Chapter 8

Examples and extensions

Theorem 2.4.1 gives estimates for operators provided the characteristic roots satisfy certain hypotheses. However, in order to test the validity of such an estimate for an arbitrary linear, constant coefficient m^{th} order strictly hyperbolic operator with lower order terms, it is desirable to find conditions on the structure of the lower order terms under which certain conditions for the characteristic roots hold. For the case $m = 2$, a complete characterisation can be given, and some extension of this is discussed in Section 8.1. However, for large m, it is difficult to do such an analysis, as no explicit formulae are available in general; nevertheless, certain conditions can be found that do make the task of checking the conditions of the characteristic roots, and these are discussed in Section 8.2, where a method is also given that can be used to find many examples. Finally, in Section 8.5, we give a few applications of these results.

8.1 Wave equation with mass and dissipation

As an example of how to use Theorem 2.4.1, here we will show that we can still have time decay of solutions if we allow the negative mass but exclude certain low frequencies for Cauchy data. This is given in (8.1.1) below. In the case of the negative mass and positive dissipation, there is an interplay between them with frequencies that we are going to exhibit. The usual non-negative and also time dependent mass and dissipation with oscillations have been considered before, even with oscillations. See, for example, [HR03] and references therein.

Let us consider second order equations of the following form

$$\begin{cases} \partial_t^2 u - c^2 \Delta u + \delta \partial_t u + \mu u = 0\,, \\ \quad u(0,x) = 0,\ u_t(0,x) = g(x)\,. \end{cases}$$

Here δ is the dissipation and μ is the mass. For simplicity, the first Cauchy data is taken to be zero. The general case when both Cauchy data are non-zero can be treated in a similar way. Let us now apply Theorem 2.4.1 to the analysis of this equation. The associated characteristic polynomial is

$$\tau^2 - c^2|\xi|^2 - i\delta\tau - \mu = 0\,,$$

and it has roots

$$\tau_\pm(\xi) = \frac{i\delta}{2} \pm \sqrt{c^2|\xi|^2 + \mu - \delta^2/4}\,.$$

Now, we have the following well-known cases, which also correspond to different cases of Theorem 2.4.1:

- $\delta = \mu = 0$. This is the wave equation.
- $\delta = 0$, $\mu > 0$. This is the Klein–Gordon equation.
- $\mu = 0$, $\delta > 0$. This is the dissipative wave equation.
- $\delta < 0$. In this case, $\operatorname{Im}\tau_-(\xi) \leq \frac{\delta}{2} < 0$ for all ξ, hence we cannot expect any decay in general.
- $\delta > 0$, $\mu > 0$. In this case the discriminant is always strictly greater than $-\delta^2/4$, and thus the roots always lie in the upper half plane and are separated from the real axis. So we have exponential decay.

  Here is the main case for us, where we can show an interesting interplay between negative mass $\mu < 0$ and how it is compensated by positive dissipation $\delta > 0$ for different frequencies:

- dissipation $\delta \geq 0$, mass $\mu < 0$. In this case, note that $\operatorname{Im}\tau_-(\xi) \geq 0$ if and only if $c^2|\xi|^2 + \mu \geq 0$, i.e. $\operatorname{Im}\tau_-(\xi) = 0$ for $|\xi| = \sqrt{-\mu}/c$. Therefore, the answer depends on the Cauchy data g. In particular, if $\operatorname{supp}\widehat{g}$ is contained in $\{c^2|\xi|^2 + \mu \geq 0\}$, then we may get decay of some type. More precisely, let $B(0,r)$ denote the open ball with radius r centred at the origin. Then we have:
  - if g is such that $\operatorname{supp}\widehat{g} \cap B(0, \frac{\sqrt{-\mu}}{c}) \neq \varnothing$, then we have no decay;
  - if there is some $\epsilon > 0$ such that $\operatorname{supp}\widehat{g} \subset \mathbb{R}^n \setminus B(0, \frac{\sqrt{-\mu}}{c} + \varepsilon)$, then the roots are either separated from the real axis (if $\delta > 0$), and we get exponential decay, or lie on the real axis (if $\delta = 0$), and we get Klein–Gordon type behaviour (since the Hessian of τ is nonsingular).

– if, for all g, $\operatorname{supp}\widehat{g} \subset \mathbb{R}^n \setminus B(0, \frac{\sqrt{-\mu}}{c}) = \left\{|\xi| \geq \frac{\sqrt{-\mu}}{c}\right\}$, then again we must consider $\delta = 0$ and $\delta > 0$ separately.

  If $\delta = 0$, then the roots lie completely on the real axis, and they meet on the sphere $|\xi| = \sqrt{-\mu}/c$. It follows from (2.4.2) (which is justified in Proposition 7.4.1) with $L = 2$ and $\ell = 1$ that, although the representation of solution as a sum of Fourier integrals breaks down at the sphere, the solution is still bounded in a $(1/t)$-neighbourhood of the sphere. In its complement we can get the decay.

  If $\delta > 0$, then the root τ_- comes to the real axis at $|\xi| = \frac{\sqrt{-\mu}}{c}$, in which case we get the decay

$$||u(t,\cdot)||_{L^q} \leq C(1+t)^{-\left(\frac{1}{p}-\frac{1}{q}\right)}||g||_{L^p}. \tag{8.1.1}$$

  Indeed, in this case the order of the root τ_- at the axis is one, i.e. estimate (2.4.1) holds with $s = 1$. Here $1/p + 1/q = 1$ and $1 \leq p \leq 2$. Note also that compared to the case of no mass when $\ell = n$, now the codimension of the sphere $\left\{\xi \in \mathbb{R}^n : |\xi| = \frac{\sqrt{-\mu}}{c}\right\}$ is $\ell = 1$. We can apply the last case of Part II of Theorem 2.4.1 with $L = 1$ and $s = \ell = 1$ which gives estimate (8.1.1).

8.2 Higher order equations

Let us now derive a simple consequence of the stability condition of $\operatorname{Im}\tau_k(\xi) \geq 0$, for all $k = 1, \ldots, m$ and $\xi \in \mathbb{R}^n$, for the coefficient of the $D_t^{m-1}u$ term in (1.0.1). In fact, this coefficient plays an important role for higher order equations and can be compared with the dissipation term in the dissipative wave equation.

Let $L = L(D_t, D_x)$ be an m^{th} order constant coefficient, linear strictly hyperbolic operator such that $\operatorname{Im}\tau_k(\xi) \geq 0$ for all $k = 1, \ldots, m$ and for all $\xi \in \mathbb{R}^n$. Recall that the characteristic polynomial corresponding to the principal part of L is of the form

$$L_m = L_m(\tau, \xi) = \tau^m + \sum_{k=1}^{m} P_k(\xi)\tau^{m-k} = 0\,,$$

where the $P_k(\xi)$ are homogeneous polynomials of order k. Then, by the strict hyperbolicity of L, L_m has real roots $\varphi_1(\xi) \leq \varphi_2(\xi) \leq \cdots \leq \varphi_m(\xi)$ (where the inequalities are strict when $\xi \neq 0$). By the Vièta formulae, observe that

$$P_1(\xi) = -\sum_{k=1}^{m} \varphi_k(\xi) \in \mathbb{R}\,. \tag{8.2.1}$$

On the other hand, the characteristic polynomial of the full operator is

$$L(\tau,\xi) = \tau^m + \sum_{k=1}^{m} P_k(\xi)\tau^{m-k} + \sum_{j=0}^{m-1} \sum_{|\alpha|+l=j} c_{\alpha,l}\xi^\alpha \tau^l = 0\,. \tag{8.2.2}$$

In particular, the coefficient of the τ^{m-1} term is

$$P_1(\xi) + c_{0,m-1} = -\sum_{k=1}^{m} \tau_k(\xi), \tag{8.2.3}$$

where the $\tau_k(\xi)$, $k = 1,\dots,m$ are the roots of (8.2.2). Comparing (8.2.1) and (8.2.3), we see that $\operatorname{Im}\left(\sum_{k=1}^m \tau_k(\xi)\right) = -\operatorname{Im} c_{0,m-1}$. Therefore, since $\operatorname{Im}\tau_k(\xi) \geq 0$ for all $k = 1,\dots,m$ and $\xi \in \mathbb{R}^n$, it follows that $\operatorname{Im} c_{0,m-1} \leq 0$, or, equivalently, $\operatorname{Re} ic_{0,m-1} \geq 0$. Furthermore, if $\operatorname{Im} c_{0,m-1} = 0$ then it must be the case that $\operatorname{Im}\tau_k(\xi) = 0$ for all $\xi \in \mathbb{R}^n$ and $k = 1,\dots,m$ since the characteristic roots are continuous. Hence we have shown the following:

Proposition 8.2.1. *Let $L = L(D_t, D_x)$ be an m^{th} order linear constant coefficient strictly hyperbolic operator such that all the characteristic roots $\tau_k(\xi)$, $k = 1,\dots,m$, satisfy $\operatorname{Im}\tau_k(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$. Then the imaginary part of the coefficient of $D_t^{m-1}u$ is non-positive. Furthermore, if in addition the (imaginary part of the) coefficient of $D_t^{m-1}u$ is zero then each of the characteristic roots lie completely on the real axis.*

If we transform our operator back to the form $L(\partial_t, \partial_x)$, this result tells us that in order for the characteristic polynomial to be stable, that is for $\operatorname{Im}\tau_k(\xi) \geq 0$ for all $k = 1,\dots,m$, $\xi \in \mathbb{R}^n$, it is necessary for the coefficient of $\partial_t^{m-1}u$ to be non-negative; this is the case for the dissipative wave equation. In some sense this may be interpreted as a *higher order dissipation*, since it is necessary for the characteristic roots to behave geometrically like those of the wave equation with a dissipative term, where they lie in the half-plane $\operatorname{Im} z \geq 0$ and lie away from $\operatorname{Im} z = 0$ for large $|\xi|$.

In the next section, we look at the case where characteristic roots must lie completely on the real axis. First, though, let us consider one case where a root lies completely on the real axis but the coefficient $c_{0,m-1}$ is nonzero, $c_{0,m-1} \neq 0$.

Consider a constant coefficient strictly hyperbolic operator of the form

$$L_m(\partial_t, \partial_x) + L_{m-1}(\partial_t, \partial_x) + L_{m-2}(\partial_t, \partial_x) = 0, \tag{8.2.4}$$

where $L_r = L_r(\partial_t, \partial_x)$ denotes a homogeneous operator of degree r with real coefficients. This is an example of a hyperbolic triple, which will be discussed in more generality in Section 8.3. Furthermore, assume that L_{m-1} is not identically zero. Let $\tau(\xi) \in \mathbb{R}$ be a characteristic root of (8.2.4) which lies completely on the real axis. So, denoting as usual $D_{x_j} = -i\partial_{x_j}$, $D_t = -i\partial_t$, we have that $\tau(\xi)$ is a root of

$$L_m(\tau, \xi) - iL_{m-1}(\tau, \xi) - L_{m-2}(\tau, \xi) = 0.$$

This means that $L_{m-1}(\xi, \tau(\xi)) = 0$, and so $\tau(\xi)$ is homogeneous of order 1, and thus for such roots Theorem 2.4.1 applies to yield results similar to those described in Section 1.2.

8.3 Hyperbolic triples

We now turn to the case when all the characteristic roots lie completely on the real axis. This section is devoted to showing some more examples of appearances of real valued non-homogeneous roots and some sufficient conditions for this. In order to study this case we first recall some results of Volevich–Radkevich [VR03] on hyperbolic pairs and triples. Throughout this section only, $L_r(\tau, \xi)$ denotes a homogeneous polynomial in τ and $\xi = (\xi_1, \dots, \xi_n)$ of order r such that $L_r(\tau, i\xi)$ has real coefficients.

Definition 8.3.1. *Suppose $L_m = L_m(\tau, \xi)$ and $L_{m-1} = L_{m-1}(\tau, \xi)$ are homogeneous polynomials as above. Furthermore, assume that the roots of L_m, $\tau_1(\xi), \dots, \tau_m(\xi)$, and those of L_{m-1}, $\sigma_1(\xi), \dots, \sigma_{m-1}(\xi)$, are real-valued (in which case we say L_m and L_{m-1} are* hyperbolic polynomials*). Then, (L_m, L_{m-1}) is called a* hyperbolic pair *if (possibly after reordering)*

$$\tau_1(\xi) \leq \sigma_1(\xi) \leq \tau_2(\xi) \leq \cdots \leq \tau_{m-1}(\xi) \leq \sigma_{m-1}(\xi) \leq \tau_m(\xi). \tag{8.3.1}$$

If, in addition, the roots of L_m, L_{m-1} are pairwise distinct for $\xi \neq 0$ (in which case they are called strictly hyperbolic polynomials*) and the inequalities in (8.3.1) are all strict, then we say (L_m, L_{m-1}) is a* strictly hyperbolic pair*.*

Definition 8.3.2. *Let*

$$L_m = L_m(\tau, \xi)\,,\ L_{m-1} = L_{m-1}(\tau, \xi)\,,\ L_{m-2} = L_{m-2}(\tau, \xi)$$

be (homogeneous) hyperbolic polynomials. If (L_m, L_{m-1}) and (L_{m-1}, L_{m-2}) are both hyperbolic pairs then we say that (L_m, L_{m-1}, L_{m-2}) is a hyperbolic triple*. If, in addition, all the polynomials and all the pairs are strictly hyperbolic (in the sense of Definition 8.3.1) then (L_m, L_{m-1}, L_{m-2}) is called a* strictly hyperbolic triple*.*

Theorem 8.3.3 ([VR03]). *Suppose that (L_m, L_{m-1}, L_{m-2}) is a strictly hyperbolic triple. Then $L_m(\tau,\xi)+L_{m-1}(\tau,\xi)+L_{m-2}(\tau,\xi) \neq 0$ for all $\operatorname{Im}\tau \leq 0$. Furthermore, any two of the polynomials L_m, L_{m-1}, L_{m-2} have no common purely imaginary zeros.*

We also recall a theorem of Hermite (see, for example, [Nis00]):

Theorem 8.3.4. *Suppose $p_m(z)$, $p_{m-1}(z)$ are real polynomials of degree $m, m-1$, respectively, and that all the zeros of $p(z) = p_m(z) - ip_{m-1}(z)$ lie in the upper half-plane (that is, if $p(z) = 0$ then $\operatorname{Im} z > 0$). Then all the zeros of $p_m(z)$ and $p_{m-1}(z)$ are real and distinct.*

Now we will give some rather constructive examples of how non-homogeneous real roots may arise, and some sufficient conditions for this.

Assume that L is of the form $L_m(D_t, D_x) + L_{m-2}(D_t, D_x)$, where the L_r are as in Definition 8.3.2 and neither is identically zero. Suppose that there exists a homogeneous operator of order $m-1$, $L_{m-1}(D_t, D_x)$, such that the characteristic polynomials $L_m(\tau,\xi)$, $L_{m-1}(\tau,\xi)$ and $L_{m-2}(\tau,\xi)$ form a strictly hyperbolic triple. Then, by Theorem 8.3.3, we have

$$L_m(\tau,\xi) + L_{m-1}(\tau,\xi) + L_{m-2}(\tau,\xi) \neq 0 \text{ for } \operatorname{Im}\tau \leq 0\,.$$

Thus, by Theorem 8.3.4, all the zeros of $L_m(\tau,\xi) + L_{m-2}(\tau,\xi)$ are real, but clearly non-homogeneous if $L_{m-2} \not\equiv 0$. So, using this construction, we can obtain examples of operators for which all the characteristic roots lie completely on the imaginary axis (so that $i\tau(\xi)$ are real, which would be the notation for the rest of this paper), but for which we cannot automatically expect the standard decay for homogeneous symbols to hold.

8.4 Strictly hyperbolic systems

Our results can also be used to establish $L^p - L^q$ decay rates for strictly hyperbolic systems. Let us briefly sketch the reduction of systems to the situation covered by results of this paper. Let

$$iU_t = A(D)U\,, \quad U(0) = U_0\,,$$

be an $m \times m$ first order strictly hyperbolic system of partial differential equations. That is, the associated system of polynomials may be written as $A(\xi) = A_1(\xi) + A_0(\xi)$, with A_1 being positively homogeneous of order one in ξ and $A_0(\xi) \in S^0_{1,0}(\mathbb{R}^n)$. If $A(\xi)$ is a matrix of first order polynomials, then A_0 is constant. It is known that $A(D)$ is hyperbolic if and only if

$\det A(D)$ is hyperbolic (see e.g. Atiyah, Bott and Gårding [ABG]). Moreover, if $\det A_1(D)$ is strictly hyperbolic, then $A(D)$ is strongly hyperbolic.

Now, the strict hyperbolicity of the operator $A(D)$ means that the roots $\varphi_1(\xi), \ldots, \varphi_m(\xi)$ of equation $\det(\varphi I - A_1(\xi)) = 0$ are all real and distinct away from the origin. Denote the roots of the equation $\det(\tau I - A(\xi)) = 0$ (which is an m^{th} order polynomial in τ with smooth coefficients) by $\tau_1(\xi), \ldots, \tau_m(\xi)$. Now, by analogy to the case of the m^{th} order scalar equation, we can, via perturbation methods, show that for large $|\xi|$ the $\tau_k(\xi)$ behave similarly to the $\varphi_k(\xi)$, in that they are distinct, analytic and belong to $S^1_{1,0}(\mathbb{R}^n)$. For bounded $|\xi|$ we will need similar regularity assumptions on the characteristic roots $\tau_k(\xi)$ as for the scalar equations. Furthermore, we assume that there exists $Q \in S^0_{1,0}(\mathbb{R}^n)$ such that $|\det Q(\xi)| \geq C > 0$ and such that

$$Q^{-1}AQ = \operatorname{diag}(\tau_1(\xi), \ldots, \tau_m(\xi)) =: T\,.$$

The existence of such Q is a very interesting question itself, especially in the presence of variable multiplicities, but we will not go into such details here. Now, we use the transformation $U = Q(D)V$, so that

$$U_t = QV_t \implies iQV_t = A(D)QV \implies iV_t = TV\;;\; U(0) = QV(0).$$

This systems decouples into m independent scalar equations:

$$\partial_t V_k = \tau_k(D)V_k, \quad k = 1, \ldots, m, \quad V_k(0) = (Q^{-1}U(0))_k$$

each of which is solved by

$$V_k(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + \tau_k(\xi)t)} \widehat{V}_k(0,\xi)\, d\xi\,.$$

Now, $Q \in S^0(\mathbb{R}^n)$, so it is a bounded map $L^q \to L^q$, $1 < q < \infty$, and we can get our estimates for V_k as in the case of m^{th} order scalar equations; thus, we can conclude that

$$\begin{aligned}\|U\|_{L^q} = \|QV\|_{L^q} \leq C\|V\|_{L^q} \\ \leq CK(t)\|V\|_{L^p} = CK(t)\|Q^{-1}U\|_{L^p} \leq CK(t)\|U\|_{L^p}\,,\end{aligned}$$

where $K(t)$ is as in Theorem 2.4.1.

8.5 Application to Fokker–Planck equation

The classical Boltzmann equation for the particle distribution function $f = f(t,x,c)$, where $x, \mathbf{c} \in \mathbb{R}^n$, $n = 1,2,3$, is

$$(\partial_t + \mathbf{c}\cdot\nabla_x)f = S(f),$$

where $S(f)$ is the so-called integral of collisions. The important special case of this equation is the Fokker–Planck equation for the distribution function of particles in Brownian motion, when the integral of collisions is linear and is given by

$$S(f) = \nabla_{\mathbf{c}} \cdot (\mathbf{c} + \nabla_{\mathbf{c}})f = \sum_{k=1}^{n} \partial_{c_k}(c_k + \partial_{c_k})f.$$

In this case the kinetic Fokker–Planck equations takes the form

$$\left(\partial_t + \sum_{k=1}^{n} c_k \partial_{x_k}\right) f(t,x,c) = \sum_{k=1}^{n} \partial_{c_k}(c_k + \partial_{c_k})f.$$

The Hermite–Grad method of dealing with Fokker–Planck equation consists in decomposing $f(t,x,\cdot)$ in the Hermite basis, i.e. writing

$$f(t,x,c) = \sum_{|\alpha|\geq 0} \frac{1}{\alpha!} m_\alpha(t,x)\psi^\alpha(c),$$

where $\psi^\alpha(c) = (2\pi)^{-n/2}(-\partial_c)^\alpha \exp(-\frac{|c|^2}{2})$ are Hermite functions. They are derivatives of the Maxwell distribution ψ^0 which annihilates the integral of collisions and form a complete orthonormal basis in the weighted Hilbert space $L^2_w(\mathbb{R}^n)$ with weight $w = 1/\psi^0$. This decomposition [1] yields the infinite system

$$\partial_t m_\beta(t,x) + \beta_k \partial_{x_k} m_{\beta-e_k}(t,x) + \partial_{x_k} m_{\beta+e_k}(t,x) + |\beta| m_\beta(t,x) = 0.$$

The Galerkin approximation f^N of the solution f is

$$f^N(t,x,c) = \sum_{0\leq|\alpha|\leq N} \frac{1}{\alpha!} m_\alpha(t,x)\psi^\alpha(c)\,,$$

with $m(t,x) = \{m_\beta(t,x) : \ 0 \leq |\beta| \leq N\}$ being the unknown function of coefficients. For $m(t,x)$ one obtains the following system of equations

$$D_t m(t,x) + \sum_j A_j D_{x_j} m(t,x) - iBm(t,x) = 0,$$

where B is a diagonal matrix, $B_{\alpha,\beta} = |\alpha|\delta_{\alpha,\beta}$, and the only non-zero elements of the matrix A_j are $a_j^{\alpha-e_j,\alpha} = \alpha_j$, $a_j^{\alpha+e_j,\alpha} = 1$. Hence, the dispersion equation for the system is

$$P(\tau,\xi) \equiv \det(\tau I + \sum_j A_j \xi_j - iB) = 0, \qquad (8.5.1)$$

[1] Thus, the convergence of the series of such decomposition is understood as a convergence of the decomposition with respect to a basis in a Hilbert space.

which we will call the N^{th} Fokker–Planck polynomial, and we have, in particular,

$$P(\tau, 0) = \det(\tau I - iB) = \tau \prod_{j=1}^{N} (\tau - ji)^{\gamma_j}, \tag{8.5.2}$$

for some powers $\gamma_j \geq 0$. Properties of this polynomial $P(\tau, \xi)$ have been extensively studied by Volevich and Radkevich in [VR04], who gave conditions and examples of situations when $\operatorname{Im} \tau_j(\xi) \geq 0$, for all $\xi \neq 0$. They also described more general (necessary) conditions in terms of coefficients of P. See also [VR03, ZR04]. In our situation here we have to take additional care of possible multiple roots, as is done in Theorem 2.3.2.

From formula (8.5.2) it follows in particular that there is a single characteristic root at the origin. Let $M = \prod_{j=1}^{N} j^{\gamma_j}$.

Let us examine the structure of the operator $P(\tau, \xi)$. It is a polynomial of order m which can be written in the form

$$P(\tau, \xi) = \sum_{j=0}^{m} (-i)^{m-j} P_j(\tau, \xi),$$

with P_j being a homogeneous polynomial of order j. Moreover, we have

$$P_0 = 0,\ P_1 = M\tau,\ P_2 = M \sum_{k=2}^{m} \frac{1}{k-1} \tau^2 - M|\xi|^2.$$

The case $n = 1$ was considered in [VR03], where one has $M = N!$

Let $P(\tau(\xi), \xi) = 0$, where $\tau(0) = 0$ is the simple root at the origin. Differentiation with respect to τ yields $\frac{\partial \tau}{\partial \xi}(0) = 0$. Differentiating again we get

$$\frac{\partial^2 \tau}{\partial \xi^2}(0) = 2iI_n.$$

So, for small frequencies we obtain the decomposition

$$\operatorname{Im} \tau(\xi) = 2|\xi|^2 + \ldots + c(\log m)||\xi||^4 + \ldots,$$

where

$$m = 1 + \gamma_1 + \ldots \gamma_N \approx c_n N,$$

and $||\xi||^4$ denotes a fourth order polynomial in ξ. We also easily have a rough estimate for M of the form

$$N^N \preceq M \preceq (N!)^N,\ (n \geq 2).$$

It follows then that for *small* frequencies we get the estimate

$$|m(t,x)| \le C(1+t)^{-n/2} + Ce^{-\varepsilon(N)t},$$

where, in general, it may be that $\varepsilon(N) \to 0$ as $N \to \infty$. For *medium* frequencies we get exponential decay in view of the result of Theorem 2.1.1, also in the case when there are multiple characteristics, where we can use Theorem 2.1.2. Here, there is an additional polynomial growth with respect to time caused by the resolution procedure of Section 7.1, but this is compensated by the exponential decay given by characteristics with strictly positive imaginary part (see Theorem 2.1.2).

Let us discuss the situation with *large* frequencies. For operators of general form, away from points where roots coincide, the roots are analytic. For large $|\xi|$, perturbation arguments of Section 3 give properties of roots $\tau_k(\xi)$ related to $\varphi_k(\xi)$, the characteristics of the principal part. Here $\tau_k(\xi)$ and $\phi_k(\xi)$ are defined as roots of equations $P(\tau,\xi)=0$ and its principal part $P_m(\varphi,\xi)=0$, respectively. Let K be the maximal order of lower order terms. Then we can summarise the following properties of P established in Section 3:

- there are no multiple roots for large ξ;
- $|\partial_\xi^\alpha \tau_k(\xi)| \le C(1+|\xi|)^{1-|\alpha|}$, i.e. $\tau_k \in S^1$;
- the exits φ_k such that $|\partial^\alpha \tau_k(\xi) - \partial^\alpha \varphi_k(\xi)| \le C(1+|\xi|)^{K+1-m-|\alpha|}$, for all $\xi \in \mathbb{R}^n$ and all multi-indices α;
- Since ϕ_k are real-valued, we get $\operatorname{Im}\tau_k \in S^{K+1-m}$. In particular, $\operatorname{Im}\tau_k \in S^0$.

The statements above are obtained by perturbation arguments and rely on the strict hyperbolicity of the principal part. However, this does not have to be the case for polynomials P that we obtain in the Galerkin approximation. Moreover, in general, it might happen that $\operatorname{Im}\tau_k(\xi) \to 0$ as $|\xi| \to \infty$, the case which is discussed in Section 6.8. To avoid these problems we impose the condition of strong stability. First, we will say that $P(\tau,\xi)$ is a *stable* polynomial if its roots $\tau(\xi)$ satisfy $\operatorname{Im}\tau(\xi) \ge 0$ for all $\xi \in \mathbb{R}^n$, and if $\operatorname{Im}\tau(\xi) = 0$ implies $\xi = 0$. Then we will say that $P(\tau,\xi)$ is *strongly stable* if, moreover, $\operatorname{Im}\tau(\xi) = 0$ implies $\xi = 0$ and $\operatorname{Re}\tau(\xi) = 0$, and if its roots $\tau(\xi)$ satisfy $\liminf_{|\xi|\to\infty} \operatorname{Im}\tau(\xi) > 0$. Thus, the condition of strong stability means that the roots $\tau(\xi)$ may become real only at the origin of the complex plane at $\xi = 0$, and that they do not approach the real axis asymptotically for large ξ.

In Section 8.3, as well as in [VR03, VR04], there are several sufficient conditions for the stability of hyperbolic polynomials. In this case we have a consequence of Theorem 2.3.2 and Remark 2.3.3 in the form of estimate (2.3.4):

Corollary 8.5.1. *Let P be a strongly stable polynomial with characteristic roots with non-negative imaginary parts. Let $1 \leq p \leq 2$ and $2 \leq q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then the solution to Cauchy problem (2.0.1) satisfies dispersive estimate (2.3.4), i.e. we have*

$$\left\| D_t^r D_x^\alpha u(t, \cdot) \right\|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{-\frac{n}{s}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{|\alpha|}{s}-\frac{rs_1}{s}} \sum_{j=0}^{m-1} \|f_j\|_{W_p^{N_p+|\alpha|+r-j}} ,$$

with $N_p \geq n(\frac{1}{p} - \frac{1}{q})$ for $1 < p \leq 2$ and $N_1 > n$ for $p = 1$.

From this, we can conclude the following estimates for solution to the Galerkin approximations of Fokker–Planck equation:

Theorem 8.5.2. *If the N^{th} Fokker–Planck polynomial P in (8.5.1) is strongly stable, we have the estimate*

$$||f_N(t,x,c)||_{L^\infty(\mathbb{R}^n_x)L^2_w(\mathbb{R}^n_c)} \leq C(1+t)^{-n/2} + C_N e^{-\epsilon(N)t},$$

with $w = \exp(-|c|^2/2)$ and $\epsilon(N) > 0$.

Here the constant C is independent of N, but, in general, we may have asymptotically that $\epsilon(N) \to 0$ as $N \to \infty$. The validity of the assumption of Theorem 8.5.2 for all N is an open problem.