Part II

LOOP SPACES OF COMPACT LIE GROUPS

Chapter 7

Loop space ΩG

Let G be a compact Lie group. Its *loop space* or *based loop space* is the space of (right) conjugacy classes of the loop group $LG = C^\infty(S^1, G)$ of the form

$$\Omega G = LG/G \ , \tag{7.1}$$

where $LG = C^\infty(S^1, G)$ is the group of smooth maps of the circle $S^1 = \{|z| = 1\} \subset \mathbb{C}$ into the group G, and G in the denominator is identified with the group of constant maps $S^1 \to g_0 \in G$.

The loop space ΩG is a homogeneous space of the Frechet Lie group LG with a natural action of LG on it by left translations. The origin (neutral element) in ΩG is given by the class $o := [\mathbf{1}] = [G]$ of constant maps.

The space ΩG may be identified (as a homogeneous space) with the subgroup L_1G of maps $\gamma \in LG$ such that

$$\gamma : 1 \in S^1 \longrightarrow \gamma(1) = e \in G \ ,$$

by associating with a class $[\gamma]$ of a loop $\gamma \in LG$ the map $\gamma(1)^{-1}\gamma \in L_1G$. Under this identification ΩG is realized as a closed submanifold (of codimension 1) of the Frechet manifold LG and so is itself a Frechet manifold. We note that this identification of ΩG with L_1G is not canonical, since G is not a normal subgroup in LG.

7.1 Complex homogeneous representation

One of the main features of the loop space ΩG, which plays a key role in the study of its Kähler geometry, is the existence of two kinds of its homogeneous representations. Namely, together with the "real" representation (7.1) of ΩG as a homogeneous space of the real Frechet Lie group LG, there exists also a "complex" representation of ΩG as a homogeneous space of the complex Frechet Lie group $LG^{\mathbb{C}} = C^\infty(S^1, G^{\mathbb{C}})$, where $G^{\mathbb{C}}$ is the complexification of the Lie group G. More precisely, we have the following representation

$$\Omega G = LG^{\mathbb{C}} \big/ L_+G^{\mathbb{C}} \ , \tag{7.2}$$

where $L^+G^{\mathbb{C}} = \mathrm{Hol}(\Delta, G^{\mathbb{C}})$ is the subgroup of maps from $LG^{\mathbb{C}}$, which extend smoothly to holomorphic (and smooth up to the boundary) maps of the unit disc $\Delta \subset \mathbb{C}$ into the group $G^{\mathbb{C}}$.

Let us explain the meaning of the representation (7.2) in the case of the unitary group $G = \mathrm{U}(n)$. In this case $G^{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$, and the equality (7.2) means that any complex non-degenerate (i.e. taking values in $\mathrm{GL}(n, \mathbb{C})$) matrix function $T(z)$ on the circle S^1 can be represented in the form

$$T(z) = U(z) \cdot H_+(z) \ , \quad z = e^{i\theta} \ , \tag{7.3}$$

where $U(z)$ is a smooth unitary (i.e. with values in $\mathrm{U}(n)$) matrix function, and $H_+(z)$ smoothly extends to a holomorphic non-degenerate matrix function in the disc Δ. It is a parametric analog of the standard representation of a matrix $T \in \mathrm{GL}(n, \mathbb{C})$ as the product of a unitary and upper-triangular matrices. The representation (7.3) would be unique, if one requires that $U \in L_1\mathrm{U}(n)$. Moreover, the product map $(U, H_+) \mapsto U \cdot H_+$ defines a diffeomorphism $\Omega\mathrm{U}(n) \times L^+\mathrm{GL}(n, \mathbb{C}) \to L\mathrm{GL}(n, \mathbb{C})$.

In the same sense we shall understand the equality (7.2) in the case of an arbitrary compact Lie group G. Namely, we have the following

Theorem 6 (Pressley–Segal)**.** *The product map*

$$\Omega G \times L^+G^{\mathbb{C}} \longrightarrow LG^{\mathbb{C}}$$

is a diffeomorphism of Frechet manifolds.

The proof of this Theorem, given in [65], uses the Grassmann realization of the loop space ΩG and will be given later in Ch. 9, after we introduce the Grassmann model of ΩG.

Remark 11. There is another approach to the proof of this Theorem, based on the Beurling–Helson theorem, describing the shift-invariant subspaces in L^2-spaces on the circle (this approach to the proof of Theorem 6 was proposed to us by A.Fedotov). We explain how to apply this theorem to the proof of Theor. 6 in the scalar case, i.e. for $G = S^1$.

Denote by H^2 the Hardy subspace in $L^2 = L^2(S^1)$, consisting of functions f, which extend holomorphically into the unit disc and have boundary values in the sense of L^2 on the circle S^1. In terms of Fourier decompositions $f \in H^2$ if and only if

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \quad , \quad \sum_{0}^{\infty} |c_n|^2 := \|f\|^2_{H^2} < \infty \ , \ z \in \Delta \ .$$

Consider the shift operator S in L^2, which is defined by the formula

$$S : f(z) \longmapsto zf(z)$$

and maps H^2 into itself.

Theorem 7 (Beurling–Helson (cf., e.g. [60]))*. Any subspace E in L^2, invariant under the shift operator S, has the following form:*

1. *If $SE = E$, then there exists a measurable subset d in S^1 such that*

$$E = \chi_d L^2 \ ,$$

where χ_d is the characteristic function of the set d.

2. *If $SE \subset E$, but $SE \neq E$, then there exists a function $\theta \in L^2$ such that $|\theta| = 1$ almost everywhere on S^1 and*

$$E = \theta H^2 .$$

We return to the relation (7.2). Consider for a function $f \in L\mathbb{C}^*$ (here $\mathbb{C}^* = (S^1)^{\mathbb{C}}$ denotes the multiplicative group of non-zero complex numbers) the subspace E in L^2 of the form

$$E = fH^2 .$$

It is invariant under the shift operator S and $SE \neq E$, if $f \notin L^+\mathbb{C}^*$. So by Beurling–Helson theorem

$$fH^2 = \theta H^2$$

for some function $\theta \in L^2$, such that $|\theta| = 1$ almost everywhere on S^1. It implies that

$$f = f \cdot 1 = \theta \cdot h$$

for some $h \in H^2$, which is already the relation, we are looking for. It only remains to show that the functions θ and h may be chosen smooth (and smoothly depending on f), so that $\theta \in \Omega S^1$ and $h \in L^+\mathbb{C}^*$. It may be done as in [65], Ch. 8 (we also discuss this point in Ch. 9).

7.2 Symplectic structure

Since ΩG is a homogeneous space of the loop group LG, it's natural to use geometric structures, invariant under the action of LG, for the study of its Kähler geometry. Such structures are uniquely determined by their values at the origin $o \in \Omega G$. By this reason we start from the description of the tangent space $T_o(\Omega G)$.

The tangent space $T_o(\Omega G)$ coincides with the quotient of the tangent space $T_{\mathbf{1}}(LG) = L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$ modulo constant maps, i.e.

$$T_o(\Omega G) = L\mathfrak{g}/\mathfrak{g} =: \Omega\mathfrak{g} .$$

It is convenient to represent vectors from the complexified tangent space

$$T_o^{\mathbb{C}}(\Omega G) = L\mathfrak{g}^{\mathbb{C}}/\mathfrak{g}^{\mathbb{C}} =: \Omega\mathfrak{g}^{\mathbb{C}}$$

by their Fourier decompositions

$$\Omega\mathfrak{g}^{\mathbb{C}} \ni \xi(z) = \sum_{k\neq 0} \xi_k z^k \ , \ z = e^{i\theta} \ ,$$

where $\xi_k \in \mathfrak{g}^{\mathbb{C}}$ (the term, corresponding to $k = 0$, is eliminated by the factorization modulo $\mathfrak{g}^{\mathbb{C}}$ in $\Omega\mathfrak{g}^{\mathbb{C}}$). A vector $\xi \in T_o^{\mathbb{C}}(\Omega G)$ belongs to the real tangent space $T_o(\Omega G)$ if and only if

$$\xi_{-k} = \bar{\xi}_k \ ,$$

where the "bar" means the complex conjugation in $\mathfrak{g}^{\mathbb{C}}$, for which $\overline{\mathfrak{g}} = \mathfrak{g}$.

We construct now an invariant (with respect to LG-action) symplectic structure on ΩG. Define first its value at the origin or, in other words, the restriction of the symplectic form to the tangent space $T_o(\Omega G) = \Omega\mathfrak{g}$, and then transport it to other points of ΩG with the help of left translations by LG.

To define a symplectic form at the origin, we should fix an invariant inner product $< \cdot\,, \cdot >$ on the Lie algebra $\mathfrak{g}$ of the group G. Let us recall basic definitions, related to this notion.

Digression 2 (Invariant inner product). The *inner product* on the Lie algebra $\mathfrak{g}$ of the Lie group G is a positively definite symmetric bilinear form on $\mathfrak{g}$. We say that it is *invariant*, if it is invariant under the adjoint action Ad of the group G on its Lie algebra $\mathfrak{g}$, defined in the following way. The group G acts on itself by inner automorphisms of the form

$$G \ni g : \; G \ni h \longrightarrow ghg^{-1} \in G \; .$$

This action fixes the identity $e \in G$ and generates an action of the group G on $T_eG = \mathfrak{g}$, called the *adjoint action* and denoted by $\mathrm{Ad}\, g : \mathfrak{g} \longrightarrow \mathfrak{g}$. Its differential is called the *adjoint representation* of the Lie algebra $\mathfrak{g}$ and has the form

$$\mathrm{ad}\,\xi : \mathfrak{g} \longrightarrow \mathfrak{g}, \qquad \mathrm{ad}\,\xi : \eta \longmapsto [\xi, \eta] \; .$$

An inner product $< \cdot\,, \cdot >$ on the Lie algebra $\mathfrak{g}$ is invariant iff

$$< (\mathrm{Ad}\, g)\eta, (\mathrm{Ad}\, g)\zeta > = < \eta, \zeta > \quad \text{for any } \eta,\, \zeta \in \mathfrak{g} \; . \tag{7.4}$$

If the group G is connected, this condition is equivalent to a relation on the Lie algebra level, obtained from (7.4) by differentiation:

$$< (\mathrm{ad}\,\xi)\eta, \zeta > + < \eta, (\mathrm{ad}\,\xi)\zeta > = 0$$

or, equivalently,

$$< [\xi, \eta], \zeta > + < \eta, [\xi, \zeta] > = 0 \; .$$

On any Lie algebra $\mathfrak{g}$ there exists an invariant symmetric bilinear form, called the *Killing form*, defined by

$$< \xi, \eta > := \mathrm{tr}(\mathrm{ad}\,\xi\, \mathrm{ad}\,\eta) \; , \quad \xi, \eta \in \mathfrak{g} \; .$$

In particular, for $G = \mathrm{GL}(n, \mathbb{C})$ we have $\mathfrak{g} = \mathrm{gl}(n, \mathbb{C})$ and $< \xi, \eta > := \mathrm{tr}(\xi\,\eta)$. The Killing form is non-degenerate, if the group G is semisimple (e.g. for $G = \mathrm{SL}(n, \mathbb{C})$). If, moreover, G is compact, then the Killing form is negatively definite. Hence, the negation of this form defines an invariant inner product on the Lie algebra $\mathfrak{g}$ of a compact semisimple Lie group G.

We return to the construction of an LG-invariant symplectic form ω on ΩG. Let us fix an invariant inner product $< \cdot\,, \cdot >$ on the Lie algebra $\mathfrak{g}$ of the group G and define the restriction of the form ω to $T_o(\Omega G) = \Omega\mathfrak{g} = L\mathfrak{g}/\mathfrak{g}$.

Using the inner product $< \cdot\,, \cdot >$, we introduce, first of all, a 2-form ω_0 on the loop algebra $L\mathfrak{g}$, by setting it equal to

$$\omega_0(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} < \xi(e^{i\theta}), \frac{d\eta(e^{i\theta})}{d\theta} > d\theta \tag{7.5}$$

on vectors $\xi = \xi(e^{i\theta})$, $\eta = \eta(e^{i\theta})$ from the loop algebra $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$.

This is a skew-symmetric bilinear form on $L\mathfrak{g}$, which is, due to the invariance of $< \cdot\, , \cdot >$, invariant under the adjoint action of the group G of constant loops on $L\mathfrak{g}$. It's evident that $\omega_0(\xi, \eta)$ is equal to zero, if at least one of the maps ξ, η is constant. So the form ω_0 can be pushed down to $\Omega\mathfrak{g}$, and the pushed-down form is already non-degenerate (to show that it is non-degenerate, consider the value of the pushed-down form $\omega_0(\cdot, \eta)$ on vector $\xi(e^{i\theta}) = \eta'(e^{i\theta}) := \frac{d\eta(e^{i\theta})}{d\theta}$). Hence, we have constructed a skew-symmetric bilinear form ω_0 on $\Omega\mathfrak{g}$, which is invariant under the adjoint action of the group G on $\Omega\mathfrak{g}$. This form can be extended (with the help of left translations) to an LG-invariant non-degenerate 2-form ω on ΩG.

It remains to check that the obtained form ω is closed on ΩG. The closedness condition (cf. Subsec. 1.2.4), due to the invariance of ω, takes on the form

$$\omega([\xi, \eta], \zeta) + \omega([\eta, \zeta], \xi) + \omega([\zeta, \xi], \eta) = 0 \ . \tag{7.6}$$

It is sufficient to check it on vectors $\xi, \eta, \zeta \in L\mathfrak{g}$. In this case the equality (7.6) means that

$$\int_0^{2\pi} \{< [\xi, \eta], \zeta' > + < [\eta, \zeta], \xi' > + < [\zeta, \xi], \eta' >\}\, d\theta = 0 \ . \tag{7.7}$$

Integrating the first integral by parts, we obtain

$$\int_0^{2\pi} < [\xi, \eta], \zeta' > \, d\theta = -\int_0^{2\pi} < [\xi', \eta], \zeta > \, d\theta - \int_0^{2\pi} < [\xi, \eta'], \zeta > \, d\theta \ . \tag{7.8}$$

Due to the invariance of the inner product $< \cdot\, , \cdot >$ on $\mathfrak{g}$

$$< [\xi', \eta], \zeta > = < \xi', [\eta, \zeta] > = < [\eta, \zeta], \xi' > \ ,$$

and the first term in the right hand side of (7.8) sum to zero together with the second integral in the formula (7.7). By the same reason

$$< [\xi, \eta'], \zeta > = < \eta', [\zeta, \xi] > = < [\zeta, \xi], \eta' >$$

and the second term in the right hand side of (7.8) sum to zero together with the third integral in the formula (7.7). It proves the validity of the equality (7.6), which implies that $d\omega(\xi, \eta, \zeta) = 0$ for all $\xi, \eta, \zeta \in L\mathfrak{g}$.

The choice of the formula (7.5) for the form ω_0 on $L\mathfrak{g}$ looks somewhat ambiguous, but it may be shown that this form is uniquely determined by the invariant inner product $< \cdot\, , \cdot >$ on $\mathfrak{g}$ in the case of a semisimple Lie group G. More precisely, we have the following

Proposition 14 (Pressley–Segal ([65])). *If the Lie group G is semisimple, then any 2-form ω_0 on $L\mathfrak{g}$, which satisfies the relation* (7.6) *and is invariant under the adjoint action of the group G on $L\mathfrak{g}$, is given by the formula* (7.5) *for some symmetric invariant bilinear form $< \cdot\, , \cdot >$ on $\mathfrak{g}$.*

Proof. We note, first of all, that bilinear invariant forms on complex semisimple Lie algebras are necessarily symmetric. More precisely, the following assertion is true.

Lemma 3. *If G is a semisimple Lie group with the Lie algebra $\mathfrak{g}$, then any complex-bilinear G-invariant form on the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is necessarily symmetric.*

In the case of a simple Lie group G the assertion of Lemma follows from the fact that there exists a unique (up to the proportionality) complex-bilinear G-invariant form on G (the Schur's lemma), namely, the Killing form. The case of a semisimple Lie group G is reduced to the considered case (cf. for details [65]).

We turn now to the proof of the Proposition. The form ω_0 on $L\mathfrak{g}$ may be extended to a complex-bilinear form $\omega_0 : L\mathfrak{g}^{\mathbb{C}} \times L\mathfrak{g}^{\mathbb{C}} \to \mathbb{C}$. Since any element $\xi \in L\mathfrak{g}^{\mathbb{C}}$ is represented by the Fourier series

$$\xi = \sum \xi_p z^p ,$$

the form ω_0 is uniquely determined by its values on monomials of the type $\xi_p z^p$, i.e. by the forms

$$\omega_{p,q}(\xi, \eta) := \omega_0(\xi z^p, \eta z^q) ,$$

defined for $p, q \in \mathbb{Z}$ and $(\xi, \eta) \in \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}}$. The forms $\omega_{p,q}$ are G-invariant and so, by Lemma, they are symmetric. Moreover, the skew-symmetricity of ω_0 implies that $\omega_{p,q} = -\omega_{q,p}$. The condition of closedness of ω_0 on $L\mathfrak{g}$, when applied to the monomials $\tilde{\xi} = \xi z^p$, $\tilde{\eta} = \eta z^q$, $\tilde{\zeta} = \zeta z^r$, has the form

$$\omega_0([\tilde{\xi}, \tilde{\eta}], \tilde{\zeta}) + \omega_0([\tilde{\eta}, \tilde{\zeta}], \tilde{\xi}) + \omega_0([\tilde{\zeta}, \tilde{\xi}], \tilde{\eta}) = 0 .$$

This equality transforms into the following relation for the forms $\omega_{p,q}$:

$$\omega_{p+q,r}([\xi, \eta], \zeta) + \omega_{q+r,p}([\eta, \zeta], \xi) + \omega_{r+p,q}([\zeta, \xi], \eta) = 0 . \tag{7.9}$$

From the symmetricity and G-invariance of the forms $\omega_{p,q}$ we obtain

$$\omega_{q+r,p}([\eta, \zeta], \xi) = \omega_{q+r,p}(\xi, [\eta, \zeta]) = \omega_{q+r,p}([\xi, \eta], \zeta)$$

and, analogously,

$$\omega_{r+p,q}([\zeta, \xi], \eta) = \omega_{r+p,q}(\eta, [\zeta, \xi]) = \omega_{r+p,q}([\xi, \eta], \zeta) .$$

Hence, the equality (7.9) may be rewritten in the form

$$\omega_{p+q,r}([\xi, \eta], \zeta) + \omega_{q+r,p}([\xi, \eta], \zeta) + \omega_{r+p,q}([\xi, \eta], \zeta) = 0,$$

equivalent in the case of a semisimple Lie algebra to the relation

$$\omega_{p+q,r} + \omega_{q+r,p} + \omega_{r+p,q} = 0 . \tag{7.10}$$

This relation for $q = r = 0$ implies that $\omega_{p,0} = 0$ for all p. Setting $r = -p - q$ in (7.10), we get

$$\omega_{p+q,-p-q} = \omega_{p,-p} + \omega_{q,-q} ,$$

whence $\omega_{p,-p} = p\omega_{1,-1}$. Setting $r = n - p - q$ in (7.10), we obtain

$$\omega_{n-p-q,p+q} = \omega_{n-p,p} + \omega_{n-q,q} \ ,$$

implying $\omega_{n-p,p} = p\omega_{n-1,1}$. Hence,

$$\omega_{n-1,1} = \frac{\omega_{0,n}}{n} = 0$$

and so $\omega_{p,q} = 0$, if $p + q \neq 0$. Thus, the form ω_0 on vectors $\xi = \sum \xi_p z^p$, $\eta = \sum \eta_q z^q$ takes the value

$$\omega_0(\xi, \eta) = \sum \omega_{p,q}(\xi_p, \eta_q) = \sum_p \omega_{p,-p}(\xi_p, \eta_{-p}) = \sum_p p\omega_{1,-1}(\xi_p, \eta_{-p}) \ .$$

On the other hand

$$\frac{i}{2\pi} \int_0^{2\pi} \omega_{1,-1}(\xi(\theta), \eta'(\theta))\, d\theta = -\sum_{p,q} \frac{1}{2\pi} \int_0^{2\pi} \omega_{1,-1}(\xi_p e^{ip\theta}, q\eta_q e^{iq\theta})\, d\theta =$$

$$= \sum_p \frac{1}{2\pi} \int_0^{2\pi} p\omega_{1,-1}(\xi_p, \eta_{-p})\, d\theta = \sum_p p\omega_{1,-1}(\xi_p, \eta_{-p}) \ . \quad (7.11)$$

So

$$\omega_0(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi(\theta), \eta'(\theta) \rangle \ d\theta$$

with the invariant inner product on the Lie algebra $\mathfrak{g}$, given by the formula

$$< \xi, \eta >:= \omega_{1,-1}(\xi, \eta) \ , \quad (7.12)$$

which concludes the proof of the Proposition. □

Remark 12. There is also a physical motivation behind the formula (7.5) for the symplectic form ω. It comes from the relation with string theory (cf. [14]). Mathematically, we consider the space $\Omega\mathbb{R}^d$ of based loops $S^1 \to \mathbb{R}^d$, taking values in the (non-compact) group $\mathbb{R}^d$ of translations of the d-dimensional Euclidean vector space. Physically, the loop space $\Omega\mathbb{R}^d$ may be interpreted as the phase space of the bosonic open string theory. In more detail, the configuration space of this theory consists of the smooth maps $q : [0, \pi] \to \mathbb{R}^d$ with all derivatives, vanishing at boundary points. The corresponding phase space consists then of pairs of maps (p, q) of the same type. The symplectic form on this phase space is given by the string analogue of the standard formula

$$\omega(\delta p, \delta q) = \frac{2}{\pi} \int_0^{\pi} \delta p(\sigma) \wedge \delta q(\sigma) d\sigma \ , \quad (7.13)$$

where δp, δq are smooth maps $[0, \pi] \to \mathbb{R}^d$ of the same type, as before, interpreted as tangent vectors to the phase space. A natural map, associating with a pair (p, q) the map $x : [-\pi, \pi] \to \mathbb{R}^d$, given by the formula

$$x(\sigma) = \begin{cases} p(\sigma) + q'(\sigma) & \text{for } 0 \leq \sigma \leq \pi \, ; \\ p(-\sigma) + q'(-\sigma) & \text{for } -\pi \leq \sigma \leq 0 \, , \end{cases}$$

identifies the introduced phase space with the space $\Omega\mathbb{R}^d$. It also converts the standard symplectic form (7.13) on the phase space of string theory into the symplectic form on $\Omega\mathbb{R}^d$, given by the formula, analogous to (7.5) (cf. [14]).

We have assigned to any invariant inner product on the Lie algebra $\mathfrak{g}$ an invariant symplectic structure ω on the loop space ΩG, determined by the formula (7.5). On the other hand, any invariant symplectic structure on the loop space ΩG uniquely determines an invariant inner product on $\mathfrak{g}$, given by the formula (7.12). As we have pointed out in Sec. 4.2, invariant bilinear forms on $\mathfrak{g}$ are parameterized by elements of the cohomology $H^3(\mathfrak{g})$.

Remark 13. We note in passing that the condition of invariance of the form ω with respect to the adjoint action of the group G on the loop algebra $L\mathfrak{g}$ is not essential and plays the role of normalization. Indeed, if ω_0 is an arbitrary 2-form on $L\mathfrak{g}$, satisfying the condition (7.6), then the form

$$g \cdot \omega_0(\xi, \eta) := \omega_0((\mathrm{Ad}\, g)\xi, (\mathrm{Ad}\, g)\eta) \quad \text{for } g \in G$$

belongs to the same cohomology class, as ω_0 (it follows from the cocycle identity (7.6)). So the form

$$\int_G g \cdot \omega_0 \, dg \ ,$$

obtained from ω_0 by averaging over the group G, belongs to the same cohomology class, as ω_0, but is already invariant under the adjoint action of constant loops.

7.3 Complex structure

A complex structure on the loop space ΩG is induced from the complex representation

$$\Omega G = LG^{\mathbb{C}}/L^+G^{\mathbb{C}} \ , \tag{7.14}$$

in which $LG^{\mathbb{C}}$ is a complex Lie Frechet group, and $L^+G^{\mathbb{C}}$ is its closed complex subgroup.

This complex structure, denoted by J^0 in the sequel, is LG-invariant, and its restriction to the tangent space $T_o^{\mathbb{C}}(\Omega G) = \Omega\mathfrak{g}^{\mathbb{C}}$ at the origin may be given by an explicit formula. Namely, if $\xi = \sum\limits_{k\neq 0} \xi_k z^k \in \Omega\mathfrak{g}^{\mathbb{C}}$, then

$$J^0\xi = -i\sum_{k>0} \xi_k z^k + i\sum_{k<0} \xi_k z^k \ . \tag{7.15}$$

The corresponding tangent space $T_o^{1,0}(\Omega G)$ of $(1,0)$-vectors consists of vectors of the form $\sum_{k<0} \xi_k z^k$, while the space $T_o^{0,1}(\Omega G)$ of $(0,1)$-vectors contains vectors of the form $\sum_{k>0} \xi_k z^k$.

It is clear from the description of $(1,0)$-vectors on ΩG that the complex structure J^0 is formally integrable in the sense of Subsec. 1.2.4, i.e. the bracket of any two $(1,0)$-vector fields on ΩG is again a $(1,0)$-vector field. But we have already pointed out in Subsec. 1.2.4 that the formal integrability of a complex structure in the infinite-dimensional case does not imply the existence of an atlas of coordinate neighborhoods and local complex coordinates on a given manifold. In order to construct local complex coordinates on ΩG, one should use the complex representation (7.14) and the Birkhoff factorization theorem. We formulate next a particular case of this theorem, sufficient for our applications.

Denote by $L^-G^{\mathbb{C}}$ a closed subgroup of $LG^{\mathbb{C}}$, consisting of maps $\gamma \in LG^{\mathbb{C}}$, which extend to holomorphic and smooth up to the circle S^1 maps of the disc Δ_- (equal to the complement of the closed unit disc $\overline{\Delta}$ on the Riemann sphere $\overline{\mathbb{C}}$). We also consider a closed subgroup $L_1^-G^{\mathbb{C}}$ of $L^-G^{\mathbb{C}}$, consisting of maps $\gamma \in L^-G^{\mathbb{C}}$, taking the value $e \in G^{\mathbb{C}}$ at infinity $\infty \in \Delta_-$.

Theorem 8 (Birkhoff theorem ([8, 9], cf. also [65], Ch.8)). *The product map*

$$L^+G^{\mathbb{C}} \times L_1^-G^{\mathbb{C}} \longrightarrow LG^{\mathbb{C}} \tag{7.16}$$

is a diffeomorphism onto a dense open subset in the identity component of $LG^{\mathbb{C}}$.

The Birkhoff theorem implies that for all $\gamma \in LG^{\mathbb{C}}$ in a neighborhood of the identity $\mathbf{1} \in LG^{\mathbb{C}}$ we have a representation

$$\gamma = \gamma_+ \cdot \gamma_- \ ,$$

where $\gamma_+ \in L^+G^{\mathbb{C}}$, $\gamma_- \in L_1^-G^{\mathbb{C}}$. The factors $\gamma_\pm$ are uniquely defined by γ and their product yields a local diffeomorphism (7.16). In particular, it implies that the *loop space ΩG is locally diffeomorphic to the complex Lie Frechet group $L_1^-G^{\mathbb{C}}$.*

7.4 Kähler structure

We show now that the loop space ΩG is a Kähler Frechet manifold. For that, according to Def. 17 from Subsec. 1.2.5, we need to demonstrate that the introduced complex and symplectic structures on ΩG are compatible.

Since both structures are LG-invariant, it's sufficient to check their compatibility only at the origin $o \in \Omega G$. Consider vectors $\xi, \eta \in T_o(\Omega G)$ with Fourier decompositions

$$\xi = \sum_{k\neq 0} \xi_k z^k, \qquad \eta = \sum_{l\neq 0} \eta_l z^l \ .$$

Then

$$\omega_o(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} \left\langle \xi(e^{i\theta}), \eta'(e^{i\theta})\right\rangle d\theta = -\frac{i}{2\pi}\int_0^{2\pi} \sum_{k,l\neq 0} \left\langle \xi_k e^{ik\theta}, l\eta_l e^{il\theta}\right\rangle d\theta =$$

$$= -\frac{i}{2\pi}\sum_{k\neq 0}\int_0^{2\pi} \langle \xi_k, k\eta_{-k}\rangle \, d\theta = -i\sum_{k\neq 0} k\,\langle \xi_k, \eta_{-k}\rangle \ , \tag{7.17}$$

where the inner product $\langle \cdot\,, \cdot\rangle$ is extended to a complex-bilinear positive definite form on $\mathfrak{g}^{\mathbb{C}}$. (Recall that the form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}^{\mathbb{C}}$ is positive definite, if $\left\langle \xi, \bar{\xi}\right\rangle > 0$ for any $\xi \in \mathfrak{g}^{\mathbb{C}}\backslash\{0\}$). The above relation implies the first property in Def. 17. To prove the second property in this definition, consider the form

$$g_o^0(\xi,\eta) := \omega_0(\xi, J^0\eta)$$

on $\Omega\mathfrak{g}$ and show that it is positively definite. Note that for $\eta = \xi$ this form can be given by the formula

$$g_o^0(\xi,\xi) = -i\sum_{k>0} k\,\langle \xi_k, i\xi_{-k}\rangle - i\sum_{k<0} k\,\langle \xi_k, -i\xi_{-k}\rangle = 2\sum_{k>0} k\,\langle \xi_k, \xi_{-k}\rangle \ .$$

Since $\xi_{-k} = \overline{\xi}_k$ and the inner product $\langle \cdot\,,\cdot\rangle$ is positively definite on $\mathfrak{g}^{\mathbb{C}}$, the form $g_o^0(\xi,\xi)$ is also positively definite. Extending it to a LG-invariant positively definite form on ΩG, we get an invariant Kähler metric g^0 on ΩG. So, we have proved that the loop space ΩG is a *Kähler Frechet manifold* with the symplectic structure ω and complex structure J^0.

7.5 Loop space ΩG as a universal flag manifold of a group G

We have pointed out in the beginning of Sec. 7.1 that one of the characteristic properties of the Kähler Frechet manifold ΩG is the existence of two different representations of ΩG:

$$\Omega G = LG/G = LG^{\mathbb{C}}/L^+G^{\mathbb{C}}$$

as a homogeneous space of the real Lie Frechet group LG and its complexification $LG^{\mathbb{C}}$.

We have seen in Ch. 3 that finite-dimensional Kähler manifolds, having a similar property, i.e. being homogeneous spaces of real compact and complex Lie groups simultaneously, are called the flag manifolds. So ΩG may be considered as an infinite-dimensional analogue of flag manifolds. Moreover, we show in this Section that in some sense it may be considered as a universal flag manifold of the group G, since all flag manifolds of G are canonically embedded into ΩG as complex submanifolds.

The real homogeneous representation of a flag manifold

$$F = G/L$$

of the group G may be interpreted otherwise as a representation of F as an orbit of the adjoint action Ad of G on its Lie algebra $\mathfrak{g}$ (or as an orbit of the coadjoint action Ad^* of G on the dual space $\mathfrak{g}^*$). Namely, the orbit of an element $\xi \in \mathfrak{g}$ with respect to the adjoint action has the form

$$G/G(\xi)\ ,$$

where the isotropy subgroup $G(\xi)$ at ξ coincides with the centralizer of ξ, i.e. with

$$G(\xi) = \{g \in G:\ \mathrm{Ad}\,g(\xi) = \xi\}\ .$$

All such orbits are flag manifolds and, conversely, any flag manifold of a compact semisimple Lie group may be represented in this form.

Consider now a natural action of S^1 on the loop space ΩG, identified with the subgroup L_1G in LG, given by the rotation of loops

$$\lambda\cdot\gamma(z) = \gamma(\lambda)^{-1}\gamma(\lambda z)\ , \quad \lambda \in S^1\ ,$$

where $\gamma \in \Omega G$. A loop γ is a fixed point of this S^1-action if and only if

$$\gamma(\lambda z) = \gamma(\lambda)\gamma(z) \quad \text{for all}\, \lambda, z \in S^1 \ .$$

In other words, γ should be a group homomorphism $S^1 \to G$. But if $\gamma : S^1 \to G$ is a homomorphism, so are all the loops, conjugate to γ, i.e. the loops of the form $\gamma_g = g\gamma g^{-1}$ for $g \in G$. The set of all such loops (the conjugacy class of the loop γ) is parameterized by points of the homogeneous space

$$F_\gamma = G/G(\gamma) \ ,$$

where $G(\gamma)$ is the centralizer of the one-parameter subgroup $\gamma(S^1)$ in G. The homogeneous space F_γ can be identified, as we have pointed out above, with a flag manifold of the group G.

So, the *set of fixed points of the S^1-action on ΩG is the disjoint union*

$$\mathrm{Fix}(S^1) = \bigcup_\gamma F_\gamma$$

of flag manifolds F_γ, where γ runs over the set of conjugacy classes of homomorphisms $S^1 \to G$. The flag manifolds F_γ are immersed into ΩG as finite-dimensional Kähler submanifolds.

Remark 14. We can say much more about the constructed embedding of flag manifolds of the group G into the loop space ΩG. Namely, denote by

$$\pi : \Omega G \longrightarrow G \ , \quad \gamma \longmapsto \gamma(-1) \ ,$$

the map, associating with a loop γ its value at the point $-1 \in S^1$. This map is an analogue of the canonical bundle $\pi : F \to N$, considered in Sec. 3.1, Rem. 5.

According to Uhlenbeck [74], the embedding of flag manifolds F of the group G into ΩG respects canonical bundles. In other words, not only the loop space ΩG may be considered as a universal flag manifold of the group G, but also the above canonical bundle $\pi : \Omega G \to G$ may be considered as a universal canonical flag bundle.

More precisely, there exists the following commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\ \Gamma\ } & \Omega G \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ N(F) & \xrightarrow{\ \gamma\ } & G \end{array}$$

where $\pi \colon F \to N(F)$ is the canonical bundle over the symmetric space $N(F)$, constructed in Sec. 3.1, and the map Γ is the embedding of a flag manifold into ΩG, constructed above.

The horizontal maps in the this diagram admit a simple description in terms of the canonical element, introduced in Sec. 3.1, Rem. 5. Namely, suppose that the group G has a trivial center, and consider the flag manifold $F = G/L = G^{\mathbb{C}}/P$ with the canonical element ξ. The triviality of the center of G implies that $\exp(2\pi\xi) = e \in G$. So we can define a map $\Gamma \colon F \to \Omega G$ by setting it equal to $\Gamma(o) :=$ a map $\{e^{it} \mapsto$

$\exp(t\xi)\}$ at the neutral element $o \in F$, and transporting it to other points of G/L with the help of left translations by G. On the other hand, there is a natural map $\gamma\colon N(F) \to G$, assigning to a point x of the inner symmetric space $N(F)$, associated with F, the element $\gamma(x)$ of the group G, generating the involution at the point x. Both maps Γ and γ are totally geodesic immersions.

Remark 15. The fixed points of the S^1-action on ΩG can be also interpreted as critical points of some Morse function on ΩG (cf. [65]). Namely, define the *energy* $E : \Omega G \to \mathbb{R}_+$ *of a loop* γ by the formula

$$E(\gamma) = \frac{1}{4\pi}\int_0^{2\pi} < \gamma(e^{i\theta})^{-1}\gamma'(e^{i\theta}), \gamma(e^{i\theta})^{-1}\gamma'(e^{i\theta}) > d\theta \ .$$

It may be shown that the Hamiltonian vector field on ΩG, corresponding to the function E, generates the above S^1-action on ΩG by rotation of loops. So the *critical points of E correspond to the fixed points of the S^1-action on ΩG, i.e. to the homomorphisms $\gamma : S^1 \to G$.*

7.6 Loop space $\Omega_T G$

According to the Borel–Weil theorem (cf. Sec. 3.3), the full flag manifold $F = G/T$ of the group G, where T is a maximal torus in G, plays a special role in the theory of irreducible representations of G. A natural analogue of the full flag manifold in the case of the loop group LG is given by the homogeneous space

$$\Omega_T G = LG/T \ .$$

We list some of the properties of this Kähler Frechet manifold.

In order to define a *symplectic structure* on $\Omega_T G$, we note that the loop group LG is diffeomorphic (as a Frechet manifold) to the direct product $G \times \Omega G$. If we identify ΩG with the subgroup $L_1 G$ in LG, then this diffeomorphism will assign to a loop $\gamma \in LG$ the element $(\gamma(1), \gamma(1)^{-1}\gamma) \in G \times \Omega G$. From the group-theoretical point of view, the loop group LG is the semidirect product of G and $L_1 G$. It follows that, as a Frechet manifold, $\Omega_T G$ is diffeomorphic to

$$\Omega_T G = LG/T = G/T \times \Omega G \ .$$

A symplectic structure on $\Omega_T G$ is generated by the symplectic structure on ΩG, introduced in Sec. 7.2, and a canonical symplectic structure on the full flag manifold G/T. Recall that, as we have remarked in the previous Sec. 7.5, the flag manifolds of the group G may be considered as orbits of the coadjoint representation of G on the dual space $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$. Such orbits have a canonical symplectic structure, given by the Kirillov form (cf. Subsec. 3.2.3).

A *complex structure* on $\Omega_T G$ is induced, as in the case of the loop space ΩG, from the "complex" representation of $\Omega_T G$ as a homogeneous space of the complexified loop group $LG^{\mathbb{C}}$, which has the form

$$\Omega_T G = LG^{\mathbb{C}} / B_+ G^{\mathbb{C}} \ , \qquad (7.18)$$

where $B_+G^{\mathbb{C}}$ is a subgroup in $L^+G^{\mathbb{C}} = \mathrm{Hol}(\Delta, G^{\mathbb{C}})$, consisting of the maps $\gamma \in \mathrm{Map}(S^1, G^{\mathbb{C}})$, which extend to holomorphic and smooth up to the circle S^1 maps $\gamma : \Delta \to G^{\mathbb{C}}$ of the unit disc, and satisfy the additional condition: $\gamma(0) \in B_+$, where B_+ is the standard Borel subgroup in G. The proof of this assertion is similar to the proof of the complex representation for the loop space ΩG (cf. [65], Ch.8).

The introduced complex structure on $\Omega_T G$ is compatible with the symplectic structure and so defines on $\Omega_T G$ the *structure of a Kähler Frechet manifold*.

Bibliographic comments

A key reference for this Chapter is the Pressley–Segal book [65]. In particular, the proof of the factorization theorem 6 is given in Ch.8 of [65]. Another method of proving this theorem, based on the Beurling–Helson characterization of shift-invariant subspaces in L^2, is due to A.Fedotov (unpublished). We present its idea in the scalar case, though the proof is valid for general matrix functions on the circle. The results in Secs. 7.2,7.3,7.4,7.6 may be found in [65]. An interpretation of the loop space ΩG as a universal flag manifold may be found in [5].

Chapter 8

Central extensions of loop algebras and loop groups

We start this Chapter by recalling a general method of constructing central extensions of Lie groups, acting on a smooth manifold. We then apply this method for the construction of central extensions of loop groups. In the last Section of this Chapter we describe the coadjoint action of the loop groups.

8.1 Central extensions and S^1-bundles

Suppose that a Lie group $\mathcal{G}$ acts by smooth transformations on a smooth simply connected manifold X. We assume that there exists a closed 2-form ω on X, which is invariant under the action of $\mathcal{G}$, such that $\omega/2\pi$ is an *integral form*. In other words, the cohomology class of $\omega/2\pi$ in $H^2(X,\mathbb{R})$ is integral, i.e. contained in $H^2(X,\mathbb{Z})$ (otherwise speaking, the integral of $\omega/2\pi$ over any 2-dimensional homology cycle from $H^2(X,\mathbb{Z})$ is an integer). We shall construct a natural S^1-bundle over X, associated with these data.

Proposition 15. *Suppose that a Lie group $\mathcal{G}$ acts by smooth transformations on a smooth simply connected manifold X. Assume that ω is a closed $\mathcal{G}$-invariant 2-form on X, such that $\omega/2\pi$ is an integral form. Then there exists a principal S^1-bundle $L \to X$ with a connection ∇, having the curvature, equal to ω.*

The S^1-bundle, which existence is asserted in the Proposition, is used extensively in algebraic geometry and geometric quantization. In geometric quantization the line bundle, associated with the S^1-bundle $L \to X$, is called the *prequantization bundle*.

Proof. In terms of Čech cohomology, any cohomology class in $H^2(X,\mathbb{Z})$ is given by an integer-valued 2-cocycle $\{\nu_{abc}\}$ with respect to an acyclic open covering $\{U_a\}$ of X:

$$U_{abc} := U_a \cap U_b \cap U_c \longmapsto \nu_{abc} \in \mathbb{Z} \ .$$

(We shall assume from now on that all open sets U_a in this covering are contractible and their intersections are connected to guarantee the acyclicity of the covering $\{U_a\}$. This can be always achieved by the refinement of the covering.)

In terms of de Rham cohomology, the integrality condition of the cohomology class, represented by the form $\omega/2\pi$, means that there exists an integral closed 2-form ν on X such that

$$\omega = 2\pi\nu + d\beta \ ,$$

where β is an arbitrary 1-form on X. The integral form ν in terms of Čech cohomology is given by an integer-valued cocycle $\{\nu_{abc}\}$. Given such a cocycle, one can recover the form ν by choosing a smooth partition of unity $\{\lambda_a\}$, subordinate to the covering $\{U_a\}$, and setting

$$\nu := \sum_{a,b,c} \nu_{abc}\lambda_a \, d\lambda_b \wedge d\lambda_c \ .$$

We define the required S^1-bundle $L \to X$ by explicit transition functions

$$\varphi_{ab} = \exp\{2\pi i \sum_c \nu_{abc}\lambda_c\}$$

with respect to the covering $\{U_a\}$. It's easy to check that $\{\varphi_{ab}\}$ is a cocycle, i.e. the following relation is satisfied on every triple intersection U_{abc}: $\varphi_{ab}\varphi_{bc}\varphi_{ca} = 1$.

Consider a connection on L, given by the collection of local 1-forms

$$\alpha_a := 2\pi \sum_{b,c} \nu_{abc}\lambda_b \, d\lambda_c \ ,$$

satisfying on double intersections $U_{ab} := U_a \cap U_b$ the relation

$$\alpha_b = \alpha_a + i\varphi_{ab}^{-1} d\varphi_{ab} \ .$$

The curvature of this connection is equal to

$$\sum_a \lambda_a \, d\alpha_a = 2\pi\nu \ .$$

So, by adding β to all forms α_a, we obtain a connection ∇ on L, given by the local 1-forms $\alpha_a + d\beta$ and having the curvature, equal to $2\pi\nu + d\beta = \omega$. □

Remark 16. In terms of the sheaf cohomology, the above proof can be rephrased as follows. Denote by $\mathcal{E}$ the sheaf of C^∞-smooth functions on X, and by $\mathcal{E}^*$ the (multiplicative) sheaf of non-vanishing C^∞-smooth functions on X. We have the following exact sequence of sheafs over X

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{E} \xrightarrow{\exp} \mathcal{E}^* \longrightarrow 0 \ ,$$

where exp is the map $f \longmapsto e^{2\pi i f}$. The corresponding long exact sequence of sheaf cohomology have the form

$$\ldots \longrightarrow H^1(X,\mathcal{E}) \longrightarrow H^1(X,\mathcal{E}^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{E}) \longrightarrow \ldots \ .$$

The cohomology $H^1(X,\mathcal{E}^*)$ can be identified with the set of isomorphism classes of complex line bundles on X, and the map $c_1 : H^1(X,\mathcal{E}^*) \longrightarrow H^2(X,\mathbb{Z})$ assigns to a complex line bundle E its 1st Chern class $c_1(E)$. Since the sheaf $\mathcal{E}$ is fine, the cohomologies on the extreme left and extreme right in the above exact sequence vanish, i.e.

$$H^1(X, \mathcal{E}) = H^2(X, \mathcal{E}) = 0 \ ,$$

and it follows that $c_1 : H^1(X, \mathcal{E}^*) \longrightarrow H^2(X, \mathbb{Z})$ is an isomorphism. Recall that the cohomology class $[\omega/2\pi]$ is integral, i.e. $[\omega/2\pi] \in H^2(X, \mathbb{Z})$. Hence, there exists a complex line bundle $L \to X$ with $c_1(L) = [\omega/2\pi]$.

We prove next that the S^1-bundle $L \to X$, constructed in the above Proposition, is uniquely defined.

Proposition 16. *If L and L' are two S^1-bundles over X with connections ∇ and ∇', having the same curvature ω, then there exists a fibrewise isomorphism $\psi : L \to L'$ such that*

$$\psi^* \nabla' = \nabla \ .$$

Such an isomorphism ψ is determined uniquely up to multiplication by an element of S^1.

Proof. Suppose that the bundle L is given by the transition functions $\{\varphi_{ab}\}$ with respect to the covering $\{U_a\}$ of the manifold X, and the bundle L' is given by the transition functions $\{\varphi'_{ab}\}$ with respect to the same covering. If $\psi : L \to L'$ is the required isomorphism, then it should be given locally by functions $\psi_a : U_a \to S^1$, such that

$$\psi_b \varphi_{ab} = \varphi'_{ab} \psi_a \tag{8.1}$$

on double intersections $U_{ab} = U_a \cap U_b$. The condition $\psi^* \nabla' = \nabla$ in terms of local representatives ∇_a, ∇'_a of connections ∇, ∇' means that

$$\nabla'_a = \nabla_a + i \psi_a^{-1} d\psi_a \ . \tag{8.2}$$

We shall construct now the isomorphism ψ, having the required properties. Since $d(\nabla'_a - \nabla_a) \equiv 0$ on U_a, there exist functions $\phi_a : U_a \to \mathbb{R}$ such that

$$d\phi_a = \nabla'_a - \nabla_a \ .$$

The local representatives of connections ∇, ∇' satisfy on double intersections U_{ab} the relations

$$\nabla_b - \nabla_a = i\varphi_{ab}^{-1} d\varphi_{ab} \ , \quad \nabla'_b - \nabla'_a = i{\varphi'}_{ab}^{-1} d\varphi'_{ab} \ , \tag{8.3}$$

which imply that

$$d\varphi_b - d\varphi_a = i d \ln \varphi'_{ab} - i d \ln \varphi_{ab} \iff -id(\varphi_b - \varphi_a) = d \ln \frac{\varphi'_{ab}}{\varphi'_{ab}} \ .$$

Hence

$$d(e^{-i\varphi_b} \varphi_{ab}) = d(\varphi'_{ab} e^{-i\varphi_a}) \ ,$$

whence

$$e^{-i\varphi_b} \varphi_{ab} = \varphi'_{ab} e^{-i\varphi_a} e^{i\mu_{ab}} \quad \text{on} \quad U_{ab} \ ,$$

where μ_{ab} is a real number.

The numbers $\{\mu_{ab}\}$ define a Čech 1-cocycle on X, hence, due to the simply connectedness of X, we can find real numbers $\{m_a\}$, such that $\mu_{ab} = m_b - m_a$. Then the functions

$$\psi_a = e^{-i(\varphi_a + m_a)}$$

satisfy the properties (8.1), (8.2), and so determine the required isomorphism $\psi : L \to L'$.

We analyze next the uniqueness of the constructed isomorphism. Suppose that there exists another isomorphism ψ' of the same type, given by local representatives $\{\psi'_a\}$ with respect to the covering $\{U_a\}$. The relations (8.1) imply that

$$\psi_b \varphi_{ab} = \varphi'_{ab} \psi_a \ , \quad \psi'_b \varphi_{ab} = \varphi'_{ab} \psi'_a \ ,$$

whence

$$\psi_b (\psi'_b)^{-1} = \psi_a (\psi'_a)^{-1} =: h$$

for all a, b, i.e. the local representatives $\{\psi_a\}$ and $\{\psi'_a\}$ differ by a global function $h : X \to S^1$. Then the relations (8.2) imply that

$$d\varphi_a = \nabla'_a - \nabla_a = i\psi_a^{-1} d\psi_a \ , \tag{8.4}$$

$$d\varphi_a = i\psi_a^{-1} h^{-1} \cdot h \, d\psi_a + i\psi_a^{-1} h^{-1} \, dh \psi_a \ , \tag{8.5}$$

whence $dh = 0$, i.e. $h = \text{const}$. □

Using these Propositions, we can construct for a group $\mathcal{G}$, acting by smooth transformations on X, its central extension $\tilde{\mathcal{G}}$, acting on the bundle $L \to X$. We assume again that we are given with a closed $\mathcal{G}$-invariant 2-form ω on X, such that $\omega/2\pi$ is an integral form. Then, by Prop. 15, we can construct the S^1-bundle $L \to X$ with the connection ∇, having the curvature, equal to ω.

Consider for a given $g \in \mathcal{G}$ the pull-back g^*L of L under the action of g and provide it with the connection $\nabla_g = g^*\nabla$, having the curvature $g^*\omega = \omega$ (recall that ω is invariant under $\mathcal{G}$). According to Prop. 16, there exists an isomorphism $\psi : L \to g^*L$ such that

$$\psi^* \nabla_g = \psi^* g^* \nabla = \nabla \ .$$

We define $\tilde{\mathcal{G}}$ as a group, consisting of all pairs (g, ψ), where $g \in \mathcal{G}$ and ψ is an isomorphism $L \to g^*L$, for which $\psi^*\nabla_g = \psi^* g^* \nabla = \nabla$. Or, equivalently, we can define $\tilde{\mathcal{G}}$ as a group, consisting of pairs (g, φ), where $g \in \mathcal{G}$ and $\varphi : L \to L$ is a fibrewise isomorphism, covering the action of g on X, and having the property that $\varphi^*\nabla = \nabla$. Note that the fibrewise map $\varphi : L \to L$ of the above type, covering the action of g on X, is uniquely determined by the element g and the image $\varphi(\lambda_0)$ of an arbitrary fixed point $\lambda_0 \in L_{x_0}$, $x_0 \in X$.

8.2 Central extensions of loop algebras and groups

Consider first *central extensions of the loop algebra* $L\mathfrak{g}$. As we have pointed out in Sec. 4.1, any such extension is determined by a cocycle $\omega \in H^2(L\mathfrak{g}, \mathbb{R})$, or, in other words, by a closed bilinear skew-symmetric form $\omega : L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{R}$. We can assume, according to Rem. 11 at the end of Sec. 7.2, that the form ω is invariant under the adjoint action of the group G. Any such form on $L\mathfrak{g}$, according to Prop. 14 from Sec. 7.2, in the case of a semisimple Lie group G is given by the formula

$$\omega(\xi,\eta) = \omega_0(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} < \xi(e^{i\theta}), \eta'(e^{i\theta}) > d\theta \ , \quad \xi,\eta \in L\mathfrak{g} \ ,$$

where $< \cdot , \cdot >$ is an invariant inner product on the Lie algebra $\mathfrak{g}$. This yields a description of all central extensions $\widetilde{L\mathfrak{g}}$ (in the case of a semisimple group G) in terms of the cohomology $H^3(\mathfrak{g},\mathbb{R})$ (cf. Ex. 28 in Sec. 4.2).

However, not every central extension of the loop algebra $L\mathfrak{g}$ generates a central extension of the loop group LG, even in the case of a simply connected group G. For that the form ω should be integral in the sense of the definition, given in the beginning of Sec. 8.1. More precisely, the following theorem is true.

Theorem 9 (Pressley–Segal [65], Theor. 4.4.1). *If the Lie group G is simply connected, then a central extension $\widetilde{L\mathfrak{g}}$ of the loop algebra $L\mathfrak{g}$ is associated with some central extension $\widetilde{LG}$ of the loop group LG if and only if the corresponding form $\omega/2\pi$ on LG (where ω is the cocycle of the central extension $\widetilde{L\mathfrak{g}}$) is an integral form. In this case the central extension $\widetilde{LG}$ is uniquely determined by the cocycle ω.*

Proof. The sufficiency of the integrality condition of the form $\omega/2\pi$ follows from the argument in the previous Section (cf. Prop. 15). Namely, we apply the construction of Prop. 15 to the case, when the group $\mathcal{G}$ is the loop group LG and the manifold X coincides also with LG. According to Sec. 8.1, we can construct for an integral form $\omega/2\pi$ a complex line bundle L over LG with a connection ∇, having the curvature, equal to ω. Then we define the central extension $\widetilde{LG}$ as the group of bundle automorphisms of L, covering left translations of LG by elements of LG.

We prove the necessity of the integrality condition in the general setting of Sec. 8.1. If a central extension

$$1 \to S^1 \to \tilde{\mathcal{G}} \to \mathcal{G} \to 1$$

of a Lie group $\mathcal{G}$ is generated by a cocycle ω on the Lie algebra $\mathfrak{G}$, then the form $\omega/2\pi$ represents the 1st Chern class of a complex line bundle over X, associated with S^1-bundle $\tilde{\mathcal{G}} \to \mathcal{G}$. Hence, it must be integral.

It remains to prove the uniqueness of the central extension $\tilde{\mathcal{G}}$ of $\mathcal{G}$, corresponding to the cocycle ω. We note first that a central extension $\tilde{\mathcal{G}} \to \mathcal{G}$ is trivial, if the corresponding cocycle ω is trivial. Indeed, in this case the principal S^1-bundle $\tilde{\mathcal{G}} \to \mathcal{G}$ has a flat connection. So we can define a splitting homomorphism $\sigma : \mathcal{G} \to \tilde{\mathcal{G}}$ by associating with an element $g \in \mathcal{G}$ the end-point of a horizontal lift of any path in $\mathcal{G}$, connecting $e \in \mathcal{G}$ with g (recall that $\mathcal{G}$ is simply connected). To prove the uniqueness in the general case, suppose that there are two central extensions $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}'$ of $\mathcal{G}$, corresponding to the same cocycle ω. Then from the two principal S^1-bundles $p : \tilde{\mathcal{G}} \to \mathcal{G}$ and $p' : \tilde{\mathcal{G}}' \to \mathcal{G}$ we can form a "difference" bundle $p'' : \tilde{\mathcal{G}}'' \to \mathcal{G}$, which is a central extension of $\mathcal{G}$, corresponding to the trivial cocycle. To define $\tilde{\mathcal{G}}''$, we first pull back $\tilde{\mathcal{G}}'$ to $\tilde{\mathcal{G}}$ by p to get a subbundle $p^*(\tilde{\mathcal{G}}')$ of the fibre product $\tilde{\mathcal{G}} \times_{\mathcal{G}} \tilde{\mathcal{G}}'$. The circle S^1 is mapped into $\tilde{\mathcal{G}} \times_{\mathcal{G}} \tilde{\mathcal{G}}'$ by the homomorphism $u \mapsto (u, u^{-1})$. We define $\tilde{\mathcal{G}}''$ as the quotient of $p^*(\tilde{\mathcal{G}}')$ by the image of this homomorphism. Now, as we have proved, the difference extension $\tilde{\mathcal{G}}''$ should be trivial, which implies that both central extensions $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}'$ are equivalent. □

Remark 17. Let us discuss in more detail the integrality condition of the form $\omega/2\pi$, required in the above Theorem. We have pointed out earlier in Sec. 7.2 that the form ω is uniquely determined by the choice of an invariant inner product on the semisimple Lie algebra $\mathfrak{g}$. If this algebra $\mathfrak{g}$ is simple, then all invariant inner products on it are proportional to each other and among those, satisfying the integrality condition, there exists a minimal one. It is called the *basic inner product* and the corresponding central extension is called the *basic central extension* of the loop group LG. The Killing form on $\mathfrak{g}$ satisfies the integrality condition and so is an integer multiple of the basic inner product. (The corresponding integer proportionality coefficient in the case of a simply laced group G coincides with the Coxeter number of G.)

The integrality condition can be also formulated in terms of co-roots $\alpha^\vee$ of the algebra $\mathfrak{g}$. Namely, the *form $\omega/2\pi$ is integral if and only if the inner product $(\alpha^\vee, \alpha^\vee)$ is an even number for all co-roots $\alpha^\vee$ of the algebra* $\mathfrak{g}$ (cf. [65], Sec. 4.4).

Remark 18 ([65], Sec. 4.11). At the end of Sec. 4.2 we have remarked that in the case of the loop algebra $L\mathfrak{g}$ there is an isomorphism

$$H^q(L\mathfrak{g}) = H^q(L\mathfrak{g}, \mathbb{R}) \longrightarrow H^q_{\text{top}}(LG, \mathbb{R}) \ .$$

This isomorphism can be used for the computation of the *cohomologies of the loop algebra* $L\mathfrak{g}$. Namely, since LG is diffeomorphic to $\Omega G \times G$, the cohomologies $H^*_{\text{top}}(LG, \mathbb{R})$ coincide with the tensor product of cohomologies $H^*_{\text{top}}(\Omega G, \mathbb{R}) \otimes H^*_{\text{top}}(G, \mathbb{R})$.

But in the case of a compact Lie group G, as we have pointed out in Sec. 4.2, we have

$$H^*_{\text{top}}(G, \mathbb{R}) \cong H^*(\mathfrak{g}) \ .$$

The cohomologies $H^*(\mathfrak{g})$ form an exterior algebra with r generators of odd-dimensional degrees, where r is the rank of G, and the generators correspond to generators of the algebra of invariant polynomials on $\mathfrak{g}$. By this correspondence we associate with an invariant polynomial of degree k a symmetric k-linear function $P : \mathfrak{g}\times\cdots\times\mathfrak{g} \to \mathbb{R}$, and use this function to define a skew-symmetric form S of degree $2k-1$, having the form

$$\begin{aligned} &S(\xi_1, \dots, \xi_{2k-1}) = \\ &= \sum_\sigma (-1)^{\text{sgn}(\sigma)} P([\xi_{\sigma(1)}, \xi_{\sigma(2)}], [\xi_{\sigma(3)}, \xi_{\sigma(4)}], \dots, [\xi_{\sigma(2k-3)}, \xi_{\sigma(2k-2)}], \xi_{\sigma(2k-1)}) \ , \end{aligned} \tag{8.6}$$

where the summation in the formula (8.6) is taken over all possible permutations σ of the set $\{1, 2, \dots, 2k-1\}$. If, in particular, $G = \mathrm{U}(n)$, then one can choose for generators of the algebra of invariant polynomials the functions $P_1, \dots, P_n$ with $P_j(A) = \text{tr}(A^j)$.

The de Rham cohomologies $H^*_{\text{top}}(\Omega G, \mathbb{R})$ (in the case of a simply connected group G) may be computed in terms of the cohomologies $H^*_{\text{top}}(G, \mathbb{R})$. Namely, the cohomologies $H^*_{\text{top}}(\Omega G, \mathbb{R})$ form an algebra, generated by polynomials of even-dimensional degrees, obtained from generators of the algebra $H^*_{\text{top}}(G, \mathbb{R})$ with the help of the transgression map. More precisely, consider the evaluation map

$$S^1 \times \Omega G \longrightarrow G \ , \quad (\theta, \gamma) \longmapsto \gamma(\theta) \in G \ .$$

The differential forms on G, which are the generators of the algebra $H^*_{\text{top}}(G, \mathbb{R})$, may be first pulled up to $S^1 \times \Omega G$ by the evaluation map, and then integrated over S^1. The obtained even-dimensional classes generate the algebra $H^*_{\text{top}}(\Omega G, \mathbb{R})$. More precisely, the image of the $(2k-1)$-form S from the formula (8.6) under the described transgression map coincides with a $(2k-2)$-form Σ on ΩG, which value at a point $\gamma \in \Omega G$ on vectors $\xi_1, \ldots, \xi_{2k-2} \in \Omega\mathfrak{g}$ is equal to

$$\Sigma_\gamma(\xi_1, \ldots, \xi_{2k-2}) = \frac{1}{2\pi} \int_0^{2\pi} S(\xi_1(\theta), \ldots, \xi_{2k-2}(\theta), \gamma(\theta)^{-1}\gamma'(\theta))\, d\theta \ .$$

8.3 Coadjoint representation of loop groups

To describe the *coadjoint representation of the loop group LG* of a compact Lie group G, we fix an invariant inner product $< \cdot , \cdot >$ on the Lie algebra $\mathfrak{g}$. It generates an inner product on the loop algebra $L\mathfrak{g}$ by the formula

$$< \xi, \eta > := \frac{1}{2\pi} \int_0^{2\pi} < \xi(\theta), \eta(\theta) > \, d\theta \ , \quad \xi, \eta \in L\mathfrak{g} \ .$$

The adjoint action of the loop algebra $L\mathfrak{g}$ on the central extension $\widetilde{L\mathfrak{g}}$ of $L\mathfrak{g}$, determined by a cocycle $\omega(\xi, \eta)$, is given by the formula

$$\eta \cdot (\xi, s) := ([\eta, \xi], \omega(\eta, \xi)) \ ,$$

where $\eta \in L\mathfrak{g}$, $(\xi, s) \in \widetilde{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{R}$. It is generated by the adjoint action of the group LG on $\widetilde{L\mathfrak{g}}$, defined by the formula

$$\gamma \cdot (\xi, s) = (\gamma \cdot \xi, s - < \gamma^{-1}\gamma', \xi >) \ ,$$

where $\gamma \in LG$, $(\xi, s) \in \widetilde{L\mathfrak{g}}$ and $\gamma \cdot \xi$ denotes the (pointwise) adjoint action of the loop group LG on its Lie algebra $L\mathfrak{g}$.

Consider the coadjoint action of the loop group LG on the dual space $(\widetilde{L\mathfrak{g}})^*$. We note, first of all, that the dual space of the Frechet space $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$ coincides with the space

$$(L\mathfrak{g})^* = \mathcal{D}'(S^1, \mathfrak{g}^*) = \mathcal{D}'(S^1) \otimes \mathfrak{g}^* \ ,$$

i.e. with the space of distributions on S^1 with values in $\mathfrak{g}^*$. Using the invariant inner product on the Lie algebra $\mathfrak{g}$, we can identify this space with the space of distributions on S^1 with values in the Lie algebra $\mathfrak{g}$. Under this identification, the "smooth" part of $(L\mathfrak{g})^*$, consisting of regular distributions in $(L\mathfrak{g})^*$, corresponds to the space $L\mathfrak{g}^* = C^\infty(S^1, \mathfrak{g}^*)$ or the space $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$.

We describe first the coadjoint action of the loop group LG on the smooth part of $(\widetilde{L\mathfrak{g}})^* = (L\mathfrak{g})^* \oplus \mathbb{R}$, which coincides with $L\mathfrak{g}^* \oplus \mathbb{R} \cong L\mathfrak{g} \oplus \mathbb{R}$. It is given by the formula

$$\gamma \cdot (\varphi, s) = (\gamma \cdot \varphi + s\gamma'\gamma^{-1}, s) \ ,$$

where $\gamma \in LG$, $(\varphi, s) \in L\mathfrak{g} \oplus \mathbb{R}$, and $\gamma \cdot \varphi$ denotes, as above, the adjoint action of the loop group LG on its Lie algebra $L\mathfrak{g}$. It's easy to see that the map $S(\gamma) := \gamma'\gamma^{-1} \in$

$L\mathfrak{g}$ defines a 1-cocycle in the space $C^1(LG, L\mathfrak{g})$ of 1-cochains on LG with values in $L\mathfrak{g}$, i.e. it satisfies the relation

$$S(\gamma_1\gamma_2) = \gamma_1 \cdot S(\gamma_2) + S(\gamma_1) \ .$$

We describe now the orbits of regular elements (φ, s) from $(L\mathfrak{g})^* \oplus \mathbb{R}$ under the action of the loop group LG. For that note that any element $(\varphi, s) \in L\mathfrak{g}\times\{s\}$, $s \neq 0$, is uniquely determined by a path $\psi : \mathbb{R} \to G$, satisfying the ordinary differential equation

$$\frac{d\psi}{dt}\psi^{-1} = -\frac{\varphi}{s} \iff \frac{d\ln\psi}{dt} = -\frac{\varphi}{s}$$

with the initial condition $\psi(0) = e$. It follows from the periodicity of φ with respect to θ that the shifted $\psi(\theta+2\pi)$ is also a solution of this equation together with $\psi(\theta)$. From the uniqueness theorem we obtain that

$$\psi(\theta + 2\pi) = \psi(\theta)M_\varphi \ ,$$

where the monodromy M_φ is defined by $M_\varphi := \psi(2\pi)$.

The coadjoint action of $\gamma \in LG$ on a regular element $(\varphi, s) \in L\mathfrak{g} \times \{s\}$ in terms of ψ corresponds to

$$\begin{array}{ccc}
\varphi & \xrightarrow{\gamma} & \widetilde{\varphi} = \gamma \cdot \varphi + s\gamma'\gamma^{-1} \\
\downarrow & & \downarrow \\
\psi & \xrightarrow{\gamma} & \widetilde{\psi}(\theta) = \gamma(\theta)\psi(\theta)\gamma(0)^{-1} \\
\downarrow & & \downarrow \\
M_\varphi & \longrightarrow & M_{\widetilde{\varphi}} = \gamma(0)M_\varphi\gamma(0)^{-1}
\end{array}$$

i.e. the coadjoint action of γ on $(\widetilde{L\mathfrak{g}})^*$ generates (in terms of the monodromy M_φ) an inner automorphism of the group G. Hence, we obtain a *1–1 correspondence between the orbits of regular elements of* $(L\mathfrak{g})^* \times \{s\}$ *with respect to the coadjoint action of the loop group LG and the conjugacy classes of elements* M_φ *in the group G*. Under this correspondence the isotropy subgroup of an element (φ, s) in the loop group LG corresponds to the centralizer of the monodromy M_φ in the group G.

We note that the orbit of an element (φ, s) is integral, if s is an integer and the corresponding conjugacy class of the monodromy $M \in G$ has the following property. The centralizer of M is a maximal torus T in G (with the Lie algebra $\mathfrak{t}$), in which terms M can be written in the form: $M = \exp\frac{\xi}{s}$ for an element $\xi \in \mathfrak{t} \subset \mathfrak{t}^*$, belonging to the lattice of characters $\widehat{T}$ (cf. [65], Sec. 4.3, for details).

Bibliographic comments

A key reference for this Chapter is the Pressley–Segal book [65]. In particular, the Propositions 15 and 16 are proved in Ch.4 (Prop. 4.5.3) of this book. The Theorem 9 on central extensions of loop groups is contained in Theor. 4.4.1 of [65]. The coadjoint representation of the loop group is described in Sec. 4.3 of [65].

Chapter 9

Grassmann realizations

In this Chapter we introduce the "widest" space of loops, to which the most part of the theory applies, namely, the Sobolev space of "half-differentiable" loops on S^1. This space contains the loop space ΩG, studied in previous sections, as a "smooth" part. In Sec. 9.2 we construct the Grassmann realization of this extended loop space and then apply the same idea to define the Grassmann realization of the "smooth" part ΩG. We end up with the postponed proof of the factorization theorem from Sec. 7.1, using the Grassmann realization of ΩG.

9.1 Sobolev space of half-differentiable loops

We consider first the *Sobolev space of* real-valued *half-differentiable functions* on S^1. This is a Hilbert space

$$V := H_0^{1/2}(S^1, \mathbb{R}) ,$$

which consists of functions $f \in L^2(S^1, \mathbb{R})$ with zero mean value over the circle, having the generalized derivative of order 1/2 in $L^2(S^1, \mathbb{R})$.

It may be shown (cf. [82]) that the Fourier series of a function $f \in H_0^{1/2}(S^1, \mathbb{R})$:

$$f(z) \equiv f(\theta) = \sum_{k \neq 0} f_k z^k \ , \quad f_k = \bar{f}_{-k} \ , \quad z = e^{i\theta} \ ,$$

converges outside a set of zero (logarithmic) capacity and has a finite *Sobolev norm of order* 1/2

$$\|f\|_{1/2}^2 = \sum_{k \neq 0} |k||f_k|^2 = 2 \sum_{k>0} k|f_k|^2 \ .$$

Moreover, by associating with a function $f \in V$ the sequence $\{f_k\}$ of its Fourier coefficients, we establish an isometric isomorphism between the Sobolev space V and the Hilbert space $\ell_2^{1/2}$ of sequences $\{f_k\} \in \ell_2$, satisfying the relations: $f_k = \bar{f}_{-k}$, $f_0 = 0$, and having a finite Sobolev norm: $\sum_{k \neq 0} |k||f_k|^2 < \infty$.

We can consider V as a natural Hilbert extension of the space $\Omega_0 := C_0^\infty(S^1, \mathbb{R})$ of smooth real-valued functions f on S^1, having the zero average over the circle. In terms of their Fourier series, the coefficients f_k of $f \in \Omega_0$ decrease faster than any power k^n with $n \in \mathbb{N}$. In fact, V coincides with the completion of Ω_0 with respect to the Sobolev norm.

The smooth part Ω_0 of V is a Kähler Frechet space, for which a complex and symplectic structures are introduced in the same way, as for the loop space ΩG with a compact Lie group G.

Namely, a *symplectic structure* on Ω_0 is given by the 2-form $\omega : \Omega_0 \times \Omega_0 \to \mathbb{R}$ of the type

$$\omega(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \xi(\theta) d\eta(\theta) \ .$$

In terms of Fourier decompositions of $\xi, \eta \in \Omega_0$:

$$\xi(z) \equiv \xi(\theta) = \sum_{k \neq 0} \xi_k z^k \ , \quad \eta(z) \equiv \eta(\theta) = \sum_{k \neq 0} \eta_k z^k \ , \quad z = e^{i\theta} \ ,$$

this form has the following expression

$$\omega(\xi, \eta) = -i \sum_{k \neq 0} k \xi_k \eta_{-k} = 2\mathrm{Im} \sum_{k > 0} k \xi_k \bar{\eta}_k \ .$$

A *complex structure* operator J^0 on Ω_0 is given by the *Hilbert transform* $J^0 \in \mathrm{End}(\Omega_0, \Omega_0)$, defined by the formula

$$(J^0 \xi)(\theta) = \frac{1}{2\pi} \,\mathrm{P.V.} \int_0^{2\pi} k(\theta, \varphi) \xi(\varphi) d\varphi \tag{9.1}$$

with the kernel

$$k(\theta, \varphi) = \cot \frac{1}{2}(\theta - \varphi)$$

(the integral is taken in the principal value sense). In terms of Fourier decompositions the operator J^0 is given by the formula

$$\xi(z) = \sum_{k \neq 0} \xi_k z^k \longmapsto (J^0 \xi)(z) = -i \sum_{k > 0} \xi_k z^k + i \sum_{k < 0} \xi_k z^k \ .$$

The introduced complex structure J^0 is compatible with the symplectic structure ω and, in particular, defines a *Kähler metric* on Ω_0 by the formula

$$g^0(\xi, \eta) := \omega(\xi, J^0 \eta)$$

or, in terms of Fourier decompositions,

$$g^0(\xi, \eta) = 2\mathrm{Re} \sum_{k > 0} k \xi_k \bar{\eta}_k = \sum_{k \neq 0} |k| \xi_k \bar{\eta}_k \ .$$

So, the space $\Omega_0 = C_0^\infty(S^1, \mathbb{R})$ is provided with the structure of a Kähler Frechet space.

The above definitions of the complex structure J^0 and symplectic structure ω on the space Ω_0 extend to its completion V. (For the complex structure operator J^0 it's evident and for the symplectic structure ω follows immediately from the Cauchy–Schwarz inequality.) So, V has the structure of a *Kähler Hilbert space*, provided with the Kähler metric

$$g^0(\xi, \eta) = \omega(\xi, J^0 \eta) = 2\mathrm{Re} \sum_{k > 0} k \xi_k \bar{\eta}_k = \sum_{k \neq 0} |k| \xi_k \bar{\eta}_k \ .$$

The complexification
$$V^{\mathbb{C}} = H_0^{1/2}(S^1, \mathbb{C})$$
of V is a complex Hilbert space and the Kähler metric g^0 on V extends to a *Hermitian inner product* on $V^{\mathbb{C}}$, given by the formula
$$< \xi, \eta >= \sum_{k\neq 0} |k| \xi_k \bar{\eta}_k \ .$$
We extend the symplectic form ω and the complex structure operator J^0 complex linearly to $V^{\mathbb{C}}$.

The space $V^{\mathbb{C}}$ can be decomposed into the direct sum of subspaces
$$V^{\mathbb{C}} = W_+ \oplus W_- \ ,$$
where $W_\pm$ is the $(\mp i)$-eigenspace of the operator $J^0 \in \operatorname{End} V^{\mathbb{C}}$. In other words,
$$W_+ = \{f \in V^{\mathbb{C}} : \ f(z) = \sum_{k>0} f_k z^k\} \ , \quad W_- = \overline{W}_+ = \{f \in V^{\mathbb{C}} : \ f(z) = \sum_{k<0} f_k z^k\} \ .$$
The subspaces $W_\pm$ are isotropic with respect to the symplectic form ω (i.e. $\omega(\xi, \eta) = 0$, if $\xi, \eta \in W_+$ or $\xi, \eta \in W_-$), and the splitting $V^{\mathbb{C}} = W_+ \oplus W_-$ is an orthogonal direct sum with respect to the Hermitian inner product $< \cdot, \cdot >$. The inner product $< \cdot, \cdot >$ has a simple expression in terms of the decomposition $V^{\mathbb{C}} = W_+ \oplus W_-$:
$$< \xi, \eta >= i\omega(\xi_+, \bar{\eta}_+) - i\omega(\xi_-, \bar{\eta}_-) \ ,$$
where $\xi_\pm$ denotes the projection of $\xi \in V^{\mathbb{C}}$ onto the subspace $W_\pm$.

The operator J^0 in terms of the decomposition $V^{\mathbb{C}} = W_+ \oplus W_-$ has the following matrix representation
$$J^0 \longleftrightarrow \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \ .$$

There is another useful realization of the space V in terms of harmonic functions (cf. [58]). Namely, the space V can be identified with the space $\mathcal{D}$ of (real-valued) harmonic functions F in the unit disc Δ, such that $F(0) = 0$, and the Dirichlet integral
$$E(F) := \frac{1}{2\pi} \int_\Delta \left(\left| \frac{\partial F}{\partial x} \right|^2 + \left| \frac{\partial F}{\partial y} \right|^2 \right) dxdy$$
is finite. In other words, $\mathcal{D}$ is the Hilbert space of harmonic functions on Δ, having their first derivatives in $L^2(\Delta)$ and satisfying the normalization condition $F(0) = 0$. The norm of $F \in \mathcal{D}$ is equal, by definition, to the square root of $E(F)$. A map $V \to \mathcal{D}$, given by the Poisson integral, establishes an isometric isomorphism of Hilbert spaces V and $\mathcal{D}$. The inverse map $\mathcal{D} \to V$ associates with a harmonic function $F \in \mathcal{D}$ its boundary values on $\partial\Delta = S^1$ in the Sobolev sense.

We define next the *Sobolev space* $H^{1/2}(S^1, GL(n, \mathbb{C}))$ *of half-differentiable matrix functions on* S^1. It consists of measurable matrix-valued functions $\gamma : S^1 \to \mathrm{GL}(n, \mathbb{C})$ of the form
$$\gamma = \sum_{k=-\infty}^{\infty} \gamma_k z^k \ , \quad z = e^{i\theta} \ ,$$

with a finite Sobolev norm of order 1/2:

$$\|\gamma\|_{1/2}^2 = \sum_{k=-\infty}^{\infty} |k| \|\gamma_k\|^2 < \infty \ .$$

Accordingly, the space $H\mathrm{GL}(n,\mathbb{C}) := H_0^{1/2}(S^1, \mathrm{GL}(n,\mathbb{C}))$ denotes the subspace of $H^{1/2}(S^1, \mathrm{GL}(n,\mathbb{C}))$, consisting of functions γ with Fourier decompositions of the form

$$\gamma = \sum_{k \neq 0} \gamma_k z^k \ .$$

We define also the group $L_{1/2}(\mathrm{GL}(n,\mathbb{C}))$ of half-differentiable matrix functions. For that we consider the Banach algebra of essentially bounded functions $\gamma \in H^{1/2}(S^1, \mathrm{GL}(n,\mathbb{C}))$, provided with the norm $\|\gamma\|_\infty + \|\gamma\|_{1/2}$. The group of invertible elements in this algebra is called the *group $L_{1/2}(GL(n,\mathbb{C}))$ of half-differentiable matrix functions on S^1*. It is a Banach Lie group.

In the same way one can define the *Sobolev space HG of half-differentiable loops* in a compact Lie group G, realized as a matrix group (i.e. a subgroup of $\mathrm{U}(n)$).

9.2 Grassmann realization

Consider first the Grassmann realization of the group $L_{1/2}(\mathrm{GL}(n,\mathbb{C}))$ of half-differentiable matrix functions on S^1.

Take for a complex Hilbert space H the space $H^{(n)} := L^2(S^1, \mathbb{C}^n)$ with a natural polarization, determined by the subspaces

$$H_+^{(n)} = \{f \in H : f(z) = \sum_{k \geq 0} f_k z^k \text{ with } f_k \in \mathbb{C}^n, z = e^{i\theta}\}$$

and

$$H_-^{(n)} = \{f \in H : f(z) = \sum_{k < 0} f_k z^k \text{ with } f_k \in \mathbb{C}^n\} \ .$$

Associate with a loop $\gamma \in L_{1/2}\mathrm{GL}(n,\mathbb{C})$ the multiplication operator

$$M_\gamma \in \mathrm{End}\, H^{(n)} \ ,$$

which acts on a vector $f \in L^2(S^1, \mathbb{C}^n)$ by the pointwise application of the matrix $\gamma(z) \in \mathrm{GL}(n,\mathbb{C})$ to the vector $f(z) \in \mathbb{C}^n$:

$$(M_\gamma f)(z) := \gamma(z) f(z) \ .$$

Proposition 17. *For any loop $\gamma \in L_{1/2}GL(n,\mathbb{C})$ the multiplication operator M_γ belongs to $GL_{HS}(H^{(n)})$ (cf. Sec. 5.2 for the definition of the Hilbert–Schmidt group GL_{HS}).*

Proof. Let

$$\gamma(z) = \sum_{k \in \mathbb{Z}} \gamma_k z^k \ ,$$

where $\gamma_k \in \mathrm{L}(n,\mathbb{C})$. We choose in $H^{(n)}$ the basis, given by the functions of the form $\epsilon_i z^p$, where $\{\epsilon_i\}$ is a fixed orthonormal basis in $\mathbb{C}^n$, $p \in \mathbb{Z}$. The operator M_γ in this basis has a matrix representation of the form

$$M_\gamma \longleftrightarrow \left(M_{p,q}\right)_{p,q\in\mathbb{Z}} \ , \ \text{where} \quad M_{p,q} = \gamma_{q-p} \in \mathrm{L}(n,\mathbb{C}) \ .$$

For $M_\gamma \in \mathrm{GL}_{\mathrm{HS}}(H^{(n)})$, it's necessary and sufficient that its components, given by the maps

$$M_\gamma^{+-} : H_+^{(n)} \to H_-^{(n)} \quad \text{and} \quad M_\gamma^{-+} : H_-^{(n)} \to H_+^{(n)} \ ,$$

are Hilbert–Schmidt operators. In terms of the matrix representation $(M_{p,q})_{p,q\in\mathbb{Z}}$ it means that the following inequalities should be satisfied

$$\sum_{p\geq 0, q<0} \|M_{p,q}\|^2 < \infty \quad \text{and} \quad \sum_{p<0, q\geq 0} \|M_{p,q}\|^2 < \infty \ .$$

These relations are equivalent to the inequality

$$\sum_{k\in\mathbb{Z}} k\|\gamma_k\|^2 < \infty \ ,$$

which is satisfied if $\gamma \in L_{1/2}\mathrm{GL}(n,\mathbb{C})$. □

The Grassmann realization of the group $L_{1/2}G$ can be constructed in the same way, when G is realized as a matrix group, i.e. a subgroup of $\mathrm{U}(n)$. For example, if G is a compact semisimple Lie group with the trivial centre, it can be identified with the identity component of the automorphism group of its Lie algebra $\mathfrak{g}$. In this case we can choose for H the Hilbert space $L^2(S^1, \mathfrak{g}^{\mathbb{C}})$, on which the loop group $L_{1/2}G$ acts by the adjoint representation. By identifying $\mathfrak{g}^{\mathbb{C}}$ with $\mathbb{C}^n$ (where n is the dimension of the Lie algebra $\mathfrak{g}$) and fixing an invariant inner product on $\mathfrak{g}$, we realize $L_{1/2}G$ as a subgroup of the loop group $L_{1/2}\mathrm{U}(n)$. Then the above embedding of $L_{1/2}\mathrm{GL}(n,\mathbb{C})$ into $\mathrm{GL}_{\mathrm{HS}}(H^{(n)})$ will map $L_{1/2}\mathrm{U}(n)$ into $\mathrm{U}_{\mathrm{HS}}(H^{(n)})$.

We shall describe now the image of the embedding of $L_{1/2}\mathrm{U}(n)$ into $\mathrm{U}_{\mathrm{HS}}(H^{(n)})$, following [65], Sec. 8.3. This embedding defines an action of $L_{1/2}\mathrm{U}(n)$ on $H^{(n)}$ and, hence, on $\mathrm{Gr}_{\mathrm{HS}}(H^{(n)})$. In particular, the image of this action contains the subspaces $W \in \mathrm{Gr}_{\mathrm{HS}}(H^{(n)})$ of the form $M_\gamma(H_+^{(n)}) := \gamma H_+^{(n)}$, where $\gamma \in L_{1/2}\mathrm{U}(n)$. They have the property that $M_z(W) := zW \subset W$, since the action of γ commutes with the multiplication by z. It turns out that the set of such subsets $W \in \mathrm{Gr}_{\mathrm{HS}}(H^{(n)})$ coincides with the image of the action of $L_{1/2}\mathrm{U}(n)$ on $\mathrm{Gr}_{\mathrm{HS}}(H^{(n)})$.

Before we prove this fact, let's introduce some necessary notations. Denote

$$\mathrm{Gr}_+(H^{(n)}) = \{W \in \mathrm{Gr}_{\mathrm{HS}}(H^{(n)}) : zW \subset W\} \ .$$

We also denote, as in Secs. 7.1,7.3, by $L_{1/2}^{\pm}\mathrm{GL}(n,\mathbb{C})$ the subgroups of $L_{1/2}\mathrm{GL}(n,\mathbb{C})$, consisting of loops γ, which are the Sobolev boundary values of holomorphic maps $\gamma : \Delta_\pm \to \mathrm{GL}(n,\mathbb{C})$.

Proposition 18 ([65])**.** *The group $L_{1/2}U(n)$ acts transitively on $Gr_+(H^{(n)})$ and the isotropy subgroup of $H_+^{(n)}$ coincides with the group $U(n)$ of constant loops.*

Proof. The assertion about the isotropy subgroup follows from a well known fact: $\gamma H_+^{(n)} = H_+^{(n)}$ if and only if $\gamma \in L_{1/2}^+\mathrm{GL}(n,\mathbb{C})$. The "if" part is evident. To prove the "only if" part, we decompose γ into the sum $\gamma = \gamma_+ + \gamma_-$ with $\gamma_\pm \in L_{1/2}^\pm\mathrm{gl}(n,\mathbb{C})$ (cf., e.g., [58], Theor. 2.1). Then the equality $\gamma H_+^{(n)} = H_+^{(n)}$ will imply that $\gamma_- H_+^{(n)} \subset H_+^{(n)}$, whence $\gamma_- \in H_+^{(n)}$, i.e. $\gamma_- = 0$. If we know that $\gamma \in L_{1/2}\mathrm{U}(n)$ belongs to $L_{1/2}^+\mathrm{GL}(n,\mathbb{C})$, then, by the symmetry principle, γ extends holomorphically to the whole Riemann sphere, which implies that $\gamma = \text{const}$.

To prove the transitivity of the action of $L_{1/2}\mathrm{U}(n)$ on $\mathrm{Gr}_+(H^{(n)})$, we note first that $W \in \mathrm{Gr}_+(H^{(n)})$ *implies that zW has codimension n in W.* Indeed, consider the commutative diagram

$$\begin{array}{ccc} zW & \longrightarrow & W \\ \downarrow & & \downarrow \\ zH_+^{(n)} & \longrightarrow & H_+^{(n)} \end{array} ,$$

where the horizontal arrows are inclusions and the vertical arrows are orthogonal projections. These projections are Fredholm operators, having their index, equal to the virtual dimension of W. Since the inclusion $zH_+^{(n)} \hookrightarrow H_+^{(n)}$ is evidently Fredholm with the index, equal to $-n$, the same is true for the inclusion $zW \hookrightarrow W$.

We choose now an orthonormal basis $\{w_1, \dots, w_n\}$ in the orthogonal complement of zW in W and form an $(n \times n)$-matrix-valued function γ on S^1 from the vector columns $w_1, \dots, w_n$. We assert that $\gamma(\theta)$ is unitary for almost all $\theta \in S^1$. Indeed, write down $w_k(\theta)$ in the form

$$w_k(\theta) = \sum_p w_{kp} e^{ip\theta} \ , \quad w_{kp} \in \mathbb{C}^n \ .$$

Then

$$< w_k(\theta), w_l(\theta) >= \sum_{p,q} < w_{kp}, w_{lq} > e^{i(q-p)\theta} = \sum_r < w_k, z^r w_l >_H e^{ir\theta} = \delta_{kl} \ ,$$

where we have denoted by $< \cdot, \cdot >_H$ the inner product in $H^{(n)}$ to distinguish it from the inner product $< \cdot, \cdot >$ in $\mathbb{C}^n$. This calculation implies that the *multiplication operator M_γ is unitary in $H^{(n)}$ and*

$$M_\gamma(H_+^{(n)} \ominus z^k H_+^{(n)}) = W \ominus z^k W \quad \text{for any } k \ .$$

It follows also that $M_\gamma(H_+^{(n)}) = W$, since $\bigcap_k z^k W = 0$ (which can be proved by the iteration of the codimension assertion).

It remains to check that $M_\gamma \in \mathrm{U}_{\mathrm{HS}}(H^{(n)})$. But the component M_γ^{+-} of this operator (we are using the same notation, as in the proof of Prop. 17) is factorized into the composition $H_+^{(n)} \to W \to H_-^{(n)}$, where the second map, given by the orthogonal projection, is a Hilbert–Schmidt operator. The same is true for the component M_γ^{-+} of M_γ. □

This proposition implies that the loop space $H\mathrm{U}(n) = L_{1/2}\mathrm{U}(n)/\mathrm{U}(n)$ can be identified with the Grassmanian $\mathrm{Gr}_+(H^{(n)})$. The same proof realizes the space $\Omega\mathrm{U}(n)$ of smooth loops in $\mathrm{U}(n)$ as a "smooth" part $\mathrm{Gr}_+^\infty(H^{(n)})$ of $\mathrm{Gr}_+(H^{(n)})$. Here,

$$\mathrm{Gr}_+^\infty(H^{(n)}) = \mathrm{Gr}^\infty(H^{(n)}) \cap \mathrm{Gr}_+(H^{(n)}) \ ,$$

and the "smooth" part $\mathrm{Gr}^\infty(H^{(n)})$ was introduced at the end of Sec. 5.2. It can be also shown that the group $L\mathrm{U}(n)$ of smooth loops acts smoothly and transitively on the Grassmanian $\mathrm{Gr}^\infty(H^{(n)})$ and the same is true for the action of $L\mathrm{GL}(n,\mathbb{C})$ on $\mathrm{Gr}^\infty(H^{(n)})$.

An embedding of the loop group LG, where G is a simply connected compact Lie group, into $\mathrm{Gr}^\infty_+(H^{(n)})$ can be constructed in a similar way, if one takes for H the Hilbert space $L^2(S^1,\mathfrak{g}^\mathbb{C})$, on which the group LG acts by the adjoint representation. Identifying $\mathfrak{g}^\mathbb{C}$ with $\mathbb{C}^n$ (where n is the dimension of the Lie algebra $\mathfrak{g}$) and fixing an invariant inner product on $\mathfrak{g}$, we can realize LG as a subgroup of $L\mathrm{U}(n)$. The action of $L\mathrm{U}(n)$ on $\mathrm{Gr}^\infty(H^{(n)})$, described above, realizes $L\mathrm{U}(n)$ as a subgroup of $\mathrm{U}_{\mathrm{HS}}(H^{(n)})$. This embedding generates an embedding of the loop space ΩG into the Grassmann manifold $\mathrm{Gr}^\infty(H^{(n)})$.

9.3 Proof of the factorization theorem

The Grassmann realization of the loop space $\Omega\mathrm{U}(n)$, constructed in the previous Section, allows us to give the postponed proof of the factorization theorem 6 from Sec. 7.1. We recall its formulation.

Theorem 10 ([65]). *The product map*

$$\Omega G\times L^+G^\mathbb{C}\longrightarrow LG^\mathbb{C}$$

is a diffeomorphism of Frechet manifolds .

We have pointed out in the proof of Prop. 18 that the complex group $L_{1/2}\mathrm{GL}(n,\mathbb{C})$ acts on the Grassmanian $\mathrm{Gr}_+(H^{(n)})$ and has the stabilizer at $H^{(n)}_+$, equal to the subgroup $L^+_{1/2}\mathrm{GL}(n,\mathbb{C})$. Since the loop group $L_{1/2}\mathrm{GL}(n,\mathbb{C})$ acts transitively on $\mathrm{Gr}_+(H^{(n)})$, we have proved that *the loop group $L_{1/2}GL(n,\mathbb{C})$ coincides with the product*

$$L_{1/2}\mathrm{GL}(n,\mathbb{C})=L_{1/2}\mathrm{U}(n)\cdot L^+_{1/2}\mathrm{GL}(n,\mathbb{C})\ .$$

The same factorization holds for the group $L\mathrm{GL}(n,\mathbb{C})$ of smooth loops. We have to show now that the multiplication map

$$\Omega\mathrm{U}(n)\times L^+\mathrm{GL}(n,\mathbb{C})\longrightarrow L\mathrm{GL}(n,\mathbb{C})$$

is a diffeomorphism. It is sufficient to prove that the map

$$u: L\mathrm{GL}(n,\mathbb{C})\longrightarrow\Omega\mathrm{U}(n)\ ,$$

assigning to a loop γ its unitary component, is smooth. This map is factorized into the composition of two maps: $\gamma\to\tilde\gamma\to u(\gamma)$. The first of them assigns to γ a loop $\tilde\gamma$, which is obtained from γ by projecting the columns $(\gamma_1,\dots,\gamma_n)$ of $\gamma\in L\mathrm{GL}(n,\mathbb{C})$ onto the orthogonal complement $W\ominus zW$ of the subspace zW in W, where $W:=\gamma H^{(n)}_+$. The second map $\tilde\gamma\to u(\gamma)$ consists of the orthonormalization of the basis $\{\tilde\gamma_1,\dots,\tilde\gamma_n\}$ of the subspace $W\ominus zW$. The second map is evidently smooth. The smoothness of the first map follows from the smoothness of the projection map

$$C^\infty(S^1,\mathbb{C}^n)\times\mathrm{Gr}^\infty(H^{(n)})\longrightarrow C^\infty(S^1,\mathbb{C}^n)\ ,$$

assigning to a smooth vector function f on S^1 its orthogonal projection $\mathrm{pr}_W f$ to a given subspace $W\in\mathrm{Gr}_\infty(H^{(n)})$.

Bibliographic comments

A key reference for this Chapter is the Pressley–Segal book [65]. In Sec. 9.1 we study the Sobolev space of half-differentiable loops on S^1. This space in the scalar case is well-known and widely used in the function theory (cf., e.g., [82]). On the other hand, it's role in geometric quantization became clear quite recently, especially after the paper of Nag and Sullivan [58]. The Grassmann realization of the group $L_{1/2}\mathrm{GL}(n,\mathbb{C})$ of half-differentiable matrix functions on S^1 (Prop. 17) is taken from [65], Sec. 6.3. The Prop. 18 is proved in [65], Theor. 8.3.2. The proof of the factorization theorem in Sec. 9.3 is taken from [65], Theor. 8.1.1.