Chapter 1

Introduction

This work is devoted to the investigation of dispersive and Strichartz estimates for general hyperbolic equations with constant coefficients. The analysis that we carry out is also applicable to hyperbolic systems either by looking at characteristics of the system directly, or first taking the determinant of the system (the dispersion relation).

There are several important motivations for the analysis. First, while hyperbolic equations of the second order (such as the wave equation, dissipative wave equation, Klein–Gordon equation, etc.) are very well studied, relatively little is known about equations of higher orders. At the same time, equations or systems of high orders naturally arise in applications. For example, Grad systems of non-equilibrium gas dynamics, when linearised near an equilibrium point, are examples of large hyperbolic systems with constant coefficients (see e.g. [Rad03], [Rad05]). Here one has to deal with hyperbolic equations of orders 13, 20, etc., depending on the number of moments in the Grad system. Moreover, there are important families of systems of size going to infinity, or even of infinite hyperbolic systems. For example, the Hermite–Grad method for the analysis of the Fokker–Planck equation for the distribution function for particles for the Brownian motion produces an infinite hyperbolic system with constant coefficients. Indeed, making the decomposition in the space of velocities into the Hermite basis, and writing equations for the space-time coefficients produces a hyperbolic system for infinitely many coefficients (see e.g. [VR03], [VR04], [ZR04], and Section 8.5). The Galerkin approximation of this system leads to a family of systems with sizes increasing to infinity. Although explicit calculations are difficult in these situations, the time decay rate of the solution can still be calculated ([Ruzh06]).

One of the main difficulties when dealing with large systems is that unlike in the case of the second order equations, in general characteristics can not be calculated explicitly. This raises a natural problem to look for properties of the equation that determine the decay rates for solutions. On one hand, it becomes clear that one has to look for geometric properties of characteristics that may be responsible for such decay rates. On the other hand, a subsequent problem arises to be able to reduce these properties from some properties of coefficients of the equation.

One encounters several difficulties on this path. One difficulty lies in the absence of general formulae for characteristic roots. For large frequencies one can use perturbation methods to deduce the necessary asymptotic properties of characteristics. However, this approach can not be used for small frequencies, where the situation becomes more subtle. For example, for small frequencies characteristics may become multiple, causing them to become irregular. This means that if we use the usual representation of solutions in terms of Fourier multipliers, phases become irregular, while amplitudes are irregular and blow up. Thus, we will need to carry out the detailed analysis of sets of possible multiplicities using the fact that they are solutions of parameter dependent polynomial equations. Another difficulty for small frequencies is that there exists a genuine interaction between time and frequencies. In the case of homogeneous symbols it can be shown (see e.g. Section 1.2) that time can be taken out of the estimates, after which low frequencies can be ignored since the corresponding operators are smoothing and their estimates are independent of time. In the case of the presence of lower order terms, the time can no longer be eliminated from the estimates, so even small frequencies become large for large times and may influence the resulting estimates.

The purpose of this work is to present a comprehensive analysis of such problems. Despite the difficulties described above, we will be able to determine what geometric properties of characteristic roots are responsible for qualitatively different time decay rates for solutions. Moreover, we will calculate these rates and relate them to geometric properties of equations. This will lead to a comprehensive picture of decay rates and orders in dispersive estimates for hyperbolic equations with constant coefficients. Such estimates lead to Strichartz estimates, for which our analysis will be applied, with further implications for the corresponding semilinear problems.

Thus, in this paper we consider a problem of determining dispersive and Strichartz estimates for general hyperbolic equations with lower order terms. Therefore, we consider the Cauchy problem for general m^{th} order constant coefficient linear strictly hyperbolic equation with solution $u = u(t,x)$:

$$\begin{cases} \overbrace{D_t^m u + \sum_{j=1}^m P_j(D_x) D_t^{m-j} u}^{\text{homogeneous principal part}} + \overbrace{\sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} D_x^\alpha D_t^r u}^{\text{general lower order terms}} = 0, \quad t > 0, \\ D_t^l u(0,x) = f_l(x) \in C_0^\infty(\mathbb{R}^n), \quad l = 0, \dots, m-1, \ x \in \mathbb{R}^n\,, \end{cases} \tag{1.0.1}$$

where $P_j(\xi)$, the polynomial obtained from the operator $P_j(D_x)$ by replacing each D_{x_k} by ξ_k, is a constant coefficient homogeneous polynomial of order j, and the $c_{\alpha,r}$ are (complex) constants. Here, as usual, $\alpha = (\alpha_1, \dots, \alpha_n)$, $D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$, $D_{x_k} = \frac{1}{i}\partial_{x_k}$ and $D_t = \frac{1}{i}\partial_t$. The full symbol of the operator in (1.0.1) will be denoted by

$$L(\tau,\xi) = \tau^m + \sum_{j=1}^m P_j(\xi)\tau^{m-j} + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r}\xi^\alpha\tau^r,$$

where $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. We will always assume that the differential operator in (1.0.1) is hyperbolic, that is for each $\xi \in \mathbb{R}^n$, the symbol of the principal part,

$$L_m(\tau,\xi) = \tau^m + \sum_{j=1}^m P_j(\xi)\tau^{m-j},$$

has m real roots with respect to τ. For simplicity, unless explicitly stated otherwise, we will also assume that the operator in (1.0.1) is *strictly hyperbolic*, that is at each $\xi \in \mathbb{R}^n \backslash \{0\}$, these roots are pairwise distinct. We denote the roots of $L_m(\tau,\xi)$ with respect to τ by $\varphi_1(\xi) \leq \cdots \leq \varphi_m(\xi)$, and if L is strictly hyperbolic the above inequalities are strict for $\xi \neq 0$.

The condition of hyperbolicity arises naturally in the study of the Cauchy problem for linear partial differential operators and it can be shown that it is a necessary condition for C^∞ well-posedness of the problem; this is discussed in [ES92] and [Hör83b], for example. Strict hyperbolicity is sufficient for C^∞ well-posedness of the Cauchy problem for such an operator with any lower order terms; if the operator is only hyperbolic (sometimes called *weakly hyperbolic*) the lower order terms must satisfy additional conditions for C^∞ well-posedness, the so-called *Levi conditions*. For this reason, we only consider strictly hyperbolic operators with lower order terms, since our main interest is to understand the influence of lower order terms on the decay properties of solutions.

The roots of the associated full characteristic polynomial $L(\tau,\xi)$ with respect to τ will be denoted by $\tau_1(\xi), \dots, \tau_m(\xi)$ and referred to as the *characteristic roots* of the full operator. Clearly, if L is a homogeneous operator then the characteristic roots $\tau_k(\xi)$, $k = 1, \ldots, m$, coincide, possibly after reordering, with the roots $\varphi_k(\xi)$, $k = 1, \ldots, m$, of the operator L_m. However, in general there is no natural ordering on the roots $\tau_k(\xi)$ as they may be complex-valued or may intersect.

The analysis here will be based on the properties of characteristic roots $\tau_k(\xi)$. If the problem (1.0.1) is strictly hyperbolic, we can derive their asymptotic properties in a general situation, necessary for our analysis. However, if the problem is only hyperbolic, functions $\tau_k(\xi)$ may develop singularities for large ξ. If this does not happen and we have the necessary information about them, we may drop the strict hyperbolicity assumption. This may be the case in some applications, for example in those arising in the analysis of the Fokker–Planck equation.

We seek *a priori* estimates for the solution $u(t, x)$ to the Cauchy problem (1.0.1), of the type

$$\|D_x^\alpha D_t^r u(t, \cdot)\|_{L^q} \leq K(t) \sum_{l=0}^{m-1} \|f_l\|_{W_p^{N_p - l}}, \tag{1.0.2}$$

where $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $N_p = N_p(\alpha, r)$ is a constant depending on p, α and r, and $K(t)$ is a function to be determined. Here $W_p^{N_p - l}$ is the Sobolev space over L^p with $N_p - l$ (fractional) derivatives.

We note that sometimes, for example in [Trè80], in the definition of a hyperbolic operator the polynomial $L(i\tau, \xi)$ is used as it is better suited to taking the partial Fourier transform in x, corresponding as it does to $L(\partial_t, D_x)$; in this case, one requires the roots with respect to τ to be purely imaginary (in the cases when we will require them to be real). However, the definition that we give above is perhaps more standard, and thus adopted here throughout.

For a hyperbolic equation with real coefficients we note that the constants $c_{\alpha,r}$ satisfy $i^{m-|\alpha|-l} c_{\alpha,r} \in \mathbb{R}$; the equation is written in the form above since our results may be used to study hyperbolic systems, which can be reduced to an m^{th} order equation with complex coefficients.

Most results presented here will apply to operators which are pseudo-differential in x and to hyperbolic systems via their dispersion equation. Moreover, most of results in this paper are in general sharp.

In this work, we place the priority on obtaining a comprehensive collection of estimates for hyperbolic equations with constant coefficients. The case of variable coefficients is also of great interest, but we leave some extensions of our analysis to this case outside the scope of this paper. Let us mention that already in the case of coefficients depending on time, some unpleasant phenomena may happen. For example, already for the second order equations the oscillations in time dependent coefficients may change the time decay rates for solutions to the corresponding Cauchy problem. For example, equations with very fast oscillations, or with increasing coefficients, have been analysed in [RY99, RY00], to mention only a few references. Results even for the wave equations with bounded coefficients may depend on the oscillations in coefficients (see e.g. [ReS05]). At the same time, many results of this paper are stable under time perturbations of coefficients. For example, in the case of equations with homogeneous symbols with time-dependent coefficients with integrable derivative, a comprehensive analysis has been carried out in [MR09], [MR09a], with the case of the wave equation considered in more detail in [MR07]. We will not deal with such questions in this paper. Let us also mention that while dispersive estimates are devoted to $L^p - L^q$ estimates for solutions, $L^p - L^p$ estimates are also of interest. A survey of L^p estimates for general non-degenerate Fourier integral operators and their dependence on the geometry can be found in [Ruzh00] in the case of real-valued phase functions, while operators with complex-valued phase functions have been analysed in [Ruzh01]. L^p–estimates for solutions to some classes of hyperbolic systems with variable multiplicities appeared in [KR07].

Let us now explain the organisation of these notes. In the following parts of the introduction we will review results for second order equations and for equations with homogeneous symbols, as well as give several more motivations for the comprehensive analysis of this paper. In Section 2 we will present results for different types of behaviour of characteristic roots, and also of corresponding phase functions in cases where we can represent solutions in terms of Fourier multipliers. Thus, in Section 2.1 we will present results without and with multiplicities, when roots are separated from the real axis, in which cases we can get exponential decay of solutions. In Section 2.2 we present results for roots with non-degeneracies, in which case we have a variety of conclusions depending on geometric properties of roots. In Section 2.3 we present results for complex roots that become real on some set. A version of this type of statements (although not in the microlocal form used here) partly appeared in [RS05], and those are improved here. In Section 2.4 we summarise the microlocal results and formulate the main theorem on dispersive estimates for general hyperbolic equations with constant coefficients. Theorem 2.4.1 is the main theorem containing a table of results, and the rest of this section is devoted to the explanation and further remarks about this table. In Section 2.5 we will outline our approach, indicating the relations between frequency regions and statements. In Section 2.6 we present results for non-homogeneous equations, as well as formulate corresponding Strichartz estimates with further applications to semilinear equations. In general, we leave such developments outside the scope of this paper since they are quite well understood (see e.g. [KT98]), once the time decay rates are determined (as we will do in Theorem 2.4.1).

The subsequent chapters contain the detailed analysis and proofs. In Section 3 we establish necessary properties of roots of hyperbolic polynomials, as well as carry out the perturbation analysis for large frequencies. In Section 4 we investigate estimates for oscillatory integrals under certain convexity assumptions on the level sets of the phase function. In Section 5 we analyse the corresponding oscillatory integrals without convexity assumption. Section 6 is devoted to dispersive estimates for solutions to the general Cauchy problem, and here we prove various parts of Theorem 2.4.1. Section 7 deals with multiple characteristics. Here we present a procedure for the resolution of multiplicities in the representation of solutions, enabling us to obtain estimates in these cases as well. Section 7.4 is devoted to multiple roots on the real axis. Here, we investigate solutions for frequencies very close to multiplicities (in some shrinking neighborhoods) as well as for larger, but still bounded, frequencies. Here we present several different versions of results dependent on possibly different assumptions. Finally, Section 8 is devoted to examples of the presented analysis with further applications. Thus, in Section 8.1 we deal with second order equations and give examples of how our results can be applied to investigate the interplay between mass, dissipation, and frequencies. Further, in Section 8.2 we discuss some conditions on coefficients of equations, and in Section 8.3 we give examples of non-homogeneous roots in terms of hyperbolic triples and Hermite's theorem. In Section 8.4 we show briefly how the results can be applied for strictly hyperbolic systems. And finally, in Section 8.5 we give an application to the Fokker–Planck equations.

We will denote various constants throughout the paper by the same letter C. Balls with radius R centred at $\xi \in \mathbb{R}^n$ will be denoted by $B_R(\xi)$. We will use the notation $\langle \xi \rangle = \sqrt{1+|\xi|^2}$, $\langle D \rangle = \sqrt{1-\Delta}$ and $|D| = |-\Delta|^{1/2}$. The Sobolev space W^l_p is then defined as the space of measurable functions for which $\langle D \rangle^l f \in L^p(\mathbb{R}^n_x)$.

We will also use the standard notation for the symbol class $S^\mu = S^\mu_{1,0}$, as a space of smooth functions $a = a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying symbolic estimates $|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{\mu-|\alpha|}$, for all $x, \xi \in \mathbb{R}^n$, and all multi-indices α, β.

If function $a = a(\xi)$ is independent of x, we will sometimes also write $a \in S^\mu_{1,0}(U)$ for an open set $U \subset \mathbb{R}^n$, if $a = a(\xi) \in C^\infty(U)$ satisfies $|\partial_\xi^\alpha a(\xi)| \leq C_\alpha(1+|\xi|)^{\mu-|\alpha|}$, for all $\xi \in U$, and all multi-indices α.

1.1 Background

The study of $L^p - L^q$ decay estimates, or Strichartz estimates, for linear evolution equations began in 1970 when Robert Strichartz published two papers, [Str70a] and [Str70b]. He proved that if $u = u(t,x)$ satisfies the Cauchy problem (that is, the initial value problem) for the homogeneous linear wave equation

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) = 0, \quad (t,x) \in \mathbb{R}^n \times (0,\infty)\,, \\ u(0,x) = \phi(x),\ \partial_t u(0,x) = \psi(x), \quad x \in \mathbb{R}^n\,, \end{cases} \tag{1.1.1}$$

where the initial data ϕ and ψ lie in suitable function spaces such as $C_0^\infty(\mathbb{R}^n)$, then the *a priori* estimate

$$\|(u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^q} \le C(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|(\nabla_x \phi, \psi)\|_{W_p^{N_p}} \tag{1.1.2}$$

holds when $n \ge 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p \le 2$ and $N_p \ge n(\frac{1}{p} - \frac{1}{q})$. Using this estimate, Strichartz proved global existence and uniqueness of solutions to the Cauchy problem for nonlinear wave equations with suitable ("small") initial data. This procedure of proving an *a priori* estimate for a linear equation and using it, together with local existence of a nonlinear equation, to prove global existence and uniqueness for a variety of nonlinear evolution equations is now standard; a systematic overview, with examples including the equations of elasticity, Schrödinger equations and heat equations, can be found in [Rac92], or in many other more recent books.

There are two main approaches used in order to prove (1.1.2); firstly, one may write the solution to (1.1.1) using the d'Alembert ($n = 1$), Poisson ($n = 2$) or Kirchhoff ($n = 3$) formulae, and their generalisation to large n,

$$u(t,x) = \begin{cases} \dfrac{1}{\prod_{j=1}^{\frac{n-1}{2}}(2j-1)} \Big[\partial_t (t^{-1}\partial_t)^{\frac{n-3}{2}} \Big(t^{n-1} \displaystyle\fint_{\partial B_t(x)} \phi \, dS\Big) \\ \qquad\qquad + (t^{-1}\partial_t)^{\frac{n-3}{2}} \Big(t^{n-1} \displaystyle\fint_{\partial B_t(x)} \psi \, dS\Big)\Big] \quad (\text{odd } n \ge 3) \\ \dfrac{1}{\prod_{j=1}^{n/2} 2j} \Big[\partial_t (t^{-1}\partial_t)^{\frac{n-2}{2}} \Big(t^n \displaystyle\fint_{B_t(x)} \frac{\phi(y)}{\sqrt{t^2 - |y-x|^2}} \, dy\Big) \\ \qquad\qquad + (t^{-1}\partial_t)^{\frac{n-2}{2}} \Big(t^n \displaystyle\fint_{B_t(x)} \frac{\psi(y)}{\sqrt{t^2 - |y-x|^2}} \, dy\Big)\Big] \quad (\text{even } n)\,, \end{cases}$$

(here $\fint$ stands for the averaged integral; for the derivation of these formulae see, for example, [Ev98]), as is done in [vW71] and [Rac92]. Alternatively, one may write the solution as a sum of Fourier integral operators:

$$u(t,x) = \mathcal{F}^{-1}\Big(\frac{e^{it|\xi|}+e^{-it|\xi|}}{2}\,\widehat{\phi}(\xi) + \frac{e^{it|\xi|}-e^{-it|\xi|}}{2|\xi|}\,\widehat{\psi}(\xi)\Big).$$

This is done in [Str70a], [Bre75] and [Pec76], for example. Using one of these representations for the solution and techniques from either the theory of Fourier integral operators ([Pec76]), Bessel functions ([Str70a]), or standard analysis ([vW71]), the estimate (1.1.2) may be obtained.

Let us now compare the time decay rate for the wave equation with equations with lower order terms. An important example is the Klein–Gordon equation, where $u = u(t,x)$ satisfies the initial value problem

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) + \mu^2 u(t,x) = 0, \quad (t,x) \in \mathbb{R}^n \times (0,\infty)\,, \\ u(0,x) = \phi(x),\ u_t(0,x) = \psi(x), \quad x \in \mathbb{R}^n\,, \end{cases} \tag{1.1.3}$$

where $\phi, \psi \in C_0^\infty(\mathbb{R}^n)$, say, and $\mu \neq 0$ is a constant (representing a *mass term*); then

$$\|(u(t,\cdot), u_t(t,\cdot), \nabla_x u(t,\cdot))\|_{L^q} \leq C(1+t)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|(\nabla_x\phi, \psi)\|_{W_p^{N_p}}, \tag{1.1.4}$$

where p, q, N_p are as before. Comparing (1.1.2) to (1.1.4), we see that the estimate for the solution to the Klein–Gordon equation decays more rapidly. The estimate is proved in [vW71], [Pec76] and [Hör97] in different ways, each suggesting reasons for this improvement: in [vW71], the function

$$v = v(x, x_{n+1}, t) := e^{-i\mu x_{n+1}} u(t,x)\,, \quad x_{n+1} \in \mathbb{R}\,,$$

is defined; using (1.1.3), it is simple to show that v satisfies the wave equation in $\mathbb{R}^{n+1}$, and thus the Strichartz estimate (1.1.2) holds for v, yielding the desired estimate for u. This is elegant, but cannot easily be adapted to other situations due to the importance of the structures of the Klein–Gordon and wave equations for this proof. In [Pec76] and [Hör97], a representation of the solution via Fourier integral operators is used and the stationary phase method then applied in order to obtain estimate (1.1.4).

Another second order problem of interest is the Cauchy problem for the dissipative wave equation,

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) + u_t(t,x) = 0\,, \quad (t,x) \in \mathbb{R}^n \times (0,\infty), \\ u(0,x) = \phi(x),\ u_t(0,x) = \psi(x), \quad x \in \mathbb{R}^n\,, \end{cases} \tag{1.1.5}$$

where $\psi, \phi \in C_0^\infty(\mathbb{R}^n)$, say. In this case,

$$\|\partial_t^r \partial_x^\alpha u(t,\cdot)\|_{L^q} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-r-\frac{|\alpha|}{2}} \|(\phi, \nabla\psi)\|_{W_p^{N_p}}\,, \tag{1.1.6}$$

with some $N_p = N_p(n, \alpha, r)$. This is proved in [Mat77] with a view to showing well-posedness of related semilinear equations. Once again, this estimate (for the solution $u(t,x)$ itself) is better than that for the solution to the wave equation; there is an even greater improvement for higher derivatives of the solution. As before, the proof of this may be done via a representation of the solution using the Fourier transform:

$$u(t,x) = \begin{cases} \mathcal{F}^{-1}\Big(\Big[\dfrac{e^{-t/2}\sinh\left(\frac{t}{2}\sqrt{1-4|\xi|^2}\right)}{\sqrt{1-4|\xi|^2}} + e^{-t/2}\cosh\left(\frac{t}{2}\sqrt{1-4|\xi|^2}\right)\Big]\widehat{\phi}(\xi) \\ \qquad\qquad + \dfrac{2e^{-t/2}\sinh\left(\frac{t}{2}\sqrt{1-4|\xi|^2}\right)}{\sqrt{1-4|\xi|^2}}\widehat{\psi}(\xi)\Big), \quad |\xi| \le 1/2, \\ \mathcal{F}^{-1}\Big(\Big[\dfrac{e^{-t/2}\sin\left(\frac{t}{2}\sqrt{4|\xi|^2-1}\right)}{\sqrt{4|\xi|^2-1}} + e^{-t/2}\cos\left(\frac{t}{2}\sqrt{4|\xi|^2-1}\right)\Big]\widehat{\phi}(\xi) \\ \qquad\qquad + \dfrac{2e^{-t/2}\sin\left(\frac{t}{2}\sqrt{4|\xi|^2-1}\right)}{\sqrt{4|\xi|^2-1}}\widehat{\psi}(\xi)\Big), \quad |\xi| > 1/2. \end{cases}$$

Matsumura divides the phase space into the regions where the solution has different properties and then uses standard techniques from analysis.

It is, therefore, motivating to ask why the addition of lower order terms improves the rate of decay of the solution to the equation; furthermore, in the first instance, we would like to understand why the improvement in the decay is the same for both the addition of a mass term and for the addition of a dissipative term. It will follow from the analysis of the paper that the quantities responsible for the decay rates for the Klein-Gordon and dissipative equations are of completely different nature. In the first instance the characteristic roots are real and lie on the real axis for all frequencies, while for the latter equation they are in the upper complex half-plane, intersect at a point, and one of them comes to the origin. From this point of view, the same decay rates in the dispersive estimate for these two equations is quite a coincidence. On the example of the dissipative equation we can see another difficulty for the analysis, namely the appearance of the multiple roots. This may lead to the loss of regularity in roots and blow-ups in the amplitudes of a representation, so we need to develop some techniques to deal with this type of situations.

These questions are even more important for equations of higher orders. Let us mention briefly an example of a system that arises as the linearisation of the 13–moment Grad system of non-equilibrium gas dynamics in two dimensions (other Grad systems are similar). The dispersion relation (the determinant) of this system is a polynomial of 9^{th} order that can be written as

$$P = Q_9 - iQ_8 - Q_7 + iQ_6 + Q_5 - iQ_4,$$

with polynomials $Q_j(\omega,\xi)$ defined by

$$\begin{aligned}
Q_9(\omega,\xi) =&|\xi|^9\omega^3\left[\omega^6 - \frac{103}{25}\omega^4 + \frac{21}{5}\omega^2\left(1-\frac{912}{2625}\alpha\beta\right) - \frac{27}{25}\left(1-\frac{432}{675}\alpha\beta\right)\right],\\
Q_8(\omega,\xi) =&|\xi|^8\omega^2\left[\frac{13}{3}\omega^6 - \frac{1094}{75}\omega^4 + \frac{1381}{125}\omega^2\left(1-\frac{2032}{6905}\alpha\beta\right) - \frac{264}{125}\left(1-\frac{143}{330}\alpha\beta\right)\right],\\
Q_7(\omega,\xi) =&|\xi|^7\omega\left[\frac{67}{9}\omega^6 - \frac{497}{25}\omega^4 + \frac{3943}{375}\omega^2\left(1-\frac{832}{3943}\alpha\beta\right) - \frac{159}{125}\left(1-\frac{48}{159}\alpha\beta\right)\right],\\
Q_6(\omega,\xi) =&|\xi|^6\left[\frac{19}{3}\omega^6 - \frac{2908}{225}\omega^4 + \frac{13}{3}\omega^2\left(1-\frac{32}{325}\alpha\beta\right) - \frac{6}{25}\right],\\
Q_5(\omega,\xi) =&|\xi|^5\omega\left[\frac{8}{3}\omega^4 - \frac{178}{45}\omega^2 + \frac{2}{3}\right],\\
Q_4(\omega,\xi) =&\frac{4}{9}|\xi|^4\omega^2(\omega^2-1),
\end{aligned}$$

where

$$\omega(\xi) = \frac{\tau(\xi)}{|\xi|},\ \alpha = \frac{\xi_1^2}{|\xi|^2},\ \beta = \frac{\xi_2^2}{|\xi|^2}.$$

A natural question of finding dispersive (and subsequent Strichartz) estimates for the Cauchy problem for operator $P(D_t, D_x)$ with symbol $P(\tau,\xi)$ becomes calculationally complicated. Clearly, in this situation it is hard to find the roots explicitly, and, therefore, we need some procedure of determining what are the general properties of the characteristics roots, and how to derive the time decay rate from these properties. Thus, in [Rad03] and [VR04] it is discussed when such polynomials are stable. In this case, the analysis of this paper will guarantee the decay rate, e.g. by applying Theorem 2.3.2 for frequencies near the origin, Theorem 2.1.2 for bounded frequencies near possible multiplicities (independent of the structure of such multiplicities), and Theorem 2.1.1 for large frequencies. In fact, once the behavior of the characteristic roots is understood, Theorem 2.4.1 will immediately show that the overall time decay rate here is the same as for the dissipative wave equation.

1.2 Homogeneous symbols

The case where the operator in (1.0.1) has homogeneous symbol has been studied extensively:

$$\begin{cases} L_m(D_x, D_t)u = 0, & (t,x)\in\mathbb{R}^n\times(0,\infty),\\ D_t^l u(0,x) = f_l(x), & l=0,\dots,m-1,\ x\in\mathbb{R}^n\,, \end{cases} \tag{1.2.1}$$

where L_m is a homogeneous m^{th} order constant coefficient strictly hyperbolic differential operator; the symbol of L_m may be written in the form

$$L_m(\tau,\xi) = (\tau - \varphi_1(\xi)) \dots (\tau - \varphi_m(\xi)), \text{ with } \varphi_1(\xi) < \dots < \varphi_m(\xi) \quad (\xi \neq 0).$$

In a series of papers, [Sug94], [Sug96] and [Sug98], Sugimoto showed how the geometric properties of the characteristic roots $\varphi_1(\xi), \dots, \varphi_m(\xi)$ affect the $L^p - L^q$ estimate. To understand this, let us summarise the method of approach.

Firstly, the solution can be written as the sum of Fourier multipliers:

$$u(t,x) = \sum_{l=0}^{m-1} [E_l(t) f_l](x), \quad \text{where } E_l(t) = \sum_{k=1}^{m} \mathcal{F}^{-1} e^{it\varphi_k(\xi)} a_{k,l}(\xi) \mathcal{F},$$

and $a_{k,l}(\xi)$ is homogeneous of order $-l$. Now, the problem of finding an $L^p - L^q$ decay estimate for the solution is reduced to showing that operators of the form

$$M_r(D) := \mathcal{F}^{-1} e^{i\varphi(\xi)} |\xi|^{-r} \chi(\xi) \mathcal{F}\,,$$

where $\varphi(\xi) \in C^\omega(\mathbb{R}^n \setminus \{0\})$ is homogeneous of order 1 and $\chi \in C^\infty(\mathbb{R}^n)$ is equal to 1 for large ξ and zero near the origin, are $L^p - L^q$ bounded for suitably large $r \geq l$. In particular, this means that, for such r, we have

$$\|E_l(1) f\|_{L^q} \leq C \|f\|_{W_p^{r-l}}\,.$$

Then it may be assumed, without loss of generality, that $t = 1$. Indeed, it can be readily checked that for $t > 0$ and $f \in C_0^\infty(\mathbb{R}^n)$, we have the equality

$$[E_l(t) f](x) = t^l [E_l(1) f(t\cdot)](t^{-1}x)\,.$$

Using this identity and denoting $f_t(\cdot) = f(t\cdot)$, we have

$$\begin{aligned}\|E_l(t) f\|_{L^q}^q &= t^{lq} \|[E_l(1) f_t](t^{-1}\cdot)\|_{L^q}^q = t^{lq} \int_{\mathbb{R}^n} |[E_l(1) f_t](t^{-1}x)|^q \, dx \\ &\overset{(x=tx')}{=} t^{lq} \int_{\mathbb{R}^n} t^n |[E_l(1) f_t](x')|^q \, dx' = t^{lq+n} \|E_l(1) f_t\|_{L^q}^q\,.\end{aligned}$$

Then, noting that a simple change of variables yields

$$\|f_t\|_{W_p^k}^p \leq C t^{kp-n} \|f\|_{W_p^k}^p\,,$$

we have,

$$\|E_l(t) f\|_{L^q} \leq C t^{l+\frac{n}{q}} \|f_t\|_{W_p^{r-l}} \leq C t^{r-n(\frac{1}{p}-\frac{1}{q})} \|f\|_{W_p^{r-l}}\,;$$

hence,

$$\|u(t,\cdot)\|_{L^q} \le Ct^{r-n(\frac{1}{p}-\frac{1}{q})} \sum_{l=0}^{m-1} \|f_l\|_{W_p^{r-l}} .$$

It has long been known that the values of r for which $M_r(D)$ is $L^p - L^q$ bounded depend on the geometry of the level set

$$\Sigma_\varphi = \{\xi \in \mathbb{R}^n \setminus \{0\} : \varphi(\xi) = 1\} .$$

In [Lit73], [Bre75], it is shown that if the Gaussian curvature of Σ_φ is never zero then $M_r(D)$ is $L^p - L^q$ bounded when $r \ge \frac{n+1}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$. This is extended in [Bre77] where it is proven that $M_r(D)$ is $L^p - L^q$ bounded provided $r \ge \frac{2n-\rho}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$, where $\rho = \min_{\xi\neq 0} \operatorname{rank} \operatorname{Hess} \varphi(\xi)$.

Sugimoto extended this further in [Sug94], where he showed that if Σ_φ is convex then $M_r(D)$ is $L^p - L^q$ bounded when $r \ge \left(n - \frac{n-1}{\gamma(\Sigma_\varphi)}\right)\left(\frac{1}{p} - \frac{1}{q}\right)$; here,

$$\gamma(\Sigma) := \sup_{\sigma\in\Sigma} \sup_{P} \gamma(\Sigma;\sigma,P) , \quad \Sigma \subset \mathbb{R}^n \text{ a hypersurface},$$

where P is a plane containing the normal to Σ at σ and $\gamma(\Sigma;\sigma,P)$ denotes the order of the contact between the line $T_\sigma \cap P$, T_σ is the tangent plane at σ, and the curve $\Sigma \cap P$. See Section 4.3 for more on this maximal order of contact.

In order to apply this result to the solution of (1.2.1), it is necessary to find a condition under which the level sets of the characteristic roots are convex. The following notion is the one that is sufficient:

Definition 1.2.1. *Let $L = L(D_t, D_x)$ be a homogeneous m^{th} order constant coefficient partial differential operator. It is said to satisfy the* convexity condition *if the matrix of the second order derivatives,* $\operatorname{Hess}\varphi_k(\xi)$*, corresponding to each of its characteristic roots $\varphi_1(\xi), \dots, \varphi_m(\xi)$, is semi-definite for $\xi \neq 0$.*

It can be shown that if an operator L does satisfy this convexity condition, then the above results can be applied to the solution and thus an estimate of the form (1.0.2) holds with

$$K(t) = (1+t)^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)} , \quad \text{with some } \gamma \le m , \qquad (1.2.2)$$

where γ can be related to the convex indices of the level sets of characteristics. Indeed, under the convexity condition one can show that ϕ_k can be made always positive or negative by adding an affine function, the corresponding level sets $\Sigma_{\phi_k} = \{\xi \in \mathbb{R}^n : \phi_k(\xi) = 1\}$ are convex for each $k = 1, \dots, m$, and that $\gamma(\Sigma_{\phi_k}) \le 2[m/2]$. So the decay in (1.2.2) is guaranteed with $\gamma = 2[m/2]$.

Finally, if this convexity condition does not hold the estimate fails; in papers [Sug96] and [Sug98] it is shown that in general, $M_r(D)$ is $L^p - L^q$ bounded when $r \geq \left(n - \frac{1}{\gamma_0(\Sigma_\varphi)}\right)\left(\frac{1}{p} - \frac{1}{q}\right)$, where

$$\gamma_0(\Sigma) := \sup_{\sigma \in \Sigma} \inf_P \gamma(\Sigma; \sigma, P) \leq \gamma(\Sigma).$$

For $n = 2$, $\gamma_0(\Sigma) = \gamma(\Sigma)$, so, the convexity condition may be lifted in that case. However, in [Sug96], examples are given when $n \geq 3$, $p = 1, 2$ where this lower bound for r is the best possible and, thus, the convexity condition is necessary for the above estimate. It turns out that the case $n \geq 3$, $1 < p < 2$ is more interesting and is studied in greater depth in [Sug98], where microlocal geometric properties must be looked at in order to obtain an optimal result.

Two remarks are worth making; firstly, the convexity condition result recovers the Strichartz decay estimate for the wave equation, since that clearly satisfies such a condition. Secondly, the convexity condition is an important restriction on the geometry of the characteristic roots that affects the $L^p - L^q$ decay rate; hence, in the case of an m^{th} order operator with lower order terms we must expect some geometrical conditions on the characteristic roots to affect the decay rate of solutions.