb}(S^1) \text{ for all } g \in G \ ,$$

modulo fractional-linear automorphisms of the disc Δ. If we denote by $\mathrm{QS}(S^1)^G$ the subset of G-invariant quasisymmetric homeomorphisms in $\mathrm{QS}(S^1)$, then

$$T(G) = \mathrm{QS}(S^1)^G/\mathrm{Möb}(S^1) \ .$$

The universal Teichmüller space $\mathcal{T}$ itself corresponds to the Fuchsian group $G = \{1\}$.

The various interpretations of the universal Teichmüller space $\mathcal{T}$, given in Sec. 11.1, are compatible with the notion of G-invariance. In particular, the Teichmüller spaces $T(G)$ admit a description in terms of G-invariant Beltrami differentials. More precisely, denote by $B(\Delta)^G$ the subspace of $B(\Delta)$, consisting of Beltrami differentials μ, satisfying the relation

$$\mu(gz)\frac{\overline{g'(z)}}{g'(z)} = \mu(z) \quad \text{almost everywhere on } \Delta \text{ for all } g \in G \ .$$

Then we'll have, as in Sec. 11.1:

$$T(G) = B(\Delta)^G / \sim \ ,$$

where $\mu \sim \nu$ iff $w_\mu = w_\nu$ on S^1 or, equivalently, $w^\mu|_{\Delta_-} = w^\nu|_{\Delta_-}$.

We can associate with a G-invariant Beltrami differential μ a Fuchsian group G_μ, conjugate to G:

$$G_\mu := w_\mu G w_\mu^{-1} \ ,$$

where w_μ is the quasiconformal homeomorphism of $\overline{\mathbb{C}}$, leaving $\Delta_\pm$ invariant (cf. Sec. 11.1).

We have a natural quasiconformal map of the Riemann surface $X := \Delta/G$ onto another Riemann surface $X_\mu := \Delta/G_\mu$. This map is a homeomorphism which is biholomorphic precisely, when $\mu \in$ Möb(S^1). Hence, one can say that the space $T(G)$ parametrizes, with the help of the map $\mu \mapsto G_\mu$, different complex structures on the Riemann surface $X := \Delta/G$, which can be obtained from the original one by quasiconformal deformations.

On the other hand, we can associate with a G-invariant Beltrami differential $\mu \in B(\Delta)^G$ another conjugated group

$$G^\mu := w^\mu G (w^\mu)^{-1} \ ,$$

operating properly discontinuously on the quasidisc $\Delta^\mu := w^\mu(\Delta)$. Here, w^μ is the quasiconformal homeomorphism of $\overline{\mathbb{C}}$, which is conformal on Δ_- (cf. Sec. 11.1). The group G^μ is a Kleinian group, called otherwise a *quasi-Fuchsian group* (cf. [49, 56]). The Riemann surface X_μ is biholomorphic to Δ^μ/G^μ (cf. [56], Theor. 1.3.5). We note also that the Riemann surface Δ_-^μ/G^μ is biholomorphic to the Riemann surface Δ_-/G, due to the conformality of w^μ on Δ_-.

The definition and main properties of the Bers embedding, given in Sec. 11.2, extend to the Teichmüller spaces $T(G)$. For the case of the unit disc $\Delta \equiv \Delta_+$ the Bers embedding is the map

$$F : B(\Delta_+)^G \longrightarrow B_2(\Delta_-)^G \ ,$$

associating with a Beltrami differential $\mu \in B(\Delta_+)^G$ the quadratic differential $S[w^\mu|_{\Delta_-}]$ on Δ_-. The image of this map is contained in the space $B_2(\Delta_-)^G$ of G-invariant holomorphic quadratic differentials in Δ_- with a finite norm

$$\|\psi\|_2 := \sup_{z\in\Delta_-} (1 - |z|^2)^2 |\psi(z)| < \infty \ .$$

The formula for the differential d_0F has the form

$$d_0F[\mu](z) = -\frac{6}{\pi}\int_{\Delta_+}\frac{\mu(\zeta)}{(\zeta-z)^4}\,d\xi d\eta\ , \quad z\in\Delta_-\ ,$$

for $\mu\in L^\infty(\Delta_+)^G$. The kernel of d_0F is given by

$$N^G \equiv (A_2(\Delta_+)^G)^\perp = \{\mu\in L^\infty(\Delta_+)^G : <\mu,\psi>=0 \text{ for all } \psi\in A_2(\Delta_+)\}\ .$$

This definition is equivalent to

$$N^G = \{\mu\in L^\infty(\Delta)^G : \int_\Delta \frac{\mu(\zeta)}{(\zeta-z)^4}\,d\xi d\eta = 0 \text{ for all } z\in\Delta_-\}\ .$$

So the tangent space of $T(G)$ at the origin coincides with the space $L^\infty(\Delta)^G/N^G$.

As in Sec. 11.2, there is the Ahlfors map $L^\infty(\Delta)^G/N^G \longrightarrow B_2(\Delta)^G$, given by

$$L^\infty(\Delta)^G \ni \mu \longmapsto \varphi[\mu](z) = \int_\Delta \frac{\overline{\mu(\zeta)}}{(1-z\bar\zeta)^4}\,d\xi d\eta\ .$$

Using this map, we can define the *Weil–Petersson metric* on $T(G)$, as in Sec. 11.2, by setting for two tangent vectors μ,ν in $T_0T(G) = L^\infty(\Delta)^G/N^G$:

$$g_G(\mu,\nu) := \int_{\Delta/G}\int_\Delta \frac{\mu(z)\overline{\nu(\zeta)}}{(1-z\bar\zeta)^4}\,d\xi d\eta\,dxdy\ . \tag{11.7}$$

As was pointed out in Sec. 11.2, the image $\varphi[\mu]\in B_2(\Delta)^G$ of the Ahlfors map for a general Fuchsian group G may not belong to the space $A_2(\Delta)^G$ of integrable holomorphic quadratic differentials, so the formula (11.7) for the metric $g_G(\mu,\nu)$ is ill-defined for general Fuchsian groups. But in the case of finite-dimensional Teichmüller spaces $T(G)$ this difficulty does not show up, since in this situation $B_2(\Delta)^G = A_2(\Delta)^G$ (cf. [56]), and the introduced metric coincides with the standard Weil–Petersson metric on the finite-dimensional Teichmüller spaces $T(G)$. Moreover, S.Nag has proved (cf. [59]) that the metric $g_G(\mu,\nu)$ on $T(G)$ can be obtained from the metric (μ,ν) on $\mathcal{T}$ by a certain reduction procedure. This procedure involves a regularization of the integral

$$(\mu,\nu) = \int_\Delta\int_\Delta \frac{\mu(z)\overline{\nu(\zeta)}}{(1-z\bar\zeta)^4}\,d\xi d\eta\,dxdy = \int_\Delta \mu\cdot\varphi[\nu]\ . \tag{11.8}$$

To define the regularization, we rewrite the integral (11.8) in the form

$$(\mu,\nu) = \lim_{r\to 1-0} g_r(\mu,\nu)$$

where

$$g_r(\mu,\nu) = \int_{\Delta_r} \mu\cdot\varphi[\nu]\ , \tag{11.9}$$

and $\Delta_r := \{z\in\Delta : |z|<r\}$, $0<r<1$.

In the case when μ, ν are G-invariant, i.e. belong to $L^\infty(\Delta)^G/N^G$, the integral (11.8) coincides with

$$n \int_{\Delta/G} \mu \cdot \varphi[\nu] = n g_G(\mu, \nu) \ ,$$

where n is the number of copies of the fundamental domain Δ/G, contained in Δ. Hence, this integral must diverge, if the group G has infinitely many elements. The integral (11.9) by the same argument is proportional to $n_r g_G(\mu, \nu)$, where n_r is the number of copies of the fundamental domain Δ/G, contained in Δ_r. It follows that the integral (11.9) may be regularized by dividing it by a quantity, proportional to n_r. More precisely, the following assertion is true .

Proposition 21 ([59]). *For any finite-dimensional Teichmüller space $T(G)$ its Weil–Petersson metric $g_G(\mu, \nu)$ may be computed by the formula*

$$\frac{g_G(\mu, \nu)}{g_G(\mu_0, \mu_0)} = \lim_{r \to 1-0} \frac{g_r(\mu, \nu)}{g_r(\mu_0, \mu_0)} \ ,$$

where $\mu, \nu \in L^\infty(\Delta)^G$, and $\mu_0 \in L^\infty(\Delta)^G/N^G$ is an arbitrary nonzero tangent vector from $T_0 T(G)$.

As we have remarked at the beginning of Sec. 11.1, the universal Teichmüller space $\mathcal{T}$ contains the homogeneous space $\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$ as its "regular" part:

$$\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1) \hookrightarrow \mathcal{T} = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1) \ .$$

In Sec. 10.3 we have defined the structure of a Kähler–Frechet manifold on $\mathcal{S}$. We recall the definition of the Kähler metric g on this space in terms of Fourier decompositions. For given tangent vectors $u, v \in T_o\mathcal{S}$ with Fourier decompositions

$$u = \sum_{n \neq -1,0,1} u_n e_n \quad \text{and} \quad v = \sum_{n \neq -1,0,1} v_n e_n \ ,$$

the value of g on these vectors is equal to

$$g(u,v) = 2\,\mathrm{Re}\left(\sum_{n=2}^{\infty} u_n \bar{v}_n \, (n^3 - n) \right) \ . \tag{11.10}$$

As we have noted before, the series on the right hand side is absolutely converging, if the vector fields u, v are of the class $C^{3/2+\epsilon}$ on S^1.

It was pointed out in [59] that the Kähler metric g on $\mathcal{S}$ coincides (up to a constant factor) with the Weil–Petersson metric (11.6) on $\mathcal{S}$, induced by the embedding $\mathcal{S} \hookrightarrow \mathcal{T}$. (Note that the metric (11.6) on the smooth part $\mathcal{S}$ of $\mathcal{T}$ is correctly defined, as we have remarked in Sec. 11.2.) Using the interpretation of tangent vectors from $T_0\mathcal{T}$, given at the end of Sec. 11.2, we can express the equality of these metrics on $\mathcal{S}$ as follows. Given two tangent vectors $u, v \in T_0\mathcal{S}$, written in the form $u = \dot{w}[\mu]\partial/\partial z$, $v = \dot{w}[\nu]\partial/\partial z$, we have

$$g(\mu, \nu) = \lambda \int_\Delta \int_\Delta \frac{\mu(z)\overline{\nu(\zeta)}}{(1 - z\bar{\zeta})^4} \, d\xi d\eta \, dx dy$$

for a suitable choice of the constant λ. By introducing this constant into the definition of the Kähler metric on $\mathcal{S}$, we can make the embedding $\mathcal{S} \hookrightarrow \mathcal{T}$ an isometry.

It is an interesting question, how the smooth part $\mathcal{S}$ is placed inside the universal Teichmüller space $\mathcal{T}$ with respect to the classical Teichmüller spaces $T(G)$. It can be shown (cf. [12]) that the quasidiscs, corresponding to all points of $T(G)$, except the origin, have fractal boundaries (i.e. boundaries of Hausdorff dimension> 1) in contrast with the qiasidiscs, corresponding to points of $\mathcal{S}$, which have C^∞-smooth boundaries.

11.4 Grassmann realization of the universal Teichmüller space

The Grassmann realization of the universal Teichmüller space $\mathcal{T}$ is based on the fact that the group $\mathrm{QS}(S^1)$ of quasisymmetric homeomorphisms of the circle acts on the Sobolev space V of half-differentiable functions on S^1 (cf. Sec. 9.2).

Suppose that $f : S^1 \to S^1$ is a homeomorphism of S^1, preserving its orientation. We define an operator T_f by the formula

$$T_f(\xi) := \xi \circ f - \frac{1}{2\pi}\int_0^{2\pi} \xi\left(f(\theta)\right) d\theta$$

for $\xi \in V$. This operator has the following remarkable property.

Proposition 22 ([58]). *The operator T_f acts on V (i.e. $T_f(\xi)$ belongs to V for any $\xi \in V$) if and only if $f \in QS(S^1)$. Moreover, if f extends to a K-quasiconformal homeomorphism of the disc Δ, then the operator norm of T_f does not exceed $\sqrt{K + K^{-1}}$.*

The proof of this assertion, given in [58], uses the interpretation of the space V in terms of harmonic functions in the disc, given at the end of Sec. 9.1.

Transformations of the form T_f with $f \in \mathrm{QS}(S^1)$ preserve the symplectic form ω, i.e. they are symplectic transformations of V.

Proposition 23 ([58]). *If $f \in QS(S^1)$, then*

$$\omega(f^*(\xi), f^*(\eta)) = \omega(\xi, \eta)$$

for any $\xi, \eta \in V$. Moreover, the complex-linear extension of the $QS(S^1)$-action on V to the complexification $V^{\mathbb{C}}$ preserves the subspace W_+ (cf. Sec. 9.1) if and only if f is a Möbius transformation, i.e. $f \in Möb(S^1)$. In the latter case, T_f acts as a unitary operator on $W_\pm$.

Proof. For homeomorphisms f of the class C^1 the first assertion is a corollary of the change of variables formula. For a general quasisymmetric homeomorphism $f \in \mathrm{QS}(S^1)$ the assertion follows from the fact (cf. [49]) that f may be uniformly approximated by real analytic quasisymmetric homeomorphisms of S^1, having the same quasiconformal constant K as f.

If the action of f on $V^{\mathbb{C}}$ preserves W_+, then it should extend to a map $\Delta \to \Delta$. This map must be a biholomorphism, since f is a homeomorphism, hence, it is a Möbius transformation. It is clear from the definition of the inner product on $V^{\mathbb{C}}$ (cf. Sec. 9.1) that such a transformation acts unitarily on $W_\pm$. □

The symplectic form ω on V is uniquely determined by the invariance property, stated in the above Proposition. In fact, a much stronger assertion is true.

Proposition 24 ([58]). *Suppose that ω_1 is a real-valued continuous bilinear skew-symmetric form on V such that*

$$\omega_1(f^*(\xi), f^*(\eta)) = \omega_1(\xi, \eta)$$

for any $f \in Möb(S^1)$ and arbitrary $\xi, \eta \in V$. Then ω_1 is a real multiple of ω, in particular, any form ω_1, satisfying the hypothesis of the Proposition, coincides necessarily with a symplectic form, invariant under quasisymmetric homeomorphisms of S^1.

Proof. Note that both forms ω and ω_1 define the duality maps

$$\Sigma : V \longrightarrow V^* \quad \text{and} \quad \Sigma_1 : V \longrightarrow V^* \ ,$$

given by

$$\Sigma(\xi) := \omega(\cdot, \xi) \ , \quad \Sigma_1(\xi) := \omega_1(\cdot, \xi)$$

for $\xi \in V$. In the case of ω the duality operator Σ coincides, in fact, with the (minus of) J^0. In particular, Σ is a bounded invertible operator, defining an isomorphism between V and its dual.

We consider an intertwining operator

$$M := \Sigma^{-1} \circ \Sigma_1 : V \longrightarrow V \ .$$

It is a bounded linear operator on V, defined by the equality

$$\omega(\xi, M\eta) = \omega_1(\xi, \eta) \ .$$

Note that M commutes with any invertible bounded linear operator on V, preserving the forms ω and ω_1. Indeed, if T is such an operator, then

$$\omega(T\xi, TM\eta) = \omega(\xi, M\eta) = \omega_1(\xi, \eta) = \omega_1(T\xi, T\eta) = \omega(T\xi, MT\eta) \ .$$

Since T is invertible, it implies that

$$\omega(\xi, TM\eta) = \omega(\xi, MT\eta)$$

for any $\xi, \eta \in V$. Since the duality operator Σ, determined by ω, is an isomorphism, the last equality implies that $TM = MT$, as asserted.

We have to show that the intertwining operator M coincides with the scalar operator $\text{const} \cdot I$. We prove it by considering the complex-linear extension of M to the complexification $V^{\mathbb{C}}$.

Consider the complexified action $f \mapsto T_f$ of the Möbius group $\text{Möb}(S^1)$ on $V^{\mathbb{C}}$. Then its restriction to W_+ can be identified with the standard unitary representation of the group $\text{SL}(2, \mathbb{R})$ on the space of L^2-holomorphic functions in the disc Δ (cf. [58], lemma 4.6), hence, it is irreducible. The same is true for the restriction of $f \mapsto T_f$ to W_-. Moreover, $W_\pm$ are the only irreducible invariant subspaces of the representation $f \mapsto T_f$ of $\text{Möb}(S^1)$ on $V^{\mathbb{C}}$.

As we have just proved, the intertwining operator M commutes with all operators $T_f : V^{\mathbb{C}} \to V^{\mathbb{C}}$ with $f \in \text{Möb}(S^1)$. Since $W_\pm$ are the only invariant subspaces for all such T_f, the operator M should map W_+ either to W_+ or W_-. If M maps W_+ into W_+, then by Schur's lemma it should be a scalar, which is real, since the operator M was real.

If the other possibility (when M maps W_+ into W_-) would realize, we would substitute M by the operator $\tilde{M}$, given by the composition of M with the complex conjugation. The operator $\tilde{M}$ would map W_+ into W_+ and commute with all operators T_f. As we have just proved, such an operator $\tilde{M}$ should be a real scalar and so coincide with M. But in this case M cannot map W_+ into W_-, so the second possibility is not realized. □

The Propositions 22 and 23 imply that the quasisymmetric homeomorphisms from $\text{QS}(S^1)$ act on the Hilbert space V by bounded symplectic operators. Hence, we have a map

$$\mathcal{T} = \text{QS}(S^1)/\text{Möb}(S^1) \longrightarrow \text{Sp}(V)/\text{U}(W_+) \ . \tag{11.11}$$

Here, by $\text{Sp}(V)$ we denote the symplectic group of V, consisting of linear bounded symplectic operators on V, and by $\text{U}(W_+)$ its subgroup, consisting of unitary operators, i.e. operators, whose complex-linear extensions to $V^{\mathbb{C}}$ preserve the subspace W_+. We describe these groups in more detail.

Recall that the complexified Hilbert space $V^{\mathbb{C}}$ is decomposed into the direct sum

$$V^{\mathbb{C}} = W_+ \oplus W_-$$

of subspaces

$$W_+ = \{f \in V^{\mathbb{C}} : \ f(z) = \sum_{k>0} x_k z^k\} \ , \quad W_- = \overline{W}_+ = \{f \in V^{\mathbb{C}} : \ f(z) = \sum_{k<0} x_k z^k\} \ .$$

In terms of this decomposition any linear operator $A : V^{\mathbb{C}} \to V^{\mathbb{C}}$ can be written in the block form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ ,$$

where

$$a : W_+ \to W_+ \ , \ b : W_- \to W_+ \ , \ c : W_- \to W_- \ , \ d : W_+ \to W_- \ .$$

In particular, the linear operators on $V^{\mathbb{C}}$, obtained by the complex-linear extensions of operators $A : V \to V$, have the block form

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \ ,$$

where we identify W_- with the complex conjugate $\overline{W}_+$.

An operator $A : V \to V$ belongs to the symplectic group $\text{Sp}(V)$, if it preserves the symplectic form ω. This condition is equivalent to the following relation:

$$A^t J^0 A = J^0 \ ,$$

where

$$J^0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \ .$$

In other words, the condition $A \in \mathrm{Sp}(V)$ can be written in the form:

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathrm{Sp}(V) \Longleftrightarrow \bar{a}^t a - b^t \bar{b} = 1 \ , \ \bar{a}^t b = b^t \bar{a} \ . \qquad (11.12)$$

Here a^t, b^t denote the transposed operators

$$a^t : W'_+ \to W'_+ \Longleftrightarrow a^t : W_- \to W_- \ , \quad b^t : W'_+ \to W'_- \Longleftrightarrow b^t : W_- \to W_+ \ ,$$

where the space W'_+, dual to W_+, is identified with W_- with the help of the inner product $< \cdot , \cdot >$ (cf. Sec. 9.1).

The unitary group $\mathrm{U}(W_+)$ is embedded into $\mathrm{Sp}(V)$ as a subgroup, consisting of block matrices

$$A = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \ .$$

We return to the map (11.11). The space

$$\mathrm{Sp}(V)/\mathrm{U}(W_+) \ ,$$

standing on the right hand side of the formula (11.11), can be considered as an infinite-dimensional Siegel disc. To justify this assertion, we should study the action of $\mathrm{QS}(S^1)$ on compatible complex structures on the space V.

As we have proved above, Möbius transformations $f \in \mathrm{Möb}(S^1)$ define, via the representation $f \mapsto T_f$, unitary operators in $\mathrm{U}(W_+)$, in particular such transformations preserve the complex structure J_0 on V. If a quasisymmetric homeomorphism f does not belong to $\mathrm{Möb}(S^1)$, it does not preserve the original complex structure J^0, transforming it into another complex structure J_f, which is also compatible with the symplectic form ω. We explain this assertion in more detail.

Any complex structure J on V, compatible with ω, determines a decomposition

$$V^{\mathbb{C}} = W \oplus \overline{W} \qquad (11.13)$$

into the direct sum of subspaces, isotropic with respect to ω. This decomposition is orthogonal with respect to the Kähler metric g_J on $V^{\mathbb{C}}$, determined by J and ω. The subspaces W and $\overline{W}$ are identified with, respectively, the $(-i)$- and $(+i)$-eigenspaces of the operator J on $V^{\mathbb{C}}$. Conversely, any decomposition (11.13) of the space $V^{\mathbb{C}}$ into the direct sum of isotropic subspaces determines a complex structure J on $V^{\mathbb{C}}$, which is equal to $-i \cdot I$ on W and $+i \cdot I$ on $\overline{W}$ and is compatible with ω.

This argument shows that the symplectic group $\mathrm{Sp}(V)$ acts transitively on the space $\mathcal{J}(V)$ of complex structures J, compatible with ω. It follows that the space $\mathrm{Sp}(V)/\mathrm{U}(W_+)$ can be identified with the space $\mathcal{J}(V)$. Otherwise, it may be considered as the space of the so called *positive polarizations* of V, i.e. decompositions (11.13) of $V^{\mathbb{C}}$ into the direct sum $V^{\mathbb{C}} = W \oplus \overline{W}$ of isotropic subspaces of $V^{\mathbb{C}}$, orthogonal with respect to the Kähler metric g_J on $V^{\mathbb{C}}$.

We are ready to give a Siegel disc interpretation of the space $\mathrm{Sp}(V)/\mathrm{U}(W_+)$. By definition, the *Siegel disc* is the set of bounded linear operators Z of the form

$$\mathcal{D} = \{Z : W_+ \to W_- \text{ is a symmetric bounded linear operator with } \bar{Z}Z < I\} \ .$$

The symmetricity of Z means, as above, that $Z^t = Z$ and the condition $\bar{Z}Z < I$ means that the symmetric operator $I - \bar{Z}Z$ is positive definite. In order to identify $\mathcal{J}(V)$ with $\mathcal{D}$, consider the action of the group $\mathrm{Sp}(V)$ on $\mathcal{D}$, given by fractional-linear transformations $A : \mathcal{D} \to \mathcal{D}$ of the form

$$Z \longmapsto (\bar{a}Z + \bar{b})(bZ + a)^{-1} ,$$

where $A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathrm{Sp}(V)$. The invertibility of the operator $bZ + a$ follows from the invertibility of the operator a (cf. (11.12)) and the inequality (cf. (11.12))

$$bZ\bar{Z}\bar{b}^t < b\bar{b}^t < a\bar{a}^t .$$

It's evident that $A : \mathbb{D} \to \mathbb{D}$. The isotropy subgroup of the point $Z = 0$ consists of the operators $A \in \mathrm{Sp}(V)$, for which $\bar{b}a^{-1} = 0$, i.e. $b = 0$. This subgroup coincides with $\mathrm{U}(W_+)$. It remains to check that the action of $\mathrm{Sp}(V)$ on $\mathbb{D}$ is transitive, i.e. to construct for a given $Z \in \mathbb{D}$ an operator A, sending $Z = 0$ to this Z. Such an operator may be given by

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \tag{11.14}$$

with $b = \bar{a}\bar{Z}$ and

$$\bar{a}^t(1 - \bar{Z}Z)a = 1 \Rightarrow (\bar{a}^t)^{-1}a^{-1} = 1 - \bar{Z}Z \Rightarrow a = (1 - \bar{Z}Z)^{-1/2} .$$

This proves that the space

$$\mathcal{J}(V) = \mathrm{Sp}(V)/\mathrm{U}(W_+)$$

may be identified with the Siegel disc $\mathcal{D}$.

In Sec. 5.1 we have introduced the Grassmanian $\mathrm{Gr}_b(V^{\mathbb{C}})$, consisting of the images of bounded linear operators $W_+ \to W$. It is clear from the given description of $\mathcal{D}$ that it is embedded in $\mathrm{Gr}_b(V^{\mathbb{C}})$ as a complex submanifold.

Summarizing the argument above, we have the following

Proposition 25 ([58]). *The map*

$$\mathcal{T} = QS(S^1)/M\ddot{o}b(S^1) \hookrightarrow Sp(V)/U(W_+) = \mathcal{D} \hookrightarrow Gr_b(V^{\mathbb{C}})$$

is an equivariant holomorphic embedding of Banach manifolds.

11.5 Grassmann realization of Diff$_+(S^1)$/Möb(S^1) and Diff$_+(S^1)/(S^1)$

We have constructed in the previous Sec. 11.4 the natural embedding

$$\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{M\ddot{o}b}(S^1) \hookrightarrow \mathrm{Sp}(V)/\mathrm{U}(W_+) = \mathcal{D} \hookrightarrow \mathrm{Gr}_b(V^{\mathbb{C}}) .$$

Recall now that in Sec. 10.3 we have identified the space $\mathcal{S}$ with the "regular" part of the universal Teichmüller space $\mathcal{T}$. Combining the above embedding

$$\mathcal{T} \hookrightarrow \mathrm{Sp}(V)/\mathrm{U}(W_+)$$

with the embedding $\mathcal{S} \hookrightarrow \mathcal{T}$, we obtain a map

$$\mathcal{S} \hookrightarrow \mathrm{Sp}(V)/\mathrm{U}(W_+) \ ,$$

yielding an embedding of $\mathcal{S}$ in the Grassmann manifold $\mathrm{Gr}_b(V^{\mathbb{C}})$.

However, this result may be significantly strengthened by replacing the Grassmann manifold $\mathrm{Gr}_b(V^{\mathbb{C}})$ with its "regular" part, namely, the Hilbert–Schmidt Grassmanian $\mathrm{Gr}_{\mathrm{HS}}(V)$, introduced in Sec. 5.2.

We recall that this Grassmanian $\mathrm{Gr}_{\mathrm{HS}}(V)$ consists of closed subspaces $W \subset V$ such that the orthogonal projection $\mathrm{pr}_+ : W \to W_+$ is a Fredholm operator, while the orthogonal projection $\mathrm{pr}_- : W \to W_-$ is a Hilbert–Schmidt operator. It was shown in Sec. 5.2 that $\mathrm{Gr}_{\mathrm{HS}}(V)$ is a Kähler Hilbert manifold, having as its local model the Hilbert space $\mathrm{HS}(W_+, W_-)$ of Hilbert–Schmidt operators. Recall (cf. Sec. 5.2) that $\mathrm{Gr}_{\mathrm{HS}}(V)$ is a homogeneous space of the Hilbert–Schmidt unitary group $\mathrm{U}_{\mathrm{HS}}(V)$, more precisely

$$\mathrm{Gr}_{\mathrm{HS}}(V) = \mathrm{U}_{\mathrm{HS}}(V)/\,\mathrm{U}(W_+) \times \mathrm{U}(W_-) \ .$$

We introduce now, by analogy with the group $\mathrm{U}_{\mathrm{HS}}(V)$, the *Hilbert–Schmidt symplectic group* $\mathrm{Sp}_{\mathrm{HS}}(V)$. Recall that the symplectic group $\mathrm{Sp}(V)$ consists of bounded linear operators $A : V^{\mathbb{C}} \to V^{\mathbb{C}}$, having the block representations of the form

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \ ,$$

where

$$\bar{a}^t a - b^t \bar{b} = 1 \ , \ \bar{a}^t b = b^t \bar{a} \ .$$

By definition, the group $\mathrm{Sp}_{\mathrm{HS}}(V) \subset \mathrm{Sp}(V)$ consists of transformations $A \in \mathrm{Sp}(V)$, for which the operator b is Hilbert–Schmidt. The unitary group $\mathrm{U}(W_+)$ is contained in $\mathrm{Sp}_{\mathrm{HS}}(V)$ as a subgroup

$$\mathrm{U}(W_+) \ni a \longmapsto A = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \ .$$

The diffeomorphism group $\mathrm{Diff}_+(S^1)$ acts on the space V by symplectic transformations, given by the same formula, as in Sec. 11.4:

$$T_f(\xi) := \xi \circ f - \frac{1}{2\pi} \int_0^{2\pi} \xi\left(f(\theta)\right) d\theta \ .$$

As before, the transformation T_f preserves the subspace $W_+ \subset V$ if and only if $f \in \mathrm{M\ddot{o}b}(S^1)$, and in this case $T_f \in \mathrm{U}(W_+)$. The correspondence $f \mapsto T_f$ defines an embedding

$$\mathcal{S} \hookrightarrow \mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_+) \ .$$

Moreover, the following result is true.

Proposition 26 ([57]). *The map*

$$\mathcal{S} \hookrightarrow Sp_{HS}(V)/\,U(W_+) = Gr_{HS}(V)$$

is an equivariant holomorphic embedding.

By analogy with Sec. 11.4, we identify the space $\mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_+)$ with the space $\mathcal{J}_{\mathrm{HS}}(V)$ of admissible complex structures on V, compatible with the symplectic form ω. As in the previous Section, it has a natural realization as a *Hilbert–Schmidt Siegel disc*

$$\mathcal{D}_{\mathrm{HS}} = \{Z : W_+ \to W_- \text{ is a symmetric Hilbert–Schmidt operator with } \bar{Z}Z < I\} \ .$$

So, the above Proposition yields a holomorphic embedding

$$\mathcal{S} \hookrightarrow \mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_+) = \mathcal{D}_{\mathrm{HS}} \ .$$

There is another interpretation of the space $\mathcal{S}$ as the space of complex structures, namely, as the space of admissible complex structures on the loop space ΩG.

There is a natural action of the diffeomorphism group of the circle $\mathrm{Diff}_+(S^1)$ on the loop group LG by the reparametrization of loops. It is given by the formula

$$f_*\gamma(\theta) := \gamma\left(f(\theta)\right) - \frac{1}{2\pi}\int_0^{2\pi} \gamma\left(f(\theta)\right) d\theta$$

for $\gamma \in LG$, $f \in \mathrm{Diff}_+(S^1)$. By identifying ΩG with the subgroup $L_1(G)$, it's evident that this action can be pushed down to the action of $\mathrm{Diff}_+(S^1)$ on the loop space ΩG.

From the definition of the symplectic structure ω on ΩG, generated by the form

$$\omega_0(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} < \xi(e^{i\theta}), \eta'(e^{i\theta}) > d\theta \ ,$$

on $L\mathfrak{g}$, it's clear (by the change of variables in the integral) that diffeomorphisms from $\mathrm{Diff}_+(S^1)$ preserve ω, i.e. generate symplectomorphisms of the manifold ΩG.

The complex structure J^0 on ΩG is given at the origin $o \in \Omega G$ by the formula

$$\xi = \sum_{k\neq 0} \xi_k z^k \in \Omega\mathfrak{g}^{\mathbb{C}} \Longrightarrow J^0_o\xi = -i\sum_{k>0}\xi_k z^k + i\sum_{k<0}\xi_k z^k \ ,$$

so the tangent subspaces, consisting of vectors of the type $(1,0)$ and $(0,1)$, have the form

$$T^{1,0}_o(\Omega G) = \{\xi = \sum_{k<0}\xi_k z^k \in \Omega\mathfrak{g}^{\mathbb{C}}\}, \quad T^{0,1}_o(\Omega G) = \{\xi = \sum_{k>0}\xi_k z^k \in \Omega\mathfrak{g}^{\mathbb{C}}\} \ .$$

A diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ transforms the complex structure J^0 into the complex structure

$$J_f := f_*^{-1} \circ J^0 \circ f_* \ ,$$

where f_* is the tangent map to f.

Proposition 27. *The complex structure J_f with $f \in \mathit{Diff}_+(S^1)$ coincides with the original complex structure J_0 if and only if $f \in \mathit{Möb}(S^1)$.*

Proof. If the diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ does not change the original complex structure, i.e. defines a biholomorphism of ΩG, provided with the complex structure J_0, it means , in particular, that it preserves the tangent space $T^{0,1}_o(\Omega G)$. Hence, such a diffeomorphism should preserve the subspace $L^+G^{\mathbb{C}}$, implying that it extends to a biholomorphism of the unit disc Δ. So, $f \in \mathrm{Möb}(S^1)$. The converse assertion is obvious. □

We shall call the complex structures J_f on ΩG:

$$J_f := f_*^{-1} \circ J^0 \circ f_* \ ,$$

obtained from J^0 by the action of the diffeomorphism group, the *admissible* complex structures on ΩG. The Proposition 27 implies that the space of admissible complex structures on ΩG can be identified with the manifold $\mathcal{S}$.

Recall that the complex structure J^0 on ΩG is invariant under the left LG-translations on the space ΩG and compatible with the symplectic structure ω (in the sense of Def. 17 from Sec. 1.2.5). Due to the invariance of ω with respect to the action of the group $\text{Diff}_+(S^1)$, the complex structures J_f are also invariant under the left LG-translations and compatible with ω. In particular, any such complex structure J_f defines a *Kähler metric* g_f on ΩG by the formula

$$g_f(\xi, \eta) := \omega(\xi, J_f \eta)$$

for any $\xi, \eta \in T_\gamma(\Omega G)$, $\gamma \in \Omega G$.

Consider now the space $\mathcal{R} = \text{Diff}_+(S^1)/(S^1)$. Combining the above embedding

$$\mathcal{S} \hookrightarrow \text{Sp}_{\text{HS}}(V)/\text{U}(W_+) = \mathcal{D}_{\text{HS}}$$

with the holomorphic map

$$\mathcal{R} = \text{Diff}_+(S^1)/(S^1) \longrightarrow \mathcal{S} \ ,$$

we obtain the Grassmann realization of the space $\mathcal{R} = \text{Diff}_+(S^1)/(S^1)$:

$$\mathcal{R} \longrightarrow \text{Sp}_{\text{HS}}(V)/\text{U}(W_+) = \mathcal{D}_{\text{HS}} \ .$$

As in the case of $\mathcal{S}$, the space $\mathcal{R}$ can be also considered as a space of complex structures on the loop space ΩG. Recall that the loop space ΩG, provided with the complex structure J_0, admits the following complex homogeneous representation

$$\Omega G = LG^{\mathbb{C}} \big/ L_+G^{\mathbb{C}} \ .$$

According to Birkhoff theorem (cf. Sec. 7.3), we can identify a neighborhood of the origin in ΩG with a neighborhood of the identity in the loop subgroup $L_1^-G^{\mathbb{C}}$. If a diffeomorphism $f \in \text{Diff}_+(S^1)$ fixes the origin in ΩG and generates a biholomorphism of

$$(\Omega G, J_0) = LG^{\mathbb{C}} \big/ L_+G^{\mathbb{C}} \ ,$$

it generates also a biholomorphism of $L_1^-G^{\mathbb{C}}$. In this case we shall say that the complex structure J_f, associated with $f \in \text{Diff}_+(S^1)$, is *equivalent* to the original complex structure J_0.

Proposition 28. *The complex structure J_f with $f \in \mathit{Diff}_+(S^1)$ is equivalent to the original complex structure J_0 in the above sense if and only if f is a rotation, i.e. $f \in S^1$.*

Proof. If the diffeomorphism $f \in \text{Diff}_+(S^1)$ generates a biholomorphism of

$$(\Omega G, J_0) = LG^{\mathbb{C}} \big/ L_+G^{\mathbb{C}} \ ,$$

fixing the origin, then it leaves the subspace $L_+G^{\mathbb{C}}$ invariant and generates a biholomorphism of $L_1^-G^{\mathbb{C}}$. The first property implies that f extends to a biholomorphism of the unit disc Δ, while the second one implies that f extends to a biholomorphism of its exterior Δ_-, fixing the infinity. Then, by Liouville theorem, $f \in S^1$. □

Bibliographic comments

A key reference for this Chapter is the Nag's book [56]. Most of the assertions in Sec. 11.1, 11.2, 11.3 may be found there. Prop. 21 is proved in the paper [59]. The Grassmann approach to the study of the universal Teichmüller space was initiated by Nag–Sullivan's paper [58]. All assertions from Sec. 11.4 may be found there. Prop. 26 is proved in [57].